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Semicontinuity of the sets of approximate solutions for parametric set optimization problems

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ABSTRACT: The aim of this paper is to study the semicontinuity of the sets of approximate solutions for parametric set optimization problems (PSOPs). We use the generalized Hiriart-Urruty oriented distance function to define a metric in the Hausdorff sense, which allows us to examine the continuity of parametric scalarization functions (PSFs). Furthermore, we explore the relationship between the solution sets for the PSOPs and the parametric equilibrium problems (PEPs). We demonstrate that the weak l-minimal approximate solution to the PSOPs is equivalent to the approximate solution of the PEPs. Finally, the semicontinuity of the solution mappings of the PSOPs is obtained by the scalarization methods.

KEYWORDS: semicontinuity, scalarization function, parametric set optimization problem, parametric equilibrium problem

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INTRODUCTION

A set-valued optimization problems is one in which both the objective and constraint functions are set-valued mappings, and the solution is a set. Since the 1990s, such problems have drawn increasing attention due to their wide applications in optimal control, variational analysis, and vector optimization [1].

The criteria commonly used to solve the above problems are the vectorial criterion and the set criterion. Research on vector optimization methods for set optimization problems (SOPs) is highly developed and has yielded significant results, as highlighted in studies [2, 3]. When set-valued optimization problems use the set criterion, they are referred to as SOPs. Kuroiwa [4] focused on each element of the image set, defining the optimal solution based on comparisons of values within the entire image set. This method relies on comparing the order of sets. There are several set order relations to compare sets, such as the upper set less relation, the lower set less relation and partial order relations based on Minkowski difference, as discussed in [5]. Building on these set relations, Preechasilp and Wangkeeree [6] explored the stability of parametric set optimization problems (PSOPs), PSOPs are a type of SOPs in which both the feasible set and the objective mapping depend on unknown parameters. However, we have noticed that research on the stability of approximate solutions for SOPs is still relatively scarce. In this study, we introduce the lower set less relation introduced in [5] to investigate the stability of approximate solutions for SOPs.

Scalarization methods are essential tools for analyzing the stability of solution mappings in SOPs. Two commonly used types of nonlinear scalarization functions include the extension of the Gerstewitz function and the generalization of the Hiriart-Urruty oriented distance function. The generalized Hiriart-Urruty oriented distance function (GHUODF) in set form has been extensively studied by numerous researchers, and various intriguing properties and conclusions have been obtained. For example, properties such as the translation property, the triangle inequality, and the characterization of the GHUODF in set form via set order relations, among others (see [7–9]). Despite these advancements, the stability of solutions in SOPs remains an active research area, with increasing attention in recent years (see, e.g., [10–15] and references therein).

The PSOPs, which involve perturbations in the feasible set and objective mappings, is of particular interest. The Gerstewitz function in set form, introduced by Hernández and Rodríguez-Marín [16], was used by Han and Huang [12] to study the upper and lower semicontinuity of the strongly approximate solution mappings for the PSOPs. Han [14] defined a Hausdorff-type distance using the Gerstewitz function in set form and established a metric between two set-valued mappings to study the semicontinuity and Lipschitz continuity of l-minimal and weak l-minimal approximate solution mappings for the PSOPs. It is natural to ask whether one can define the Hausdorff-type distance in terms of the GHUODF in set form and use it to investigate the semicontinuity of approximate solution mappings for the PSOPs. Inspired by the work of Han [14], we define the Hausdorff-type distance using the GHUODF in set form, as proposed by Ha [17], and establish the Hausdorff-type metric between two set-valued mappings with respect to the GHUODF of set form. The first aim of this paper is to study the continuity of the GHUODF of set form with parametric.

Furthermore, we will apply the GHUODF of set form to the well-known equilibrium problems [18–20] to study the semicontinuity of the solution mappings of SOPs. Inspired by the work of Anh [20], we consider whether it is possible to establish the equivalence between the solution sets of the parametric equilibrium problems (PEPs) and the PSOPs based on the GHUODF of set form. Unlike the research method of Han [14], we first establish the equivalence relation, and then study the semicontinuity of the solution mappings of the PSOPs by leveraging the semicontinuity of the solution mappings of the PEPs. In this way, studying the semicontinuity of solution mappings of the PSOPs is equivalent to studying the semicontinuity of the solution mappings of the PEPs. Therefore, the second aim of this paper is to study the semicontinuity of the solution mappings of the PSOPs.

PRELIMINARIES

Let X, Y, U and V be real-normed linear spaces. The parametrics λ and μ are crucial to the PSOPs, PEPs and APEPs addressed in the subsequent sections of the paper. K is called a cone in Y if $\lambda x \in K$ for all $x \in K$ and $\lambda \geqslant 0$. The convex cone $K \in Y$ defines a preorder $\preccurlyeq_K$ induced by K for any $x, y \in Y$,

$$x \preccurlyeq_K y \Longleftrightarrow y - x \in K.$$

Assume that Y^* is the dual space of Y, and the dual cone K^* of K is defined as

$$K^* = \{y^* \in Y : \langle y^*, k\rangle \geqslant 0,\ \forall\ k \in K\}.$$

A cone K is convex if $K + K \subseteq K$ and K is solid if $\operatorname{int} K \neq \emptyset$, where $\operatorname{int} K$ is the topological interior of the cone K. We denote the family of nonempty subsets of Y by $P_0(Y)$. Further, for $A \in P_0(Y)$, we denote the complement of A by A^c. $\mathbb{R}^n$ is the n dimensional Euclidean space. Let

$$\begin{aligned} \mathbb{R}^n_+ &= \{x \in \mathbb{R}^n :\ x_i \geqslant 0,\ i = 1, \dots, n\}, \\ \mathbb{R}^n_{++} &= \{x \in \mathbb{R}^n :\ x_i > 0,\ i = 1, \dots, n\}. \end{aligned} \quad (1)$$

It is said that a nonempty set $A \subseteq Y$ is K-proper if $A + K \neq Y$, K-bounded if for each neighbourhood O of zero in Y there is some positive number t such that $A \subseteq tO + K$, K-closed if $A + K$ is a closed set and K-compact if any cover of A of the form $\{O_\alpha + K \mid \alpha \in I,\ O_\alpha \text{ are open}\}$ admits a finite subcover. It has been documented that if A is K-compact, then A is K-bounded and K-closed [21].

Let $A, B \in P_0(Y)$, K be solid, $\varepsilon \geqslant 0$, and $e \in \operatorname{int} K$. We consider the following set relations on Y, the lower relation " $\preccurlyeq^l_K$ ", the weak lower relation " $\prec^l_K$ ", the ε-lower relation " $\preccurlyeq^l_{\varepsilon,K}$ " and the weak ε-lower relation " $\prec^l_{\varepsilon,K}$ ". These different ordering relations can be clearly distinguished in Table 1; see [4, 14].

Let (X, d) be a metric space, A and B be two nonempty subsets of X. The Hausdorff distance between A and B is defined by

$$H(A, B) := \max\{g(A, B), g(B, A)\},$$

where

$$g(A, B) := \sup_{a \in A} d(a, B) \text{ with } d(a, B) = \inf_{b \in B} d(a, b).$$

Let $F : X \rightrightarrows Y$ be a set-valued mapping and S be a nonempty subset in X. We consider the following SOPs:

$$\begin{cases} \min & F(x) \\ \text{s.t.} & x \in S. \end{cases}$$

In the following we retrospect the concept of the solutions for the SOPs with regard to the set order relation "$\prec^l_K$" and "$\preccurlyeq^l_K$".

Definition 1 ([14]) For $\varepsilon \geqslant 0$, an element $x_0 \in S$ is said to be

(i) l-minimal solution of SOPs if, for $x \in S$, $F(x) \preccurlyeq^l_K F(x_0)$ implies $F(x_0) \preccurlyeq^l_K F(x)$.
(ii) weak l-minimal solution of SOPs if, for $x \in S$, $F(x) \prec^l_K F(x_0)$ implies $F(x_0) \prec^l_K F(x)$.
(iii) l-minimal approximate solution of SOPs if, for $x \in S$, $F(x) \preccurlyeq^l_{\varepsilon,K} F(x_0)$ implies $F(x_0) \preccurlyeq^l_{\varepsilon,K} F(x)$.
(iv) weak l-minimal approximate solution of SOPs if, for $x \in S$, $F(x) \prec^l_{\varepsilon,K} F(x_0)$ implies $F(x_0) \prec^l_{\varepsilon,K} F(x)$.

$E_l(F, S)$, $W_l(F, S)$, $E_l(\varepsilon, F, S)$, and $W_l(\varepsilon, F, S)$ are defined as the l-minimal solution set, weak l-minimal solution set, l-minimal approximate solution set, and weak l-minimal approximate solution set of SOPs, respectively. According to [22], for any $\varepsilon \geqslant 0$, it holds that $E_l(\varepsilon, F, S) \subseteq W_l(\varepsilon, F, S)$.

The oriented distance function and its generalization are important tools for studying SOPs. Next, we recall their definitions and related lemmas.

Definition 2 ([23]) Let $A \in P_0(Y)$. A function $\triangle_A : Y \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\triangle_A(y) := d_A(y) - d_{A^c}(y), \quad \forall\ y \in Y,$$

is said to be an oriented distance function, where $d_A(y) := \inf_{a \in A} \parallel y - a \parallel$ is the distance function from $y \in Y$ to the set A.

Lemma 1 ([7, 23]) *Let $A \subseteq Y$ be nonempty and $A \neq Y$. The following assertions hold:*

(i) *$\triangle_A(\cdot)$ is real-valued and 1-Lipschitzian.*
(ii) *If A is a closed convex cone, the $\triangle_{-A}$ nondecreasing with respect to the ordering induced by A, i.e., if y_1, $y_2 \in Y$ and $y_2 - y_1 \in A$, then $\triangle_{-A}(y_1) \leqslant \triangle_{-A}(y_2)$.*

Table 1 Set relations.

	Relation	
ε	Lower relation	Weak lower relation
$\varepsilon=0$	$A \preccurlyeq^l_K B \Longleftrightarrow B \subseteq A+K$	$A \prec^l_K B \Longleftrightarrow B \subseteq A+\operatorname{int}K$
$\varepsilon \geqslant 0$	$A \preccurlyeq^l_{\varepsilon,K} B \Longleftrightarrow B \subseteq A+K+\varepsilon e$	$A \prec^l_{\varepsilon,K} B \Longleftrightarrow B \subseteq A+\operatorname{int}K+\varepsilon e$

(iii) $\triangle_A(-y)=\triangle_{-A}(y)$ *for all* $y\in Y$.
(iv) *If* A *is a convex cone and* $\operatorname{int}A\neq\varnothing$, *then* $\triangle_A(y):=\sup_{y*\in S(A^*)}\langle -y^*,y\rangle$ *for all* $y\in Y$, *where* $S(A^*):=\{y^*\in A^* \mid \|y^*\|=1\}$.

Definition 3 ([17]) Let A, B be nonempty subsets of Y. Define the GHUODF $D_K:P_0(Y)\times P_0(Y)\to\mathbb{R}\cup\{\pm\infty\}$ by
$$D_K(A,B):=\sup_{b\in B}\inf_{a\in A}\triangle_{-K}(a-b).$$

The following lemmas will play a crucial role in the subsequent sections of this paper.

Lemma 2 ([7, 8]) *Let* $A,B\in P_0(Y)$. *The following assertions hold:*
(i) *Suppose* $A_1,A_2\in P_0(Y)$, *where* A_2 *and* B_2 *are* K*-compact and* K *is solid. If* $A_1\prec^l_K A_2$, *then* $D_K(A_1,B)<D_K(A_2,B)$.
(ii) *Assume that* $e\in K$, $\varepsilon\in R$, $d_{-K}(-e)=d_{Y\setminus -K}(-e)=1$ *and* A *and* B *are* K*-proper and* K*-bounded. Then,* $D_K(A+\varepsilon e,B)=D_K(A,B)+\varepsilon$.
(iii) *If* A *is* K*-proper, then* $D_K(A,A)=0$.
(iv) *If* B *is* K*-compact and* K *is solid, then* $A\prec^l_K B \Longleftrightarrow D_K(A,B)<0$.
(v) *If* A *is* K*-closed, then* $A\preccurlyeq^l_K B \Longleftrightarrow D_K(A,B)\leqslant 0$.
(vi) *If* $A,B,C\in P_0(Y)$ *are* K*-proper and* K*-compact, then* $D_K(A,B)\leqslant D_K(A,C)+D_K(C,B)$.
(vii) $D_K(\cdot,B)$ *is* l*-monotone increasing on* $P_0(Y)$.

Lemma 3 ([22]) *Assume that* $x_0\in S$ *and* $F(x_0)$ *is* K*-compact.*
(i) *If* $\varepsilon>0$ *then* $x_0\in E_l(\varepsilon,S)$ *if and only if there does not exist* $y\in S$ *satisfying* $F(y)\preccurlyeq^l_{\varepsilon,K}F(x_0)$.
(ii) *If* $\varepsilon>0$ *then* $x_0\in W_l(\varepsilon,S)$ *if and only if there does not exist* $y\in S$ *satisfying* $F(y)\prec^l_{\varepsilon,K}F(x_0)$.

The following definition recalls the notions of semicontinuity for set-valued mappings.

Definition 4 ([24]) A set-valued mapping $F:X\rightrightarrows Y$ is said to be
(i) Hausdorff upper semicontinuous (H-$u.s.c.$) at $x_0\in X$ if, for any neighborhood V of $0\in Y$, there exists a neighborhood $U(x_0)$ of x_0 such that for every $x\in U(x_0)$, $F(x)\subseteq F(x_0)+V$.
(ii) upper semicontinuous ($u.s.c.$) at $x_0\in X$ if, for any neighborhood V of $F(x_0)$, there exists a neighborhood $U(x_0)$ of x_0 such that for every $x\in U(x_0)$, $F(x)\subseteq V$.
(iii) K-upper semicontinuous (K-$u.s.c.$) at $x_0\in X$ if, for any neighborhood V of $F(x_0)$, there exists a neighborhood $U(x_0)$ of x_0 such that for every $x\in U(x_0)$, $F(x)\subseteq V+K$.
(iv) lower semicontinuous ($l.s.c.$) at $x_0\in X$ if, for any $y\in F(x_0)$ and any neighborhood V of y, there exists a neighborhood $U(x_0)$ of x_0 such that for every $x\in U(x_0)$, $F(x)\cap V\neq\varnothing$.
(v) K-lower semicontinuous (K-$l.s.c.$) at $x_0\in X$ if, for any $y\in F(x_0)$ and any neighborhood V of y, there exists a neighborhood $U(x_0)$ of x_0 such that for every $x\in U(x_0)$, $F(x)\cap(V-K)\neq\varnothing$.

We define F as (H-$u.s.c.$), ($u.s.c.$), (K-$u.s.c.$), ($l.s.c.$) and (K-$l.s.c.$) on $S\subseteq X$ if it satisfies these conditions at each point $x\in S$: (H-$u.s.c.$), ($u.s.c.$), (K-$u.s.c.$), ($l.s.c.$) and (K-$l.s.c.$), respectively. We consider F to be continuous on S if it is both ($u.s.c.$) and ($l.s.c.$) on S. We say that F is K-continuous on S if it is both (K-$u.s.c.$) and (K-$l.s.c.$) on S.

Lemma 4 ([24]) *Let* $F:X\rightrightarrows Y$ *be a set-valued mapping. For any given* $\mu_0\in X$, *if* $F(\mu_0)$ *is compact, then* F *is* $u.s.c.$ *at* $\mu_0\in X$ *if and only if for any sequence* $\{\mu_n\}\subseteq X$ *with* $\mu_n\to\mu_0$ *and for any* $x_n\in F(\mu_n)$, *one can find a subsequence* $\{x_{n_k}\}$ *of* x_n *and a point* $x_0\in F(\mu_0)$ *such that* $x_{n_k}\to x_0$.

Lemma 5 ([25]) *A set-valued mapping* $F:X\rightrightarrows Y$ *is* $l.s.c.$ *at* $\mu_0\in X$ *if and only if for any sequence* $\{\mu_n\}\subseteq X$ *with* $\mu_n\to\mu_0$ *and for any* $x_0\in F(\mu_0)$, *there exists* $x_n\in F(\mu_n)$ *such that* $x_n\to x_0$.

Lemma 6 ([26, 27]) *Let* $F:X\rightrightarrows Y$ *be a set-valued mapping. The following assertions are valid:*
(i) $F(\cdot)$ *is upper semicontinuous, then* $F(\cdot)$ *is Hausdorff upper semicontinuous.*
(ii) $F(\cdot)$ *is lower semicontinuous if and only if it is Hausdorff lower semicontinuous.*

Lemma 7 ([14]) *Let* S *be a nonempty subset of a topological space, and* $\Phi:X\times S\rightrightarrows Y$ *and* $F:X\rightrightarrows Y$ *be two set-valued mappings. Assume that*
(i) *for any* $\alpha\in S$, $\Phi(\cdot,\alpha)$ *is lower semicontinuous at* $\mu_0\in X$ *and* $F(\mu_0)=\bigcup_{\alpha\in S}\Phi(\mu_0,\alpha)$;
(ii) *there exists a neighborhood* $U(\mu_0)$ *of* μ_0 *such that* $\Phi(\mu,\alpha)\subseteq F(\mu)$ *for any* $(\mu,\alpha)\in U(\mu_0)\times S$.
Then $F(\cdot)$ *is lower semicontinuous at* $\mu_0\in X$.

Among the three definitions presented below, $\xi(x,y)$, $H(\varepsilon,x)$, and $Q(\varepsilon,x)$ are essential for the subsequent proof of the semicontinuity of the approximate solution set.

We define the function $\xi : X \times X \to \mathbb{R} \cup \{\pm\infty\}$ and the set-valued mappings $H : \mathbb{R}_+ \times X \rightrightarrows S$ and $Q : \mathbb{R}_{++} \times X \rightrightarrows S$ as follows:

$$\xi(x, y) = D_K(F(x), F(y)), \quad \forall\ (x, y) \in X \times X.$$

$$\begin{aligned} H(\varepsilon, x) &= \{v \in S \mid D_K(F(v), F(x)) \\ &\qquad \leqslant D_K(F(y), F(x)) + \varepsilon,\ \forall\ y \in S\} \\ &= \{v \in S \mid \xi(v, x) \leqslant \xi(y, x) + \varepsilon,\ \forall\ y \in S\}, \\ &\qquad\qquad \forall\ (\varepsilon, x) \in \mathbb{R}_+ \times X. \end{aligned}$$

$$\begin{aligned} Q(\varepsilon, x) &= \{v \in S \mid D_K(F(v), F(x)) \\ &\qquad < D_K(F(y), F(x)) + \varepsilon,\ \forall\ y \in S\} \\ &= \{v \in S \mid \xi(v, x) < \xi(y, x) + \varepsilon,\ \forall\ y \in S\}, \\ &\qquad\qquad \forall\ (\varepsilon, x) \in \mathbb{R}_{++} \times X. \end{aligned}$$

Proposition 1 *Assume that $F(x)$ is nonempty and K-compact for any $x \in D \subseteq X$, $d_{-K}(-e) = d_{Y\setminus -K}(-e) = 1$, and $S \subseteq D$.*
(i) *If $\varepsilon \geqslant 0$, then*

$$\bigcup_{x\in D} H(\varepsilon, x) = W_l(\varepsilon, F, S).$$

(ii) *If $\varepsilon > 0$, then*

$$\bigcup_{x\in D} Q(\varepsilon, x) = E_l(\varepsilon, F, S).$$

Proof: (i) We first prove that

$$\bigcup_{x\in D} H(\varepsilon, x) \subseteq W_l(\varepsilon, F, S).$$

Let $v_0 \in \bigcup_{x\in D} H(\varepsilon, x)$. Then there exists $x_0 \in D$ such that $v_0 \in H(\varepsilon, x_0)$. Hence,

$$D_K(F(v_0), F(x_0)) \leqslant D_K(F(y), F(x_0)) + \varepsilon,\ \forall y \in S. \quad (2)$$

We now show that $v_0 \in W_l(\varepsilon, F, S)$. Suppose, on the contrary, that $v_0 \notin W_l(\varepsilon, F, S)$. Then, by Lemma 3(ii), there exists $\bar{y} \in S$ such that

$$F(\bar{y}) \prec^l_{\varepsilon,K} F(v_0).$$

Equivalently,

$$F(\bar{y}) + \varepsilon e \prec^l_K F(v_0).$$

Since $F(x)$ is nonempty and K-compact, we conclude from (i) and (ii) of Lemma 2 that

$$\begin{aligned} D_K(F(\bar{y}), F(x_0)) + \varepsilon &= D_K(F(\bar{y}) + \varepsilon e, F(x_0)) \\ &< D_K(F(v_0), F(x_0)), \end{aligned}$$

which contradicts (2). Hence, $v_0 \in W_l(\varepsilon, F, S)$, and so, $\bigcup_{x\in D} H(\varepsilon, x) \subseteq W_l(\varepsilon, F, S)$.

Conversely, let $\bar{x} \in W_l(\varepsilon, F, S)$. Then, by Lemma 3 (ii), we have $F(y) \not\prec^l_{\varepsilon,K} F(\bar{x})$ for any $y \in S$. By the converse of Lemma 2(iv), it follows that $D_K(A, B) \geqslant 0$. Since $\varepsilon \geqslant 0$, applying Lemma 2(ii) gives

$$D_K(F(y), F(\bar{x})) + \varepsilon = D_K(F(y) + \varepsilon e, F(\bar{x})) \geqslant 0. \quad (3)$$

Moreover, by Lemma 2(iii), we obtain $D_K(F(\bar{x}), F(\bar{x})) = 0$. This, together with (3), implies that

$$D_K(F(y), F(\bar{x})) + \varepsilon \geqslant D_K(F(\bar{x}), F(\bar{x})), \quad \forall\ y \in S.$$

This shows that

$$\bar{x} \in H(\varepsilon, \bar{x}) \subseteq \bigcup_{x\in S} H(\varepsilon, x) \subseteq \bigcup_{x\in D} H(\varepsilon, x),$$

and then $W_l(\varepsilon, F, S) \subseteq \bigcup_{x\in D} H(\varepsilon, x)$.

(ii) We first prove that $\bigcup_{x\in D} Q(\varepsilon, x) \subseteq E_l(\varepsilon, F, S)$. Let $v_0 \in \bigcup_{x\in D} Q(\varepsilon, x)$. Then there exists $x_0 \in D$ such that $v_0 \in Q(\varepsilon, x_0)$, and so

$$D_K(F(v_0), F(x_0)) < D_K(F(y), F(x_0)) + \varepsilon,\ \forall y \in S. \quad (4)$$

It is easy to get $v_0 \in E_l(\varepsilon, F, S)$. In fact, suppose that $v_0 \notin E_l(\varepsilon, F, S)$. It follows from Lemma 3(i) that there exists $\bar{y} \in S$ such that $F(\bar{y}) \preccurlyeq^l_{\varepsilon,K} F(v_0)$. This implies that can be further obtained that $F(\bar{y}) + \varepsilon e \preccurlyeq^l_K F(v_0)$. Since $F(x)$ is nonempty and K-compact, we conclude from Lemma 2(ii) and (vii) that

$$\begin{aligned} D_K(F(\bar{y}), F(x_0)) + \varepsilon &= D_K(F(\bar{y}) + \varepsilon e, F(x_0)) \\ &\leqslant D_K(F(v_0), F(x_0)), \end{aligned}$$

which contradicts (4). Hence $v_0 \in E_l(\varepsilon, F, S)$ and so $\bigcup_{x\in D} Q(\varepsilon, x) \subseteq E_l(\varepsilon, F, S)$.

On the other hand, let $\bar{x} \in E_l(\varepsilon, F, S)$, it follows from Lemma 3(i) that $F(y) \not\preccurlyeq^l_{\varepsilon,K} F(\bar{x})$ for any $y \in S$. By the converse of Lemma 2(v), it follows that that $D_K(A, B) > 0$. Since $\varepsilon \geqslant 0$, applying Lemma 2(ii) gives

$$D_K(F(y), F(\bar{x})) + \varepsilon = D_K(F(y) + \varepsilon e, F(\bar{x})) > 0. \quad (5)$$

By Lemma 2(iii), we obtain $D_K(F(\bar{x}), F(\bar{x})) = 0$. This, combined with (5), implies that

$$D_K(F(y), F(\bar{x})) + \varepsilon > D_K(F(\bar{x}), F(\bar{x})), \quad \forall\ y \in S.$$

This shows that

$$\bar{x} \in Q(\varepsilon, \bar{x}) \subseteq \bigcup_{x\in S} Q(\varepsilon, x) \subseteq \bigcup_{x\in D} Q(\varepsilon, x),$$

and then $E_l(\varepsilon, F, S) \subseteq \bigcup_{x\in D} Q(\varepsilon, x)$. This completes the proof. □

Remark 1 By employing the GHUODF and its properties, we adopt a scalarization approach to obtain results similar to those presented by Han [14].

CONTINUITY OF PARAMETRIC SCALARIZATION FUNCTIONS

In this section, we introduce a continuity notion for PSFs. Motivated by Han [14], we consider the following Hausdorff-type distance defined for nonempty K-bounded sets A and B. Let G_1 and G_2 be two normed vector spaces, and let S_1 and S_2 be two nonempty subsets of X.

Definition 5 Let A and B be nonempty subsets of Y. A Hausdorff-type distance between sets A and B, denoted by $h_K(A,B)$ is defined as follows:

$$h_K(A,B) = \max\{D_K(A,B), D_K(B,A)\}.$$

Remark 2 The Definition 3.1 in [14] differs from this, there is no need to assume that A and B are K-bounded subsets of Y and an order relation.

Remark 3 It is noteworthy that, unlike the classical Hausdorff distance $H(A,B)$, the Hausdorff-type distance $h_K(A,B)$ captures the relationship between two sets using the more intricate GHUODF. Additionally, the space on which it is defined is a normed linear space, which inherently has a more complex structure than a general metric space. From this perspective, the study of the Hausdorff-type distance $h_K(A,B)$ has both significant theoretical implications and practical applications.

The following example illustrates the validity of Definition 5.

Example 1 Let $Y=\mathbb{R}^2$, $K=\mathbb{R}^2_+$, $A=\{(2,3),(-1,0)\}$, $B=\{(-2,0)\}$, $y^*\in K^*$, $a\in A$, $b\in B$ and $S(K^*):=\{y^*\in K^* \mid \|y^*\|=1\}$. Clearly, $K^*=\mathbb{R}^2_+$ and $S(K^*)$ is the set of points on the circumference of a quarter of the unit circle. According to (iii) and (iv) of Lemma 1, one has

$$\begin{aligned}\triangle_{-K}(a-b) = \triangle_K(b-a) &= \sup_{y^*\in S(K^*)} \langle -y^*, b-a\rangle \\ &= \sup_{y^*\in S(K^*)} \langle y^*, a-b\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}D_K(A,B) &= \sup_{b\in B}\inf_{a\in A} \triangle_{-K}(a-b) \\ &= \sup_{b\in B}\inf_{a\in A}\sup_{y^*\in S(K^*)} \langle y^*, a-b\rangle \\ &= \sup_{b\in B}\inf_{a\in A}\left\langle \frac{a-b}{\|a-b\|}, a-b\right\rangle \\ &= \left\langle \frac{(-1,0)-(-2,0)}{\|(-1,0)-(-2,0)\|}, (-1,0)-(-2,0)\right\rangle \\ &= 1.\end{aligned}$$

Similarly,

$$D_K(B,A) = \sup_{a\in A}\inf_{b\in B}\sup_{y^*\in S(K^*)} \langle y^*, b-a\rangle = 5.$$

In conclusion,

$$h_K(B,A) = \max\{D_K(A,B), D_K(B,A)\} = 5.$$

Definition 6 Let X be a normed vector space.

(i) A collection of set-valued mappings Ω_1 is defined as follows:

$$\Omega_1 = \{\Phi : S_1 \rightrightarrows Y \mid \Phi(x) \text{ is nonempty for any } x\in S_1\}.$$

Let $\Phi_1,\Phi_2\in\Omega_1$. The metric $d_1(\Phi_1,\Phi_2)$ between Φ_1 and Φ_2 is defined by

$$d_1(\Phi_1,\Phi_2) = \sup_{x\in S_1} h_K(\Phi_1(x),\Phi_2(x)).$$

(ii) A collection of set-valued mappings Ω_2 is defined as follows:

$$\Omega_2 = \{\Psi : S_2 \rightrightarrows Y \mid \Psi(x) \text{ is nonempty for any } x\in S_2\}.$$

$$d_2(\Psi_1,\Psi_2) = \sup_{x\in S_2} h_K(\Psi_1(x),\Psi_2(x)).$$

Remark 4 Unlike [14], it is not necessary to assume that $\Phi(x)$ and $\Psi(x)$ are K-bounded here.

Let $\mathscr{A}\in\Omega_1$ and $\mathscr{B}\in\Omega_2$. We define $\tau : S_1\times S_2 \to \mathbb{R}\cup\{\pm\infty\}$ by

$$\begin{aligned}\tau(\lambda,\mu) &= D_K(\mathscr{A}(\lambda),\mathscr{B}(\mu)) \\ &= \sup_{b\in\mathscr{B}(\mu)}\inf_{a\in\mathscr{A}(\lambda)} \triangle_{-K}(a-b),\ \forall(\lambda,\mu)\in S_1\times S_2.\end{aligned}$$

Inspired by the studies [13, 14], the following two lemmas provide the foundation for establishing the continuity of the PSFs.

Lemma 8 *Let $\mathscr{A}\in\Omega_1$, $\mathscr{B}\in\Omega_2$, K be a closed and convex cone. If $\mathscr{A}(\lambda)$ and $\mathscr{B}(\mu)$ are K-continuous and K-compact values, then $\tau(\cdot,\cdot)$ is continuous on $S_1\times S_2$.*

Proof: For any given $(\lambda_0,\mu_0)\in S_1\times S_2$, we prove that $\tau(\cdot,\cdot)$ is continuous at (λ_0,μ_0). Let $r=\tau(\lambda_0,\mu_0)=D_K(\mathscr{A}(\lambda_0),\mathscr{B}(\mu_0))$. For any $\varepsilon>0$, $b\in\mathscr{B}(\mu_0)$, there exists $a_b\in\mathscr{A}(\lambda_0)$ such that

$$\triangle_{-K}(a_b-b) < \inf_{a\in\mathscr{A}(\lambda_0)} \triangle_{-K}(a-b)+\varepsilon. \quad (6)$$

From Lemma 1(i), we obtain a neighbourhood U_b of b anda neighbourhood U_{a_b} of a_b such that

$$\triangle_{-K}(y-x) < \triangle_{-K}(a_b-b)+\varepsilon, \quad \forall\ (x,y)\in U_b\times U_{a_b}. \quad (7)$$

The following point is easily understood:

$$\triangle_{-K}(y-x) < \triangle_{-K}(a_b-b)+\varepsilon, \\ \forall\ (x,y)\in (U_b+K)\times(U_{a_b}-K). \quad (8)$$

In fact, for any $(x,y)\in(U_b+K)\times(U_{a_b}-K)$, there are $x_0\in U_b$, $k_1\in K$, $y_0\in U_{a_b}$, and $k_2\in K$ such that

$x = x_0 + k_1$ and $y = y_0 - k_2$, implying that $(y_0 - x_0) - (y - x) = k_1 + k_2 \in K$. From Lemma 1(ii) and (7), we have $\triangle_{-K}(y-x) \leqslant \triangle_{-K}(y_0 - x_0) < \triangle_{-K}(a_b - b) + \varepsilon$. Therefore, (8) holds. It is clear that $\mathscr{B}(\mu_0) \subseteq \bigcup_{b\in\mathscr{B}(\mu_0)} (U_b + K)$. Since $\mathscr{B}(\mu_0)$ is K-compact, there exists $\{b_i : i = 1, 2, \ldots, n\} \subseteq \mathscr{B}(\mu_0)$ such that $\mathscr{B}(\mu_0) \subseteq \bigcup_{i=1}^{n}(U_{b_i} + K)$. Noting that $\mathscr{B}(\mu_0)$ is K-$u.s.c$ at μ_0, there exists a neighbourhood $U^1_{\mu_0}$ of $\mathscr{B}(\mu_0)$ such that

$$\mathscr{B}(\mu) \subseteq \bigcup_{i=1}^{n}(U_{b_i} + K) + K \subseteq \bigcup_{i=1}^{n}(U_{b_i} + K), \quad \forall\, \mu \in U^1_{\mu_0}. \tag{9}$$

Since $\mathscr{A}(\lambda_0)$ is K-$l.s.c$ at λ_0, there exists a neighbourhood $U^1_{\lambda_0}$ of λ_0 such that

$$\mathscr{A}(\lambda) \bigcap (U_{a_{b_i}} - K) \neq \varnothing,\ \forall i \in \{1, 2, \ldots, n\},\ \forall \lambda \in U^1_{\lambda_0}. \tag{10}$$

Let $(\lambda, \mu) \in U^1_{\lambda_0} \times U^1_{\mu_0}$. For any $\beta \in \mathscr{B}(\mu)$, it follows from (9) that there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $\beta \in (U_{b_{i_0}} + K)$. From (10), there exists $\alpha \in \mathscr{A}(\lambda) \bigcap (U_{a_{b_{i_0}}} - K)$. By (6) and (8), we have

$$\begin{aligned}\inf_{a\in\mathscr{A}(\lambda_0)} \triangle_{-K}(a - \beta) \leqslant \triangle_{-K}(\alpha - \beta) &< \triangle_{-K}(a_{b_{i_0}} - b_{i_0}) + \varepsilon \\ &< \inf_{a\in\mathscr{A}(\lambda_0)} \triangle_{-K}(a - b_{i_0}) + 2\varepsilon \\ &\leqslant D_K(\mathscr{A}(\lambda_0), \mathscr{B}(\mu_0)) + 2\varepsilon.\end{aligned}$$

By the arbitrariness of $\beta \in \mathscr{B}(\mu)$, we have

$$\begin{aligned}D_K(\mathscr{A}(\lambda), \mathscr{B}(\mu)) &\leqslant D_K(\mathscr{A}(\lambda_0), \mathscr{B}(\mu_0)) + 2\varepsilon \\ &= r + 2\varepsilon, \quad \forall\, (\lambda, \mu) \in U^1_{\lambda_0} \times U^1_{\mu_0}.\end{aligned} \tag{11}$$

On the other hand, by the definition of $D_K(\mathscr{A}(\lambda_0), \mathscr{B}(\mu_0))$, there exists $\bar{b} \in \mathscr{B}(\mu_0)$ such that

$$r - \varepsilon < \inf_{a\in\mathscr{A}(\lambda_0)} \triangle_{-K}(a - \bar{b}). \tag{12}$$

For any $a \in \mathscr{A}(\lambda_0)$, Lemma 1(i) implies the existence of a neighbourhood U_a of a and a neighbourhood $U^a_{\bar{b}}$ of $\bar{b}$ such that

$$\triangle_{-K}(a - \bar{b}) - \varepsilon < \triangle_{-K}(y - x), \quad \forall\, (x, y) \in U^a_{\bar{b}} \times U_a. \tag{13}$$

Combining Lemma 1(ii) with (13) yields

$$\begin{aligned}&\triangle_{-K}(a - \bar{b}) - \varepsilon < \triangle_{-K}(y - x), \\ &\qquad \forall\, (x, y) \in (U^a_{\bar{b}} - K) \times (U_a + K).\end{aligned} \tag{14}$$

Since $\mathscr{A}(\lambda_0)$ is K-compact and satisfies $\mathscr{A}(\lambda_0) \subseteq \bigcup_{a\in\mathscr{A}(\lambda_0)} (U_a + K)$, there exists $\{a_i : i = 1, 2, \ldots, m\} \subseteq \mathscr{A}(\lambda_0)$ such that $\mathscr{A}(\lambda_0) \subseteq \bigcup_{i=1}^{m}(U_{a_i} + K)$. Noting that $\mathscr{A}(\cdot)$ is K-$u.s.c$ at λ_0, there exists a neighbourhood $U^2_{\lambda_0}$ of λ_0 such that

$$A(\lambda) \subseteq \bigcup_{i=1}^{m}(U_{a_i} + K) + K \subseteq \bigcup_{i=1}^{m}(U_{a_i} + K), \quad \forall\, \lambda \in U^2_{\lambda_0}. \tag{15}$$

Similarly, since $\mathscr{B}(\cdot)$ is K-$l.s.c$ at μ_0, there exists a neighbourhood $U^2_{\mu_0}$ of μ_0 such that

$$\mathscr{B}(\mu) \cap \bigcap_{i=1}^{m}(U^{a_i}_{\bar{b}} + K) \neq \varnothing, \quad \forall\, \mu \in U^2_{\mu_0}. \tag{16}$$

Now take any $(\lambda, \mu) \in U^2_{\lambda_0} \times U^2_{\mu_0}$. For any $z \in \mathscr{A}(\lambda)$, by (15), we can see that there exists $\{a_{i_0} : i_0 = 1, 2, \ldots, m\}$ such that $z \in U_{a_{i_0}} + K$. It follows from (16) that there exists $b' \in \bigcap_{i=1}^{m}(U^{a_i}_{\bar{b}} + K)$. Combining (12) and (14) gives

$$\begin{aligned}r - 2\varepsilon &< \inf_{a\in\mathscr{A}(\lambda_0)} \triangle_{-K}(a - \bar{b}) - \varepsilon \\ &\leqslant \triangle_{-K}(a_{i_0} - \bar{b}) - \varepsilon < \triangle_{-K}(z - b').\end{aligned}$$

Since $z \in \mathscr{A}(\lambda)$ was arbitrary, we obtain $r - 2\varepsilon \leqslant \inf_{a\in\mathscr{A}(\lambda_0)} \triangle_{-K}(a - b')$, and so

$$\begin{aligned}r - 2\varepsilon &\leqslant \inf_{a\in\mathscr{A}(\lambda_0)} \triangle_{-K}(a - b') \\ &\leqslant D_K(\mathscr{A}(\lambda), \mathscr{B}(\mu)), \quad \forall\, (\lambda, \mu) \in U^2_{\lambda_0} \times U^2_{\mu_0}.\end{aligned} \tag{17}$$

Finally, let $U_{\lambda_0} = U^1_{\lambda_0} \cap U^2_{\lambda_0}$ and $U_{\mu_0} = U^1_{\mu_0} \cap U^2_{\mu_0}$. Then, from (11) and (17), we obtain

$$r - 2\varepsilon \leqslant D_K(\mathscr{A}(\lambda), \mathscr{B}(\mu)) \leqslant r + 2\varepsilon, \quad \forall\, (\lambda, \mu) \in U_{\lambda_0} \times U_{\mu_0},$$

which proves that $\tau(\cdot, \cdot)$ is continuous at (λ_0, μ_0). □

Remark 5 Han [13] established the continuity of the Gerstewitz function in set form, while here we obtain the continuity of the GHUODF.

Lemma 9 *Assume that A, B and C are nonempty, K-proper and K-bounded. Then*

$$\begin{aligned}|D_K(A, B) - D_K(C, B)| &\leqslant h_K(A, C), \\ |D_K(A, B) - D_K(A, C)| &\leqslant h_K(B, C).\end{aligned}$$

Proof: It follows from Lemma 2(vi) that

$$D_K(A, B) \leqslant D_K(A, C) + D_K(C, B), \tag{18}$$

and so

$$D_K(A, B) - D_K(C, B) \leqslant D_K(A, C). \tag{19}$$

Swapping A with C in (19), we have $D_K(C, B) - D_K(A, B) \leqslant D_K(C, A)$. This together with (19) implies that

$$\begin{aligned}|D_K(A, B) - D_K(C, B)| &\leqslant \max\{D_K(A, C), D_K(C, A)\} \\ &= h_K(A, C).\end{aligned}$$

As a result of (18), we have

$$D_K(A,B) - D_K(A,C) \leqslant D_K(C,B). \tag{20}$$

Switching B with C in (20), we have $D_K(A,C) - D_K(A,B) \leqslant D_K(B,C)$. Combining this with (20), we get

$$\begin{aligned}|D_K(A,B) - D_K(A,C)| &\leqslant \max\{D_K(C,B), D_K(B,C)\}\\ &= h_K(B,C).\end{aligned}$$

This completes the proof. □

In the following, we introduce the concept of the PSFs and explore its continuity conditions. The definition of the PSFs is provided immediately after.

Definition 7 Let $\mathscr{A}_\lambda \in \Omega_1$ for any $\lambda \in G_1$ and $\mathscr{B}_\alpha \in \Omega_2$ for any $\alpha \in G_2$. The PSFs $f : S_1 \times S_2 \times G_1 \times G_2$ is defined by

$$f(x,y,\lambda,\alpha) = D_K(\mathscr{A}_\lambda(x), \mathscr{B}_\alpha(y)),$$
$$\forall\ (x,y,\lambda,\alpha) \in S_1 \times S_2 \times G_1 \times G_2.$$

Theorem 1 *Let $(x_0, y_0, \lambda_0, \alpha_0) \in S_1 \times S_2 \times G_1 \times G_2 \to \mathbb{R}$. If the following statement holds true:*

(i) *$\mathscr{A} \in \Omega_1$ and $\mathscr{A}_{\lambda_0}(\cdot)$ is K-proper and K-continuous on S_1 with K-compact values;*
(ii) *$\mathscr{B} \in \Omega_2$ and $\mathscr{B}_{\alpha_0}(\cdot)$ is K-proper and K-continuous on S_2 with K-compact values;*
(iii) *$\lim_{\lambda \to \lambda_0} d_1(\mathscr{A}_\lambda, \mathscr{A}_{\lambda_0}) = 0$ and $\lim_{\alpha \to \alpha_0} d_2(\mathscr{B}_\alpha, \mathscr{B}_{\alpha_0}) = 0$.*

Then, $f(\cdot,\cdot,\cdot,\cdot)$ is continuous at $(x_0, y_0, \lambda_0, \alpha_0)$.

Proof: Let $\{(x_n, y_n, \lambda_n, \alpha_n)\} \subseteq S_1 \times S_2 \times G_1 \times G_2$ with $\{(x_n, y_n, \lambda_n, \alpha_n)\} \to (x_0, y_0, \lambda_0, \alpha_0)$. It suffices to show that $f(x_n, y_n, \lambda_n, \alpha_n) \to f(x_0, y_0, \lambda_0, \alpha_0)$. It follows from Lemma 8 that

$$f(x_n, y_n, \lambda_0, \alpha_0) \to f(x_0, y_0, \lambda_0, \alpha_0).$$

Then for any $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$|f(x_n, y_n, \lambda_0, \alpha_0) - f(x_0, y_0, \lambda_0, \alpha_0)| \leqslant \frac{\varepsilon}{3},\ \forall n \geqslant n_1. \tag{21}$$

We conclude from Lemma 9 that

$$\begin{aligned}&|D_K(\mathscr{A}_{\lambda_n}(x_n), \mathscr{B}_{\alpha_n}(y_n)) - D_K(\mathscr{A}_{\lambda_0}(x_n), \mathscr{B}_{\alpha_n}(y_n))|\\ &\qquad \leqslant h_K(\mathscr{A}_{\lambda_n}(x_n), \mathscr{A}_{\lambda_0}(x_n)),\end{aligned} \tag{22}$$

and

$$\begin{aligned}&|D_K(\mathscr{A}_{\lambda_0}(x_n), \mathscr{B}_{\alpha_n}(y_n)) - D_K(\mathscr{A}_{\lambda_0}(x_n), \mathscr{B}_{\alpha_0}(y_n))|\\ &\qquad \leqslant h_K(\mathscr{B}_{\alpha_n}(y_n)), \mathscr{B}_{\alpha_0}(y_n)).\end{aligned} \tag{23}$$

For the above $\varepsilon > 0$, due to $\lim_{\lambda \to \lambda_0} d_1(\mathscr{A}_\lambda, \mathscr{A}_{\lambda_0}) = 0$, there exists $n_2 \in \mathbb{N}$ such that

$$\begin{aligned}h_K(\mathscr{A}_{\lambda_n}(x_n), \mathscr{A}_{\lambda_0}(x_n)) &\leqslant d_1(\mathscr{A}_\lambda, \mathscr{A}_{\lambda_0})\\ &\leqslant \frac{\varepsilon}{3}, \quad \forall n \geqslant n_2.\end{aligned} \tag{24}$$

Similarly, it follows from $\lim_{\alpha \to \alpha_0} d_2(\mathscr{B}_\alpha, \mathscr{B}_{\alpha_0}) = 0$ that there exists $n_3 \in \mathbb{N}$ such that

$$\begin{aligned}h_K(\mathscr{B}_{\alpha_n}(y_n)), \mathscr{B}_{\alpha_0}(y_n) &\leqslant d_2(\mathscr{B}_\alpha, \mathscr{B}_{\alpha_0})\\ &\leqslant \frac{\varepsilon}{3}, \quad \forall\ n \geqslant n_3.\end{aligned} \tag{25}$$

From (22), (23), (24) and (25), for any $n \geqslant \max\{n_2, n_3\}$, we have

$$\begin{aligned}&|f(x_n, y_n, \lambda_n, \alpha_n) - f(x_n, y_n, \lambda_0, \alpha_0)|\\ &= |D_K(\mathscr{A}_{\lambda_n}(x_n), \mathscr{B}_{\alpha_n}(y_n)) - D_K(\mathscr{A}_{\lambda_0}(x_n), \mathscr{B}_{\alpha_0}(y_n))|\\ &\leqslant |D_K(\mathscr{A}_{\lambda_n}(x_n), \mathscr{B}_{\alpha_n}(y_n)) - D_K(\mathscr{A}_{\lambda_0}(x_n), \mathscr{B}_{\alpha_n}(y_n))|\\ &\quad + |D_K(\mathscr{A}_{\lambda_0}(x_n), \mathscr{B}_{\alpha_n}(y_n)) - D_K(\mathscr{A}_{\lambda_0}(x_n), \mathscr{B}_{\alpha_0}(y_n))|\\ &\leqslant \frac{2}{3}\varepsilon.\end{aligned} \tag{26}$$

On the basis of (21) and (26), for any $n \geqslant \max\{n_1, n_2, n_3\}$, we have

$$\begin{aligned}&|f(x_n, y_n, \lambda_n, \alpha_n) - f(x_0, y_0, \lambda_0, \alpha_0)|\\ &\qquad \leqslant |f(x_n, y_n, \lambda_n, \alpha_n) - f(x_n, y_n, \lambda_0, \alpha_0)|\\ &\qquad\quad + |f(x_n, y_n, \lambda_0, \alpha_0) - f(x_0, y_0, \lambda_0, \alpha_0)|\\ &\qquad \leqslant \varepsilon,\end{aligned}$$

which means that $f(x_n, y_n, \lambda_n, \alpha_n) \to f(x_0, y_0, \lambda_0, \alpha_0)$. This completes the proof. □

Remark 6 Condition (iii) in Theorem 1 is essential, as it guarantees the validity of the inequalities (24)–(25).

Remark 7 Theorem 1 provides a theoretical foundation for the continuity of the PSFs. The result of this theorem indicates that when parametrics undergo small changes, the PSFs remains continuous.

THE CORRESPONDENCE OF THE SOLUTIONS TO TWO KINDS OF PROBLEMS

In this section, we investigate the semicontinuity of l-minimal approximate solution mapping and the weak l-minimal approximate solution mapping for PSOPs. Let $K : V \rightrightarrows Y$ be a set-valued mapping, and let S be a nonempty subset of X. Define

$$\Omega = \{F : S \rightrightarrows Y \mid F(x) \text{ is nonempty for any } x \in K(\lambda)\}.$$

Definition 8 Let $F_\mu \in \Omega$ and $\mu \in U$. The function $\varphi : K(\lambda) \times K(\lambda) \times U \to \mathbb{R} \cup \{\pm\infty\}$ is defined as

$$\varphi(x,y,\mu) = D_K(F_\mu(x), F_\mu(y)),$$
$$\forall\ (x,y,\mu) \in K(\lambda) \times K(\lambda) \times U.$$

Consider the PEPs as follows:

$$\begin{cases}\text{find} & x \in K(\lambda)\\ \text{s.t.} & \varphi(x,y,\mu) \geqslant 0 \text{ for all } y \in K(\lambda).\end{cases}$$

Let $F_\mu : X \rightrightarrows Y$ be a set-valued mapping. Consider the following PSOPs:

$$\begin{cases}\min & F_\mu(x)\\ \text{s.t.} & x \in K(\lambda),\end{cases}$$

where $(\lambda,\mu) \in V \times U$.

Definition 9 Let $E_l : \mathbb{R}_+ \times V \times U \rightrightarrows X$ and $W_l : \mathbb{R}_+ \times V \times U \rightrightarrows X$ represent solution mappings of l-minimal approximate solution and weak l-minimal approximate for PSOPs, respectively, i.e.,

$$\begin{aligned}&E_l(\varepsilon,\lambda,\mu) = E_l(\varepsilon, F_\mu, K(\lambda))\\ &= \Big\{x \in K(\lambda) \mid y \in K(\lambda)\\ &\qquad \text{and } F_\mu(y) \preccurlyeq^l_{\varepsilon,K} F_\mu(x) \text{ implies } F_\mu(x) \preccurlyeq^l_{\varepsilon,K} F_\mu(y)\Big\},\end{aligned}$$

and

$$\begin{aligned}&W_l(\varepsilon,\lambda,\mu) = W_l(\varepsilon, F_\mu, K(\lambda))\\ &= \{x \in K(\lambda) \mid y \in K(\lambda)\\ &\qquad \text{and } F_\mu(y) \prec^l_{\varepsilon,K} F_\mu(x) \text{ implies } F_\mu(x) \prec^l_{\varepsilon,K} F_\mu(y)\}.\end{aligned}$$

The following lemma will be crucial in proving the relationship between the approximate solution set of the PSOPs and the approximate solution set of the PEPs in Corollary 1 later.

Lemma 10 *Assume that K is solid, $F \in \Omega$ and $F(x)$ has K-proper and K-compact values for any $x \in K(\lambda)$.*

(i) *Then, $x \in K(\lambda)$ is a weak l-minimal approximate solution of PSOPs for weak ε-lower relation " $\prec^l_{\varepsilon,K}$ " if and only if $D_K(F_\mu(y), F_\mu(x)) + \varepsilon \geqslant 0$ for all $y \in K(\lambda)$.*

(ii) *Then, $x \in K(\lambda)$ is a l-minimal approximate solution of PSOPs for ε-lower relation " $\preccurlyeq^l_{\varepsilon,K}$ " if and only if $D_K(F_\mu(y), F_\mu(x)) + \varepsilon > 0$ for all $y \in K(\lambda)$.*

Proof: (i) Since $x \in K(\lambda)$ is a weak l-minimal approximate solution of PSOPs, it follows from Lemma 3(ii) that there does not exist $y \in K(\lambda)$ such that $F_\mu(y) \prec^l_{\varepsilon,K} F_\mu(x)$. This implies that $F_\mu(y) \nprec^l_{\varepsilon,K} F_\mu(x)$ for any $y \in K(\lambda)$. Inspired by the proof of Proposition 1, we have $F_\mu(y) + \varepsilon e \nprec^l_K F_\mu(x)$.

By considering the converse of statement (iv) in Lemma 2, we can further conclude that $D_K(F_\mu(y) + \varepsilon e, F_\mu(x)) \geqslant 0$. According to Lemma 2(ii), we have:

$$D_K(F_\mu(y) + \varepsilon e, F_\mu(x)) = D_K(F_\mu(y), F_\mu(x)) + \varepsilon \geqslant 0.$$

On the other hand, by Lemma 2(ii), we also obtain:

$$D_K(F_\mu(y) + \varepsilon e, F_\mu(x)) = D_K(F_\mu(y), F_\mu(x)) + \varepsilon \geqslant 0.$$

By considering the converse of statement (iv) in Lemma 2, we conclude that $F_\mu(y) + \varepsilon e \nprec^l_K F_\mu(x)$. This implies that $F_\mu(y) \nprec^l_{\varepsilon,K} F_\mu(x)$ for any $y \in K(\lambda)$, which is equivalent to saying that there does not exist $y \in K(\lambda)$ satisfying $F_\mu(y) \prec^l_{\varepsilon,K} F_\mu(x)$. Therefore, $x \in K(\lambda)$ is a weak l-minimal approximate solution of PSOPs.

(ii) Since $x \in K(\lambda)$ is an l-minimal approximate solution of PSOPs, it follows from Lemma 3(i) that there does not exist $y \in K(\lambda)$ such that $F_\mu(y) \preccurlyeq^l_{\varepsilon,K} F_\mu(x)$. This implies that $F_\mu(y) \npreccurlyeq^l_{\varepsilon,K} F_\mu(x)$ for any $y \in K(\lambda)$. Inspired by the proof of Proposition 1, we have $F_\mu(y) + \varepsilon e \npreccurlyeq^l_K F_\mu(x)$. By considering the converse of statement (v) in Lemma 2, we can further conclude that $D_K(F_\mu(y) + \varepsilon e, F_\mu(x)) > 0$. According to Lemma 2(ii), we have: $D_K(F_\mu(y) + \varepsilon e, F_\mu(x)) = D_K(F_\mu(y), F_\mu(x)) + \varepsilon \geqslant 0$.

On the other hand, by Lemma 2(ii), we have

$$D_K(F_\mu(y) + \varepsilon e, F_\mu(x)) = D_K(F_\mu(y), F_\mu(x)) + \varepsilon > 0.$$

By considering the converse of statement (v) in Lemma 2, we can conclude that $F_\mu(y) + \varepsilon e \npreccurlyeq^l_K F_\mu(x)$. This implies that $F_\mu(y) \npreccurlyeq^l_{\varepsilon,K} F_\mu(x)$ for any $y \in K(\lambda)$, which is equivalent to stating that there does not exist $y \in K(\lambda)$ satisfying $F_\mu(y) \preccurlyeq^l_{\varepsilon,K} F_\mu(x)$. Therefore, $x \in K(\lambda)$ is a l-minimal approximate solution of PSOPs. This completes the proof. □

The solution set of the PEPs is defined as follows.

Definition 10 Let $P_e : \mathbb{R}_+ \times V \times U \rightrightarrows X$ and $P_s : \mathbb{R}_{++} \times V \times U \rightrightarrows X$ represent solution mappings of approximate and strict approximate for PEPs, respectively, i.e.,

$$\begin{aligned}P_e(\varepsilon,\lambda,\mu) &= P_e(\varepsilon, F_\mu, K(\lambda))\\ &= \{x \in K(\lambda) \mid D_K(F_\mu(y), F_\mu(x)) + \varepsilon \geqslant 0,\\ &\qquad \forall (y,x,\mu) \in K(\lambda) \times K(\lambda) \times U\}\\ &= \{x \in K(\lambda) \mid \varphi(y,x,\mu) + \varepsilon \geqslant 0, \forall y \in K(\lambda)\},\end{aligned}$$

and

$$\begin{aligned}P_s(\varepsilon,\lambda,\mu) &= P_s(\varepsilon, F_\mu, K(\lambda))\\ &= \{x \in K(\lambda) \mid D_K(F_\mu(y), F_\mu(x)) + \varepsilon > 0,\\ &\qquad \forall\ (y,x,\mu) \in K(\lambda) \times K(\lambda) \times U\}\\ &= \{x \in K(\lambda) \mid \varphi(y,x,\mu) + \varepsilon > 0, \forall y \in K(\lambda)\}.\end{aligned}$$

The subsequent outcome is inspired by Lemma 10.

Corollary 1 *Assume that $F \in \Omega$ has K-proper and K-compact values. Then,*

(i) *x is a weak l-minimal approximate solution of PSOPs for weak ε-lower relation " $\prec^l_{\varepsilon,K}$ " if and only if x is approximate solution of PEPs.*

(ii) *x is a l-minimal approximate solution of PSOPs for ε-lower relation " $\preccurlyeq^l_{\varepsilon,K}$ " if and only if x is strict approximate solution of PEPs.*

Remark 8 The preceding theorem implies that the weak l-minimal approximate solution set of PSOPs is equivalent to approximate solution set of PEPs, and l-minimal approximate solution set of PSOPs is equivalent to the strict approximate solution set of PEPs. Since $E_l(\varepsilon, F, S) \subseteq W_l(\varepsilon, F, S)$ has been established, it follows that $P_s(\varepsilon, \lambda, \mu) \subseteq P_e(\varepsilon, \lambda, \mu)$ can be inferred. The same conclusion can be derived from Definition 10.

SEMICONTINUOUS OF SOLUTION MAPPINGS TO PARAMETRIC SET OPTIMIZATION PROBLEMS

Let S be a nonempty subset of X. To investigate the semicontinuity of solution maps for PSOPs, consider the APEPs given by the following system:

$$\begin{cases} \text{find} & x \in K(\lambda) \\ \text{s.t.} & \varphi(x, y, \mu) + \varepsilon \geqslant 0 \text{ for all } y \in K(\lambda). \end{cases}$$

Denote the solution set to APEPs by

$$\begin{aligned} P_e(\varepsilon, \lambda, \mu) &= P_e(\varepsilon, F_\mu, K(\lambda)) \\ &= \{x \in K(\lambda) \mid D_K(F_\mu(y), F_\mu(x)) + \varepsilon \geqslant 0, \\ &\qquad \forall\, (y, x, \mu) \in K(\lambda) \times K(\lambda) \times U\} \\ &= \{x \in K(\lambda) \mid \varphi(y, x, \mu) + \varepsilon \geqslant 0, \forall y \in K(\lambda)\}, \end{aligned}$$

and always assume that $P_e(\varepsilon, \lambda, \mu) \neq \varnothing$ for all $(\varepsilon, \lambda, \mu) \in \mathbb{R}_+ \times V \times U$.

Definition 11 The set-valued mapping $H : \mathbb{R}_+ \times V \times U \times S \rightrightarrows S$ is define as follows

$$\begin{aligned} H(\varepsilon, \lambda, \mu, x) &= \{v \in K(\lambda) \mid D_K(F_\mu(v), F_\mu(x)) \\ &\qquad \leqslant D_K(F_\mu(y), F_\mu(x)) + \varepsilon,\ \forall\ y \in K(\lambda)\} \\ &= \{v \in K(\lambda) \mid \xi(v, x, \mu) \leqslant \xi(y, x, \mu) + \varepsilon, \forall y \in K(\lambda)\}. \end{aligned}$$

Theorem 2 *Let $(\varepsilon_0, \lambda_0, \mu_0) \in \mathbb{R}_+ \times V \times U$ and K be solid. Assume that*
(i) *$K(\lambda_0)$ is nonempty and compact, and $K(\cdot)$ is continuous at λ_0;*
(ii) *$F_{\mu_0}(\cdot)$ is K-proper and K-continuous on S with K-compact values;*
(iii) $\lim_{\lambda \to \lambda_0} d_1(F_\mu, F_{\mu_0}) = 0$.
Then, $P_e(\cdot, \cdot, \cdot)$ is lower semicontinuous at $(\varepsilon_0, \lambda_0, \mu_0)$.

Proof: From Remark 8 we have that weak l-minimal approximate solution set of PSOPs is equivalent to solution set of APEPs, i.e., $W_l(\cdot, \cdot, \cdot) = P_e(\cdot, \cdot, \cdot)$. As in Proposition 1, we have

$$\bigcup_{x \in D} H(\varepsilon, x) = W_l(\varepsilon, F, S).$$

By the implication of Lemma 7, we only need to prove $H(\cdot, \cdot, \cdot, x)$ is $l.s.c.$ at $(\varepsilon_0, \lambda_0, \mu_0)$ for any $x \in S$. If there exists $x_0 \in S$ such that $H(\cdot, \cdot, \cdot, x_0)$ is not $l.s.c.$ at $(\varepsilon_0, \lambda_0, \mu_0)$. The definition of lower semicontinuous implies the existence of $v_0 \in H(\varepsilon_0, \lambda_0, \mu_0, x_0)$, a neighborhood O of $0 \in X$, and a sequence $(\varepsilon_n, \lambda_n, \mu_n) \subseteq \mathbb{R}_+ \times V \times U$ with $(\varepsilon_n, \lambda_n, \mu_n) \to (\varepsilon_0, \lambda_0, \mu_0)$ such that

$$(v_0 + O) \cap H(\varepsilon_n, \lambda_n, \mu_n, x_0) = \varnothing, \text{ for all } n \in \mathbb{N}. \tag{27}$$

Condition one states that $K(\cdot)$ is lower semicontinuous at λ_0 and $v_0 \in K(\lambda_0)$. According to Lemma 5, there exists $v_n \in K(\lambda_n)$ such that $v_n \to v_0$, and so $v_n \in v_0 + O$ for n large enough. This together with (27) implies that $v_n \notin H(\varepsilon_n, \lambda_n, \mu_n, x_0)$. Then there exists $y_n \in K(\lambda_n)$ such that

$$\xi(v_n, x_0, \mu_n) > \xi(y_n, x_0, \mu_n) + \varepsilon_n. \tag{28}$$

And as $K(\cdot)$ is upper semicontinuous at λ_0 and $y_n \in K(\lambda_n)$. By Lemma 4, one has $y_0 \in K(\lambda_0)$ and a subsequence $\{y_{n_k}\}$ of y_n such that $y_{n_k} \to y_0$. Without loss of generality, we assume that $y_n \to y_0$. It follows from Theorem 1 that $\xi(y_n, x_0, \mu_n) \to \xi(y_0, x_0, \mu_0)$ and $\xi(v_n, x_0, \mu_n) \to \xi(v_0, x_0, \mu_0)$. Combining this with the inequality (28), we obtain

$$\xi(v_0, x_0, \mu_0) > \xi(y_0, x_0, \mu_0) + \varepsilon_0. \tag{29}$$

However, due to the ability of $v_0 \in H(\varepsilon_0, \lambda_0, \mu_0, x_0)$ to derive $\xi(v_0, x_0, \mu_0) \leqslant \xi(y, x_0, \mu_0) + \varepsilon_0$, which contradicts (29). That is, $H(\cdot, \cdot, \cdot, x)$ is $l.s.c.$ at $(\varepsilon_0, \lambda_0, \mu_0)$ for any $x \in S$. The theorem has been wholly proven. □

Remark 9 From Lemma 6(ii), we are able to obtain $P_e(\cdot, \cdot, \cdot)$ is Hausdorff lower semicontinuous at $(\varepsilon_0, \lambda_0, \mu_0)$.

Theorem 3 *Let $(\varepsilon_0, \lambda_0, \mu_0) \in \mathbb{R}_+ \times V \times U$. Assume that*
(i) *$K(\lambda_0)$ is nonempty and compact, and $K(\cdot)$ is continuous at λ_0;*
(ii) *$F_{\mu_0}(\cdot)$ is K-continuous on S;*
(iii) $\lim_{\lambda \to \lambda_0} d_1(F_\mu, F_{\mu_0}) = 0$.
Then, $W_l(\cdot, \cdot, \cdot)$ is Hausdorff upper semicontinuous at $(\varepsilon_0, \lambda_0, \mu_0)$.

Proof: From Remark 8 we have $W_l(\cdot, \cdot, \cdot) = P_e(\cdot, \cdot, \cdot)$. We simply need to prove that $P_e(\cdot, \cdot, \cdot)$ is upper semicontinuous at $(\varepsilon_0, \lambda_0, \mu_0)$. Inspired by Lemma 6, we only need to show that $P_e(\cdot, \cdot, \cdot)$ is upper semicontinuous at $(\varepsilon_0, \lambda_0, \mu_0)$. Suppose, by way of contradiction, that this is not true. Then we can find a neighborhood O_0 of $P_e(\varepsilon_0, \lambda_0, \mu_0)$ and a sequence $\{(\varepsilon_n, \lambda_n, \mu_n)\}$ with $(\varepsilon_n, \lambda_n, \mu_n) \to (\varepsilon_0, \lambda_0, \mu_0)$ such that $x_n \in P_e(\varepsilon_n, \lambda_n, \mu_n)$ but $x_n \notin O_0$ for all n. Evidently, $x_n \in K(\lambda_n)$. For the sequence $\{x_n\}$, by hypothesis (i) and Lemma 4, there is a sequence $\{x_{n_k}\}$ of x_n such that $x_{n_k} \to x_0 \in K(\lambda_0)$. Without loss of generality, we assume that $x_n \to x_0$. If x_0 is not an element of $P_e(\varepsilon_0, \lambda_0, \mu_0)$, then there is $y_0 \in K(\lambda_0)$ such that

$$\varphi(y_0, x_0, \mu_0) + \varepsilon_0 < 0. \tag{30}$$

By hypothesis (i) and Lemma 5, there is $y_n \in K(\lambda_n)$ such that the sequence y_n converges to y_0. Since $x_n \in P_e(\varepsilon_n, \lambda_n, \mu_n)$, We are able to obtain $\varphi(y_n, x_n, \mu_n) + \varepsilon_n \geqslant 0$. This combines with the upper semicontinuity of φ implies that $\varphi(y_0, x_0, \mu_0) + \varepsilon_0 \geqslant 0$. This contradicts (30). Consequently, $x_n \in P_e(\varepsilon_n, \lambda_n, \mu_n)$ and $x_n \in O_0$ for all n. In conclusion, $P_e(\cdot,\cdot,\cdot)$ is upper semicontinuous at $(\varepsilon_0,\lambda_0,\mu_0)$, by Lemma 6(i), $P_e(\cdot,\cdot,\cdot)$ is Hausdorff upper semicontinuous at $(\varepsilon_0,\lambda_0,\mu_0)$. That is, $W_l(\cdot,\cdot,\cdot)$ is Hausdorff upper semicontinuous at $(\varepsilon_0,\lambda_0,\mu_0)$. □

Remark 10 Since $P_s(\varepsilon,\lambda,\mu)\subseteq P_e(\varepsilon,\lambda,\mu)$, under the same assumptions as Theorems 2–3, $P_s(\cdot,\cdot,\cdot)$ is also lower semicontinuous and Hausdorff upper semicontinuous, respectively.

CONCLUSION

This paper establishes the semicontinuity and Hausdorff semicontinuity of the sets of approximate solutions for PSOPs. The main contributions are as follows: We have established the continuity of the PSFs, as well as the equivalence theorem of solutions for PSOPs and PEPs. There is still a lack of in-depth research on the directional derivatives and subdifferential properties of the GHUODF, as well as the connectedness and contractibility of the solution set of SOPs. The above directions will become the key contents of future research.

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