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Uniqueness of meromorphic functions concerning derivatives and fixed points

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ABSTRACT: In this paper, we study the uniqueness of meromorphic function concerning derivatives and fixed points. We mainly prove: Let n, k be two positive integers with $n > 3k+8$, and let f and g be two meromorphic functions all whose zeros and poles have multiplicity at least n. If $f^{(k)}$ and $g^{(k)}$ share z CM, f and g share ∞ IM, then
(1) $k=1$, either $f(z)=c_1\,\mathrm{e}^{cz^2}$, $g(z)=c_2\,\mathrm{e}^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4c_1c_2c^2=-1$, or $f\equiv g$;
(2) $k\geqslant 2$, $f\equiv g$.
The result improves some results due to Fang-Qiu [*J Math Anal Appl*, 2002], Zhang [*J Southeast Univ*, 2004], Xu-Lű-Yi [*Comput Math Appl*, 2010] and Zhang [*Comput Math Appl*, 2008].

KEYWORDS: meromorphic functions, derivatives, fixed points, multiplicity

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INTRODUCTION AND MAIN RESULTS

In this paper, meromorphic always means meromorphic in the whole complex plane. We use the following standard notations in value distribution theory [1–4]: $T(r,f), N(r,f), m(r,f), \ldots$.

We denote by $S(r,f)$ any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$ possible outside of an exceptional set $E\in(0,\infty)$ with finite measure. A meromorphic function α is said to be a small function of f if it satisfies $T(r,\alpha)=S(r,f)$.

Let f and g be two nonconstant meromorphic functions, and let α be a small function of both f and g. If $f-\alpha$ and $g-\alpha$ have the same zeros counting multiplicities (ignoring multiplicities), then we call that f and g share α CM (IM). Moreover, we denote by $N(r,\alpha)$ be the counting function for common zeros of both $f-\alpha$ and $g-\alpha$ with the same multiplicities and the multiplicity is counted. If

$$N\left(r,\tfrac{1}{f-\alpha}\right)+N\left(r,\tfrac{1}{g-\alpha}\right)-2N(r,\alpha)\leqslant S(r,f)+S(r,g),$$

then we call that f and g share α CM almost. If

$$\overline{N}\left(r,\tfrac{1}{f-\alpha}\right)+\overline{N}\left(r,\tfrac{1}{g-\alpha}\right)-2\overline{N}(r,\alpha)\leqslant S(r,f)+S(r,g),$$

then we call that f and g share α IM almost.

Let f and g share 1 IM almost. We denote by $\overline{N}_L\left(r,\tfrac{1}{f-1}\right)$ the counting function for 1-points of both f and g about which f has larger multiplicity than g, with multiplicity being not counted. Similarly, we have the notation $\overline{N}_L\left(r,\tfrac{1}{g-1}\right)$. Especially, if f and g share 1 CM, then $\overline{N}_L\left(r,\tfrac{1}{f-1}\right)=\overline{N}_L\left(r,\tfrac{1}{g-1}\right)=0$.

We denote by $N_{(k}(r,f)$ the counting function for poles of f with multiplicity at least k, and by $\overline{N}_{(k}(r,f)$ the corresponding one for which multiplicity is not counted. Set $N_k(r,f)=\overline{N}(r,f)+\overline{N}_{(2}(r,f)+\cdots+\overline{N}_{(k}(r,f)$.

Fang-Hua [5], Yang-Hua [6] obtained the following unicity theorem.

Theorem A *Let f and g be two nonconstant entire functions, and let $n(\geqslant 6)$ be a positive integer. If f^nf' and g^ng' share 1 CM, then either $f(z)=c_1\,\mathrm{e}^{cz}$, $g(z)=c_2\,\mathrm{e}^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2=-1$, or $f\equiv tg$, where t is a constant such that $t^{n+1}=1$.*

In 2002, Fang-Qiu [7] proved the following theorem.

Theorem B *Let f and g be two nonconstant entire functions, and let $n(\geqslant 6)$ be a positive integer. If f^nf' and g^ng' share z CM, then either $f(z)=c_1\,\mathrm{e}^{cz^2}$, $g(z)=c_2\,\mathrm{e}^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2=-1$, or $f\equiv tg$, where t is a constant such that $t^{n+1}=1$.*

In 2002, Fang [8] obtained the following result.

Theorem C *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n>2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z)=c_1\,\mathrm{e}^{cz}$, $g(z)=c_2\,\mathrm{e}^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k}=1$, or $f\equiv tg$, where t is a constant such that $t^n=1$.*

In 2004, Zhang [9] proved the following theorem.

Theorem D *Let n, k be two positive integers with $n > 2k+4$, and let f and g be two nonconstant entire functions all whose zeros have multiplicity at least n. If $f^{(k)}$ and $g^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k c_1 c_2 c^{2k} = 1$, or $f \equiv g$.*

In 2008, Zhang [10] proved the following theorem.

Theorem E *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n > 2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, then*
(1) $k = 1$, either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^n(nc)^2 = -1$, or $f \equiv g$;
(2) $k \geqslant 2$, $f \equiv tg$, where t is a constant such that $t^n = 1$.

From above theorems, we naturally ask the following problem.

Problem 1. Are Theorems A–E valid or not for meromorphic functions?

In 2010, Xu-Lű-Yi [11] studied the case of meromorphic functions and obtained the following theorems.

Theorem F *Let f and g be two nonconstant meromorphic functions, and let n, k be two positive integers with $n > 3k+8$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, f and g share ∞ IM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f \equiv tg$, where t is a constant such that $t^n = 1$.*

Theorem G *Let f and g be two nonconstant meromorphic functions, and let n, k be two positive integers with $n > 3k+10$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ IM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^n(nc)^2 = -1$, or $f \equiv tg$, where t is a constant such that $t^n = 1$.*

In this paper, we extend and improve Theorem F and Theorem G, and prove the following results.

Theorem 1 *Let n, k be two positive integers with $n > 3k+6$, and let f and g be two meromorphic functions all whose zeros and poles have multiplicity at least n. If $f^{(k)}$ and $g^{(k)}$ share 1 CM, f and g share ∞ IM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k c_1 c_2 c^{2k} = 1$, or $f \equiv g$.*

Remark 1 By Theorem 1, we get Theorem F, and improve the condition that $n > 3k+8$ to $n > 3k+6$.

Theorem 2 *Let n, k be two positive integers with $n > 3k+5$, and let f and g be two nonconstant meromorphic functions all whose zeros and poles have multiplicity at least n. If $f^{(k)}$ and $g^{(k)}$ share 1 CM, f and g share ∞ CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)c^{2k} = 1$, or $f \equiv g$.*

Theorem 3 *Let n, k be two positive integers with $n > 3k+8$, and let f and g be two nonconstant meromorphic functions all whose zeros and poles have multiplicity at least n. If $f^{(k)}$ and $g^{(k)}$ share z CM, f and g share ∞ IM, then*
(1) $k = 1$, either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4c_1c_2c^2 = -1$, or $f \equiv g$;
(2) $k \geqslant 2$, $f \equiv g$.

Remark 2 By Theorem 3, we get Theorem G, and improve the condition that $n > 3k+10$ to $n > 3k+8$.

Theorem 4 *Let n, k be two positive integers with $n > 3k+7$, and let f and g be two nonconstant meromorphic functions all whose zeros and poles have multiplicity at least n. If $f^{(k)}$ and $g^{(k)}$ share z CM, f and g share ∞ CM, then*
(1) $k = 1$, either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4c_1c_2c^2 = -1$, or $f \equiv g$;
(2) $k \geqslant 2$, $f \equiv g$.

LEMMAS

Lemma 1 ([1]) *Let f be a nonconstant meromorphic function, let k be a positive integer, and let c be a nonzero constant. Then*

$$\begin{aligned} T(r,f) \leqslant\ & \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)}-c}\right) \\ & - N\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f) \\ \leqslant\ & \overline{N}(r,f) + N_{k+1}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-c}\right) \\ & - N_0\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f), \end{aligned}$$

where $N_0\left(r,\frac{1}{f^{(k+1)}}\right)$ is the counting function for which $f^{(k+1)} = 0$ and $f(f^k - c) \neq 0$.

Lemma 2 ([1]) *Let f be a nonconstant meromorphic function, and let k be a positive integer. Then*

$$m\left(r,\frac{f^{(k)}}{f}\right) = S(r,f).$$

Lemma 3 ([1]) *Let f be a nonconstant meromorphic function, and let α_1, α_2, α_3 (one may be ∞) be three distinct small functions of f. Then*

$$\begin{aligned} T(r,f) \leqslant\ & \overline{N}\left(r,\frac{1}{f-\alpha_1}\right) + \overline{N}\left(r,\frac{1}{f-\alpha_2}\right) \\ & + \overline{N}\left(r,\frac{1}{f-\alpha_3}\right) + S(r,f). \end{aligned}$$

Lemma 4 ([2]) *Let k be a positive integer, let f be a meromorphic function such that $f^{(k)} \not\equiv 0$. Then*

$$T\left(r, f^{(k)}\right) \leqslant T(r, f) + k\overline{N}(r, f) + S(r, f),$$
$$N\left(r, \tfrac{1}{f^{(k)}}\right) \leqslant N\left(r, \tfrac{1}{f}\right) + k\overline{N}(r, f) + S(r, f).$$

Lemma 5 ([2]) *Let f be a nonconstant meromorphic function, and let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where $a_n (\neq 0)$, $a_{n-1}, \ldots, a_0$ are constants. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 6 ([2]) *Let $f_i (i = 1, 2, 3)$ be nonconstant meromorphic functions such that $f_1 + f_2 + f_3 \equiv 1$. If $f_i (i = 1, 2, 3)$ are linearly independent, then*

$$T(r, f_i) \leqslant \sum_{i=1}^{3} N_2\left(r, \frac{1}{f_i}\right) + \sum_{i=1}^{3} \overline{N}(r, f_i) + o(T(r)),$$

where $T(r) = \max_{1\leqslant i\leqslant 3} \{T(r, f_i)\}$ and $r \notin E$.

Lemma 7 ([2]) *Let f be a nonconstant meromorphic function, let $n (\geqslant 2)$ be a positive integer, and let α_1, $\alpha_2, \ldots, \alpha_n$ be distinct small functions of f. Then*

$$m\left(r, \frac{1}{f - \alpha_1}\right) + \cdots + m\left(r, \frac{1}{f - \alpha_n}\right) \leqslant m\left(r, \frac{1}{f - \alpha_1} + \cdots + \frac{1}{f - \alpha_n}\right) + S(r, f).$$

Lemma 8 ([2]) *Let n, k be two positive integers with $n > k + 5$, let f and g be two meromorphic functions all whose zeros and poles have multiplicity at least n, and let $f^{(k)} \not\equiv 0$ and $g^{(k)} \not\equiv 0$. If $f^{(k)}$ and $g^{(k)}$ share z CM, then*

$$T(r, f) = O(T(r, g)), \qquad T(r, g) = O(T(r, f)).$$

Proof: From the condition of Lemma 8, we obtain

$$\begin{aligned} T(r, f) &\geqslant n \log r + O(1), \\ T(r, g) &\geqslant n \log r + O(1). \end{aligned} \tag{1}$$

By Lemma 2, Lemma 4, Lemma 7 and Nevanlinna's first fundamental theorem, we have

$$\begin{aligned} &m\left(r, \tfrac{1}{f}\right) + m\left(r, \tfrac{1}{f^{(k)} - z}\right) \\ &\leqslant m\left(r, \tfrac{1}{f^{(k)}}\right) + m\left(r, \tfrac{1}{f^{(k+1)} - 1}\right) + S(r, f) \\ &\leqslant m\left(r, \tfrac{1}{f^{(k+2)}}\right) + S(r, f) \\ &\leqslant T\left(r, f^{(k+2)}\right) - N\left(r, \tfrac{1}{f^{(k+2)}}\right) + S(r, f) \\ &\leqslant T\left(r, f^{(k)}\right) + 2\overline{N}(r, f) - N\left(r, \tfrac{1}{f^{(k+2)}}\right) + S(r, f). \end{aligned} \tag{2}$$

From (2) and Nevanlinna's first fundamental theorem, we have

$$\begin{aligned} &T(r, f) + T\left(r, f^{(k)} - z\right) \\ &\leqslant T\left(r, f^{(k)}\right) + 2\overline{N}(r, f) + N\left(r, \tfrac{1}{f}\right) \\ &\quad + N\left(r, \tfrac{1}{f^{(k)} - z}\right) - N\left(r, \tfrac{1}{f^{(k+2)}}\right) + S(r, f) \\ &\leqslant T\left(r, f^{(k)}\right) + 2\overline{N}(r, f) + (k+2)\overline{N}\left(r, \tfrac{1}{f}\right) \\ &\quad + N\left(r, \tfrac{1}{f^{(k)} - z}\right) + S(r, f). \end{aligned} \tag{3}$$

It follows from (1) that

$$\begin{aligned} T\left(r, f^{(k)} - z\right) &\geqslant T\left(r, f^{(k)}\right) - \log r \\ &\geqslant T\left(r, f^{(k)}\right) - \frac{1}{n} T(r, f). \end{aligned} \tag{4}$$

Hence, by (3), (4), and $f^{(k)}$ and $g^{(k)}$ share z CM, we have

$$\begin{aligned} \tfrac{n-1}{n} T(r, f) &\leqslant 2\overline{N}(r, f) + (k+2)\overline{N}\left(r, \tfrac{1}{f}\right) \\ &\quad + N\left(r, \tfrac{1}{g^{(k)} - z}\right) + S(r, f) \\ &\leqslant \tfrac{2}{n} N(r, f) + \tfrac{k+2}{n} N\left(r, \tfrac{1}{f}\right) \\ &\quad + T\left(r, \tfrac{1}{g^{(k)} - z}\right) + S(r, f) \\ &\leqslant \tfrac{k+4}{n} T(r, f) + T\left(r, \tfrac{1}{g^{(k)} - z}\right) + S(r, f). \end{aligned} \tag{5}$$

From (1), (5), Nevanlinna's first fundamental theorem and Lemma 4, we get

$$\begin{aligned} \tfrac{n-k-5}{n} T(r, f) &\leqslant T\left(r, \tfrac{1}{g^{(k)} - z}\right) + S(r, f) \\ &\leqslant T\left(r, g^{(k)} - z\right) + S(r, f) \\ &\leqslant T\left(r, g^{(k)}\right) + \log r + S(r, f) \\ &\leqslant T(r, g) + k\overline{N}(r, g) + \log r + S(r, f) + S(r, g) \\ &\leqslant T(r, g) + \tfrac{k}{n} N(r, g) + \tfrac{1}{n} T(r, g) + S(r, f) + S(r, g) \\ &\leqslant \tfrac{n+k+1}{n} T(r, g) + S(r, f) + S(r, g). \end{aligned}$$

Then

$$T(r, f) \leqslant \tfrac{n+k+1}{n-k-5} T(r, g) + S(r, f) + S(r, g).$$

By $n > k + 5$, we get $T(r, f) = O(T(r, g))$. Similarly, we obtain $T(r, g) = O(T(r, f))$. □

By the proof of Lemma 8, we have the following results.

Lemma 9 *Let n, k be two positive integers with $n > k + 2$, let f and g be two nonconstant meromorphic functions all whose zeros and poles have multiplicity at least n, and let $f^{(k+1)} \not\equiv 0$ and $g^{(k+1)} \not\equiv 0$. If $f^{(k)}$ and $g^{(k)}$ share 1 CM. Then*

$$T(r, f) = O(T(r, g)), \quad T(r, g) = O(T(r, f)).$$

Lemma 10 ([12]) *Let f be a nonconstant entire function, and let $k(\geqslant 2)$ be a positive integer. If $ff^{(k)} \neq 0$, then $f(z) = \mathrm{e}^{az+b}$, where $a(\neq 0)$, b are two constants.*

Lemma 11 ([9]) *Let f and g be two nonconstant entire functions, and let k be a positive integer. If all the zeros of both f and g are of multiplicity at least $k+1$, and $f^{(k)}g^{(k)} \equiv 1$, then $f(z) = c_1 \mathrm{e}^{cz}$, $g(z) = c_2 \mathrm{e}^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)c^{2k} = 1$.*

Lemma 12 ([13]) *Let f be a meromorphic function, and let $k(\geqslant 2)$ be a positive integer. If f and $f^{(k)}$ have finitely many zeros, then $f = R\mathrm{e}^P$, where R is a rational function and P is a polynomial.*

Lemma 13 *Let k be a positive integer, let f and g be two nonconstant entire functions whose zeros are of multiplicity at least $k+1$. If $f^{(k)}g^{(k)} \equiv z^2$, then*

(1) $k=1$, either $f(z) = c_1\mathrm{e}^{cz^2}$, $g(z) = c_2\mathrm{e}^{-cz^2}$ or $f(z) = \frac{Cz^2}{2}$, $g(z) = \frac{z^2}{2C}$, where c_1, c_2, c and C are nonzero constants satisfying $4c_1c_2c^2 = -1$;

(2) $k \geqslant 2$, $f(z) = \frac{Cz^{k+1}}{(k+1)!}$, $g(z) = \frac{z^{k+1}}{C(k+1)!}$, where C is a nonzero constant.

Proof: In the following, we consider two cases.

Case 1. $k=1$. Then $f'g' \equiv z^2$. Since all zeros of f and g are of multiplicity at least 2, f and g are two entire functions, then we know that $f = z^l\mathrm{e}^{\alpha}$, $g = z^m\mathrm{e}^{\beta}$, where $l, m \in \{0,2,3\}$ such that $l+m \leqslant 4$, and α, β are two entire functions.

Next, we consider five subcases.

Case 1.1. $l=0$, $m=0$. Then $f = \mathrm{e}^{\alpha}$, $g = \mathrm{e}^{\beta}$. It follows

$$f'g' = \alpha'\beta'\mathrm{e}^{\alpha+\beta} \equiv z^2. \tag{6}$$

Now, by (6) we consider three subcases.

Case 1.1.1. $\alpha' = z^2\mathrm{e}^{\gamma_1}$, $\beta' = \mathrm{e}^{\delta_1}$, where γ_1, δ_1 are two entire functions. Obviously, $T(r,\gamma_1') = m(r,\gamma_1') = m\left(r, \frac{(\mathrm{e}^{\gamma_1})'}{\mathrm{e}^{\gamma_1}}\right) = S(r,\mathrm{e}^{\gamma_1})$. Similarly, $T(r,\delta_1') = S(r,\mathrm{e}^{\delta_1})$.

By (6) we have $\mathrm{e}^{\alpha+\beta+\gamma_1+\delta_1} \equiv 1$. It follows $\alpha'+\beta'+\gamma_1'+\delta_1' \equiv 0$. Then,

$$z^2\mathrm{e}^{\gamma_1} + \mathrm{e}^{\delta_1} + \gamma_1' + \delta_1' \equiv 0. \tag{7}$$

We claim $\gamma_1'+\delta_1' \equiv 0$. Suppose, on the contrary, that $\gamma_1'+\delta_1' \not\equiv 0$. By (6), (7) and Lemma 3 we have

$$\begin{aligned} T(r,\mathrm{e}^{\delta_1}) &\leqslant \overline{N}(r,\mathrm{e}^{\delta_1}) + \overline{N}\left(r,\frac{1}{\mathrm{e}^{\delta_1}}\right) \\ &\quad + \overline{N}\left(r,\frac{1}{\mathrm{e}^{\delta_1}+\gamma_1'+\delta_1'}\right) + S(r,\mathrm{e}^{\delta_1}) \\ &\leqslant \overline{N}\left(r,\frac{1}{z^2\mathrm{e}^{\gamma_1}}\right) + S(r,\mathrm{e}^{\delta_1}) \\ &\leqslant \log r + S(r,\mathrm{e}^{\delta_1}). \end{aligned}$$

Hence, e^{δ_1} is a constant, which yields δ_1 is a constant. If γ_1 is a constant, then $\gamma_1'+\delta_1' \equiv 0$, a contradiction. If γ_1 is a nonconstant, then

$$\begin{aligned} m(r,z^2\mathrm{e}^{\gamma_1}) &\leqslant m(r,\mathrm{e}^{\gamma_1}) + m(r,z^2) \\ &= m(r,\mathrm{e}^{\gamma_1}) + S(r,\mathrm{e}^{\gamma_1}). \end{aligned} \tag{8}$$

$$\begin{aligned} m(r,\mathrm{e}^{\gamma_1}) &\leqslant m(r,z^2\mathrm{e}^{\gamma_1}) + m\left(r,\frac{1}{z^2}\right) \\ &= m(r,z^2\mathrm{e}^{\gamma_1}) + S(r,\mathrm{e}^{\gamma_1}). \end{aligned} \tag{9}$$

By (7)–(9) and $m(r,\gamma_1') = S(r,\mathrm{e}^{\gamma_1})$, we have

$$\begin{aligned} T(r,\mathrm{e}^{\gamma_1}) &= m(r,\mathrm{e}^{\gamma_1}) \\ &= m(r,z^2\mathrm{e}^{\gamma_1}) + S(r,\mathrm{e}^{\gamma_1}) \\ &= m(r,\mathrm{e}^{\delta_1}+\gamma_1') = S(r,\mathrm{e}^{\gamma_1}), \end{aligned}$$

a contradiction.

Hence, $\gamma_1'+\delta_1' \equiv 0$. That is $\gamma_1+\delta_1 = C$, where C is a constant. Then by (7) we have $z^2\mathrm{e}^{2\gamma_1} + \mathrm{e}^{C} \equiv 0$, a contradiction.

Case 1.1.2. $\alpha' = z\mathrm{e}^{\gamma_2}$, $\beta' = z\mathrm{e}^{\delta_2}$, where γ_2, δ_2 are two entire functions. By (6) we have $z\mathrm{e}^{\gamma_2} + z\mathrm{e}^{\delta_2} + \gamma_2' + \delta_2' \equiv 0$. Using the same argument as used in Case 1.1.1, we know $\gamma_2'+\delta_2' \equiv 0$. That is $z\mathrm{e}^{\gamma_2} + z\mathrm{e}^{\delta_2} \equiv 0$. Hence, we know that γ_2, δ_2 are two constants. Thus $\alpha' = 2cz$, $\beta' = -2cz$, where c is a nonzero constant. It follows $\alpha(z) = cz^2 + \log c_1$, $\beta(z) = -cz^2 + \log c_2$, where c_1, c_2 are two nonzero constants. Therefore, $f(z) = \mathrm{e}^{\alpha(z)} = c_1\mathrm{e}^{cz^2}$, $g(z) = \mathrm{e}^{\beta(z)} = c_2\mathrm{e}^{-cz^2}$, where c, c_1, c_2 satisfying $4c_1c_2c^2 = -1$.

Case 1.1.3. $\alpha' = \mathrm{e}^{\gamma_3}$, $\beta' = z^2\mathrm{e}^{\delta_3}$, where γ_3, δ_3 are two entire functions. Using the same argument as used in Case 1.1.1, we get a contradiction.

Case 1.2. $l=2$, $m=0$. Then $f = z^2\mathrm{e}^{\alpha}$, $g = \mathrm{e}^{\beta}$. It follows from $f'g' \equiv z^2$ that $z(2+z\alpha')\beta'\mathrm{e}^{\alpha+\beta} \equiv z^2$. Therefore, we know that there exist two entire functions γ_4 and δ_4 such that $2+z\alpha' = \mathrm{e}^{\gamma_4}$, $\beta' = z\mathrm{e}^{\delta_4}$. It follows $\mathrm{e}^{\gamma_4} + z^2\mathrm{e}^{\delta_4} + z\gamma_4' + z\delta_4' \equiv 2$. Using the same argument as used in Case 1.1, we obtain a contradiction.

Case 1.3. $l=2$, $m=2$. Then $f = z^2\mathrm{e}^{\alpha}$, $g = z^2\mathrm{e}^{\beta}$. It follows from $f'g' \equiv z^2$ that

$$z^2(2+z\alpha')(2+z\beta')\mathrm{e}^{\alpha+\beta} \equiv z^2. \tag{10}$$

By (10) we know that there exist two entire functions γ_5 and δ_5 such that

$$2+z\alpha' = \mathrm{e}^{\gamma_5}, \quad 2+z\beta' = \mathrm{e}^{\delta_5}. \tag{11}$$

By (10) and (11) we deduce

$$\gamma_5' + \delta_5' + \alpha' + \beta' = 0. \tag{12}$$

It follows from (11) and (12) that $\mathrm{e}^{\gamma_5} + \mathrm{e}^{\delta_5} + z(\gamma_5'+\delta_5') \equiv 4$. Using the same argument as used in Case 1.1, we know that both γ_5 and δ_5 are constants. Thus $\alpha'+\beta' \equiv 0$.

It follows from (10) that $z^2(4-z^2\alpha'^2)\,e^{\alpha+\beta}\equiv z^2$. Hence, $\alpha'=0$, $\beta'=0$. By (10) we have $f(z)=\frac{Cz^2}{2}$, $g(z)=\frac{z^2}{2C}$, where C is a nonzero constant.

Case 1.4. $l=3$, $m=0$. Then $f=z^3\,e^{\alpha}$, $g=e^{\beta}$. It follows from $f'g'\equiv z^2$ that $z^2(3+z\alpha')\beta'\,e^{\alpha+\beta}\equiv z^2$. Using the same argument as used in Case 1.2, we obtain a contradiction.

Case 1.5. Either $l=0$, $m=2$ or $l=0$, $m=3$. Using the same argument as used in Case 1.2, we obtain a contradiction.

Case 2. $k\geqslant 2$. Since f and g are two nonconstant entire functions, then by $f^{(k)}g^{(k)}\equiv z^2$ we know that $f^{(k)}$ and $g^{(k)}$ have zeros only at $z=0$. It follows from the zeros of f and g are of multiplicity at least $k+1$ that f and g have zeros only at $z=0$. Thus, by Lemma 12 we deduce that $f=z^l\,e^P$, $g=z^m\,e^Q$, where $l,m\in\{0,k+1,k+2\}$, such that $l+m\leqslant 2k+2$, and P, Q are polynomials.

By $f^{(k)}g^{(k)}\equiv z^2$ we have

$$\left(z^l(P')^k+H_k\right)\left(z^m(Q')^k+R_k\right)e^{P+Q}\equiv z^2, \tag{13}$$

where H_k, R_k are two polynomials with $\deg H_k=l+k\deg P'-1$, $\deg R_k=l+k\deg Q'-1$.

In the following, we consider three subcases.

Case 2.1. $P'\equiv 0$, $Q'\equiv 0$. Then $f(z)=az^l$, $g(z)=bz^m$, where a, b are two nonzero constants. By $f^{(k)}g^{(k)}\equiv z^2$, we know that $f=az^{k+1}$, $g=bz^{k+1}$. Thus $f^{(k)}g^{(k)}=ab\,((k+1)!)^2z^2\equiv z^2$, which yields $ab((k+1)!)^2=1$. Therefore, $f(z)=\frac{Cz^{k+1}}{(k+1)!}$, $g(z)=\frac{z^{k+1}}{C(k+1)!}$, where C is a nonzero constant.

Case 2.2. Either $P'\equiv 0$, $Q'\not\equiv 0$ or $P'\not\equiv 0$, $Q'\equiv 0$. Without loss of generality, we consider the case of $P'\equiv 0$, $Q'\not\equiv 0$.

By (13) we have $l(l-1)\cdots(l-k+1)z^{l-k}\,e^P\,(z^m(Q')^k+R_k)\,e^Q\equiv z^2$. Obviously, $l(l-1)\cdots(l-k+1)z^{l-k}\,e^P(z^m(Q')^k+R_k)$ is a nonzero polynomial be written as U_k. It follows $T(r,e^Q)=T\left(r,\frac{z^2}{U_k}\right)=O(\log r)=S(r,e^Q)$, a contradiction.

Case 2.3. $P'\not\equiv 0$, $Q'\not\equiv 0$. By (13) we deduce

$$l+m+k(\deg P'+\deg Q')=2. \tag{14}$$

Next, we consider three subcases.

Case 2.3.1. $\deg P'+\deg Q'\geqslant 2$. By $k\geqslant 2$, $l+m\geqslant 2$ and (14), we obtain a contradiction.

Case 2.3.2. $\deg P'+\deg Q'=1$. Without loss of generality, we consider the case of $\deg P'=1$, $\deg Q'=0$. Set

$$P=az^2+bz+d,\quad Q=qz+r, \tag{15}$$

where $a(\neq 0), b, d, q$, and r are constants.

By (13) and (15) we have $f^{(k)}g^{(k)}=T_k\,e^{az^2+(b+q)z+d+r}\equiv z^2$, where T_k is a polynomial. Using the same argument as used in Case 2.2, we get a contradiction.

Case 2.3.3. $\deg P'=\deg Q'=0$. By (14) we have $l+m=2$. From $l,m\in\{0,k+1,k+2\}$ and $k\geqslant 2$, we get a contradiction. □

PROOF OF THEOREM 1

Set

$$\phi=\frac{f^{(k+1)}}{f^{(k)}(f^{(k)}-1)}-\frac{g^{(k+1)}}{g^{(k)}(g^{(k)}-1)}. \tag{16}$$

Now, we consider two cases.

Case 1. $\phi\equiv 0$. It follows from (16) that

$$\frac{f^{(k)}-1}{f^{(k)}}\equiv C\,\frac{g^{(k)}-1}{g^{(k)}}, \tag{17}$$

where C is a nonzero constant.

Next, we consider two subcases.

Case 1.1. $C=1$. By (17) we have $f^{(k)}\equiv g^{(k)}$. Hence $f\equiv g+P$, where P is a polynomial with $\deg P\leqslant k-1$.

If $P\equiv 0$, then $f\equiv g$.

If $P\not\equiv 0$, then we obtain

$$\frac{f}{P}-\frac{g}{P}=1. \tag{18}$$

Since f and g are two nonconstant meromorphic functions whose zeros and poles multiplicities at least n, then we have

$$\begin{aligned}T(r,f)&\geqslant n\log r+O(1),\\ T(r,g)&\geqslant n\log r+O(1).\end{aligned} \tag{19}$$

By (19) and Nevanlinna's first fundamental theorem we get

$$\begin{aligned}T\left(r,\tfrac{f}{P}\right)&\leqslant T(r,f)+T(r,P)+O(1)\\ &\leqslant T(r,f)+(k-1)\log r+O(1)\\ &\leqslant T(r,f)+(k-1)T(r,f)+O(1)\\ &\leqslant kT(r,f)+O(1).\end{aligned}$$

It follows

$$S\left(r,\tfrac{f}{P}\right)=S(r,f). \tag{20}$$

By (18)–(20) and Nevanlinna's first and second fundamental theorems we have

$$\begin{aligned}T(r,f)&\leqslant T\left(r,\tfrac{f}{P}\right)+T(r,P)+O(1)\\ &\leqslant \overline{N}\left(r,\tfrac{f}{P}\right)+\overline{N}\left(r,\tfrac{P}{f}\right)+\overline{N}\left(r,\frac{1}{\frac{f}{P}-1}\right)\\ &\quad+T(r,P)+S(r,f)\\ &\leqslant \overline{N}(r,f)+\overline{N}\left(r,\tfrac{1}{P}\right)+\overline{N}\left(r,\tfrac{1}{f}\right)+\overline{N}\left(r,\tfrac{P}{g}\right)\\ &\quad+T(r,P)+S(r,f)\\ &\leqslant \overline{N}(r,f)+\overline{N}\left(r,\tfrac{1}{f}\right)+\overline{N}\left(r,\tfrac{1}{g}\right)\\ &\quad+2T(r,P)+S(r,f)\\ &\leqslant \tfrac{1}{n}N(r,f)+\tfrac{1}{n}N\left(r,\tfrac{1}{f}\right)+\tfrac{1}{n}N\left(r,\tfrac{1}{g}\right)\\ &\quad+\tfrac{2k-2}{n}T(r,f)+S(r,f)\\ &\leqslant \tfrac{2k}{n}T(r,f)+\tfrac{1}{n}T(r,g)+S(r,f).\end{aligned} \tag{21}$$

Similarly,

$$T(r,g) \leqslant \frac{2k}{n}T(r,g)+\frac{1}{n}T(r,f)+S(r,g). \qquad (22)$$

Since $n>3k+6$, then by (21) and (22) we have $T(r,f)+T(r,g)\leqslant S(r,f)+S(r,g)$, a contradiction.

Case 1.2. $C\neq 1$. By (17) we have

$$\frac{1}{f^{(k)}}-\frac{C}{g^{(k)}}=1-C. \qquad (23)$$

Since f and g share ∞ IM, then by (23) we know that $f^{(k)}\neq\infty$ and $g^{(k)}\neq\infty$. Thus, $\frac{1}{f^{(k)}}\neq 0$ and $g^{(k)}\neq\frac{C}{C-1}$.

By Lemma 1 and Nevanlinna's first fundamental theorem, we obtain

$$\begin{aligned} T(r,g) &\leqslant \overline{N}(r,g)+N\left(r,\tfrac{1}{g}\right)+N\left(r,\tfrac{1}{g^{(k)}-\frac{C}{C-1}}\right)\\ &\quad -N\left(r,\tfrac{1}{g^{(k+1)}}\right)+S(r,g)\\ &\leqslant N\left(r,\tfrac{1}{g}\right)-N\left(r,\tfrac{1}{g^{(k+1)}}\right)+S(r,g)\\ &\leqslant (k+1)\overline{N}\left(r,\tfrac{1}{g}\right)+S(r,g)\\ &\leqslant \tfrac{k+1}{n}N\left(r,\tfrac{1}{g}\right)+S(r,g)\\ &\leqslant \tfrac{k+1}{n}T(r,g)+S(r,g). \end{aligned}$$

It follows from $n>3k+6$ that $T(r,g)\leqslant S(r,g)$, a contradiction.

Case 2. $\phi\not\equiv 0$. By Lemma 4, we have

$$\begin{aligned} T\left(r,f^{(k)}\right) &\leqslant T(r,f)+k\overline{N}(r,f)+S(r,f)\\ &\leqslant (k+1)T(r,f)+S(r,f). \end{aligned}$$

Thus $S\left(r,f^{(k)}\right)=S(r,f)$. Similarly, we obtain $S\left(r,g^{(k)}\right)=S(r,g)$.

Let z_0 be a pole of f with multiplicity l_1. By f and g share ∞ IM, we know that z_0 is a pole of g with multiplicity l_2. Set $l=\min\{l_1,l_2\}$. By (16) we deduce that z_0 is a zero of ϕ with multiplicity $\geqslant l+k-1$. It follows from Lemma 2 and Nevanlinna's first fundamental theorem that

$$\begin{aligned} \overline{N}(r,f)=\overline{N}(r,g) &\leqslant \tfrac{1}{l+k-1}N\left(r,\tfrac{1}{\phi}\right)\\ &\leqslant \tfrac{1}{l+k-1}T(r,\phi)+O(1)\\ &\leqslant \tfrac{1}{l+k-1}N(r,\phi)+\tfrac{1}{l+k-1}m(r,\phi)+O(1)\\ &\leqslant \tfrac{1}{l+k-1}N(r,\phi)+S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right)\\ &\leqslant \tfrac{1}{l+k-1}N(r,\phi)+S(r,f)+S(r,g)\\ &\leqslant \tfrac{1}{l+k-1}\left[\overline{N}\left(r,\tfrac{1}{f^{(k)}}\right)+\overline{N}\left(r,\tfrac{1}{g^{(k)}}\right)\right]\\ &\quad +S(r,f)+S(r,g). \end{aligned} \qquad (24)$$

By Lemma 4 we have

$$\begin{aligned} \overline{N}\left(r,\tfrac{1}{f^{(k)}}\right) &= N\left(r,\tfrac{1}{f^{(k)}}\right)-\left[N\left(r,\tfrac{1}{f^{(k)}}\right)-\overline{N}\left(r,\tfrac{1}{f^{(k)}}\right)\right]\\ &\leqslant N\left(r,\tfrac{1}{f}\right)+k\overline{N}(r,f)\\ &\quad -\left[N\left(r,\tfrac{1}{f^{(k)}}\right)-\overline{N}\left(r,\tfrac{1}{f^{(k)}}\right)\right]+S(r,f)\\ &\leqslant (k+1)\overline{N}\left(r,\tfrac{1}{f}\right)+k\overline{N}(r,f)+S(r,f)\\ &\leqslant \tfrac{2k+1}{n}T(r,f)+S(r,f). \end{aligned} \qquad (25)$$

Similarly,

$$\overline{N}\left(r,\tfrac{1}{g^{(k)}}\right)\leqslant \tfrac{2k+1}{n}T(r,g)+S(r,g). \qquad (26)$$

By (24)—(26), we get

$$\begin{aligned} &\overline{N}(r,f)=\overline{N}(r,g)\\ &\leqslant \tfrac{2k+1}{n(l+k-1)}[T(r,f)+T(r,g)]+S(r,f)+S(r,g). \end{aligned} \qquad (27)$$

Set

$$\varphi=\frac{f^{(k+2)}}{f^{(k+1)}}-2\frac{f^{(k+1)}}{f^{(k)}-1}-\frac{g^{(k+2)}}{g^{(k+1)}}+2\frac{g^{(k+1)}}{g^{(k)}-1}. \qquad (28)$$

Let z_0 be a common simple zero of $f^{(k)}-1$ and $g^{(k)}-1$. By computation we have $\varphi(z_0)=0$. Suppose that $\varphi\not\equiv 0$. It follows from Lemma 2 and Nevanlinna's first fundamental theorem that

$$\begin{aligned} &N_{1)}\left(r,\tfrac{1}{f^{(k)}-1}\right)=N_{1)}\left(r,\tfrac{1}{g^{(k)}-1}\right)\leqslant N\left(r,\tfrac{1}{\varphi}\right)\\ &\quad\leqslant T(r,\varphi)+O(1)\leqslant N(r,\varphi)+S(r,f)+S(r,g), \end{aligned} \qquad (29)$$

where $N_{1)}\left(r,\frac{1}{f^{(k)}-1}\right)$ is the counting function of simple zeros of $f^{(k)}-1$. Similarly, we have the notation $N_{1)}\left(r,\frac{1}{g^{(k)}-1}\right)$. Since $f^{(k)}$ and $g^{(k)}$ share 1 CM, and f and g share ∞ IM, then by (28) we have

$$\begin{aligned} N(r,\varphi) &\leqslant \tfrac{1}{2}\overline{N}(r,f)+\tfrac{1}{2}\overline{N}(r,g)+\overline{N}\left(r,\tfrac{1}{f}\right)\\ &\quad +\overline{N}\left(r,\tfrac{1}{g}\right)+N_0\left(r,\tfrac{1}{f^{(k+1)}}\right)+N_0\left(r,\tfrac{1}{g^{(k+1)}}\right), \end{aligned} \qquad (30)$$

where $N_0\left(r,\frac{1}{f^{(k+1)}}\right)$ is the counting function for which $f^{(k+1)}=0$ and $f(f^{(k)}-1)\neq 0$. Similarly, we have the notation $N_0\left(r,\frac{1}{g^{(k+1)}}\right)$.

By Lemma 1, we obtain

$$\begin{aligned} T(r,f) &\leqslant \overline{N}(r,f)+N_{k+1}\left(r,\tfrac{1}{f}\right)+\overline{N}\left(r,\tfrac{1}{f^{(k)}-1}\right)\\ &\quad -N_0\left(r,\tfrac{1}{f^{(k+1)}}\right)+S(r,f), \end{aligned} \qquad (31)$$

$$\begin{aligned} T(r,g) &\leqslant \overline{N}(r,g)+N_{k+1}\left(r,\tfrac{1}{g}\right)+\overline{N}\left(r,\tfrac{1}{g^{(k)}-1}\right)\\ &\quad -N_0\left(r,\tfrac{1}{g^{(k+1)}}\right)+S(r,g). \end{aligned} \qquad (32)$$

From $f^{(k)}$ and $g^{(k)}$ share 1 CM, we have

$$\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)=N_{1)}\left(r,\frac{1}{f^{(k)}-1}\right)+\frac{1}{2}\left[N\left(r,\frac{1}{f^{(k)}-1}\right)+N\left(r,\frac{1}{g^{(k)}-1}\right)\right]. \quad (33)$$

It follows from (31)–(33) that

$$\begin{aligned}T(r,f)+T(r,g)\leqslant{}&\overline{N}(r,f)+\overline{N}(r,g)\\&+\tfrac{k+1}{n}\left[N\left(r,\tfrac{1}{f}\right)+N\left(r,\tfrac{1}{g}\right)\right]\\&+\tfrac{1}{2}\left[N\left(r,\tfrac{1}{f^{(k)}-1}\right)+N\left(r,\tfrac{1}{g^{(k)}-1}\right)\right]\\&-\left[N_0\left(r,\tfrac{1}{f^{(k+1)}}\right)+N_0\left(r,\tfrac{1}{g^{(k+1)}}\right)\right]\\&+N_{1)}\left(r,\tfrac{1}{f^{(k)}-1}\right)+S(r,f)+S(r,g). \quad (34)\end{aligned}$$

By Lemma 4 and Nevanlinna's first fundamental theorem we have

$$\begin{aligned}&N\left(r,\tfrac{1}{f^{(k)}-1}\right)+N\left(r,\tfrac{1}{g^{(k)}-1}\right)\\&\leqslant T(r,f^{(k)})+T(r,g^{(k)})+O(1)\\&\leqslant T(r,f)+T(r,g)+k\overline{N}(r,f)+k\overline{N}(r,g)\\&\quad+S(r,f)+S(r,g) \quad (35)\\&\leqslant T(r,f)+T(r,g)+\tfrac{k}{n}N(r,f)+\tfrac{k}{n}N(r,g)\\&\quad+S(r,f)+S(r,g).\end{aligned}$$

Since f and g share ∞ IM, then by (27), (29), (30), (34) and (35), we obtain

$$T(r,f)+T(r,g)\leqslant\left[\tfrac{3k+4}{n}+\tfrac{12k+6}{n(l+k-1)}\right][T(r,f)+T(r,g)]+S(r,f)+S(r,g). \quad (36)$$

By (36) and $l\geqslant n>3k+6$ we obtain $T(r,f)+T(r,g)\leqslant S(r,f)+S(r,g)$, a contradiction. Thus $\varphi\equiv 0$. That is

$$\frac{f^{(k+2)}}{f^{(k+1)}}-2\frac{f^{(k+1)}}{f^{(k)}-1}\equiv\frac{g^{(k+2)}}{g^{(k+1)}}-2\frac{g^{(k+1)}}{g^{(k)}-1}.$$

It follows

$$\frac{1}{f^{(k)}-1}=\frac{a}{g^{(k)}-1}+b, \quad (37)$$

where $a(\neq 0)$, b are two finite complex numbers.

Next, we consider two subcases.

Case 2.1. $b\neq 0$. Since f and g share ∞ IM, we know that $f^{(k)}$ and $g^{(k)}$ share ∞ IM. It follows from (37) that $f^{(k)}\neq\infty$, $g^{(k)}\neq\infty$. Hence $\frac{1}{f^{(k)}-1}\neq 0$. By (37) we have $g^{(k)}\neq\frac{b-a}{b}$.

Now, we consider two subcases.

Case 2.1.1. $b\neq a$. By Lemma 1, we have

$$\begin{aligned}T(r,g)\leqslant{}&\overline{N}(r,g)+N\left(r,\tfrac{1}{g}\right)+N\left(r,\frac{1}{g^{(k)}-\frac{b-a}{b}}\right)\\&-N\left(r,\tfrac{1}{g^{(k+1)}}\right)+S(r,g)\leqslant(k+1)\overline{N}\left(r,\tfrac{1}{g}\right)+S(r,g)\\&\leqslant\tfrac{k+1}{n}N\left(r,\tfrac{1}{g}\right)+S(r,g)\leqslant\tfrac{k+1}{n}T(r,g)+S(r,g). \quad (38)\end{aligned}$$

It follows from $n>3k+6$ and (38) that $T(r,g)\leqslant S(r,g)$, a contradiction.

Case 2.1.2. $b=a$. If $b\neq -1$, then using the same argument as used in Case 2.1.1, we get a contradiction. If $b=-1$, then by (37) we have $f^{(k)}g^{(k)}=1$. It follows from Lemma 11 that $f(z)=c_1\,\mathrm{e}^{cz}$, $g(z)=c_2\,\mathrm{e}^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)c^{2k}=-1$.

Case 2.2. $b=0$. We consider two subcases.

Case 2.2.1. $a=1$. By (37) we have $f^{(k)}\equiv g^{(k)}$. Using the same argument as used in Case 1.1, we get $f\equiv g$.

Case 2.2.2. $a\neq 1$. By (37) we obtain $(af-g)^{(k)}\equiv a-1$. It follows $af-g\equiv Q$, where Q is polynomial with $\deg Q=k$. Using the same argument as used in Case 1,1, we have a contradiction.

This completes the proof of Theorem 1.

PROOF OF THEOREM 2

Imitating the proof of Theorem 1, we can prove Theorem 2 only by replacing (30) with the following inequality.

$$N(r,\varphi)\leqslant\overline{N}\left(r,\tfrac{1}{f}\right)+\overline{N}\left(r,\tfrac{1}{g}\right)+N_0\left(r,\tfrac{1}{f^{(k+1)}}\right)+N_0\left(r,\tfrac{1}{g^{(k+1)}}\right).$$

Hence, we omit the details.

PROOF OF THEOREM 3

It follows from $f^{(k)}$ and $g^{(k)}$ share z CM, f and g share ∞ IM that

$$H=\frac{f^{(k)}-z}{g^{(k)}-z}, \quad (39)$$

where $H(\not\equiv 0,\infty)$ is a meromorphic function.

By (39) we have

$$\overline{N}(r,H)\leqslant\overline{N}_L(r,f),\quad \overline{N}\left(r,\tfrac{1}{H}\right)\leqslant\overline{N}_L(r,g). \quad (40)$$

Set

$$f_1=\frac{f^{(k)}}{z},\quad f_2=H,\quad f_3=-\frac{Hg^{(k)}}{z}. \quad (41)$$

Obviously, $f_1+f_2+f_3\equiv 1$.

By (39)–(41) and Lemma 4 we have

$$T(r,f_1)+T(r,f_2)+T(r,f_3)\leqslant O(T(r,f)+T(r,g)). \quad (42)$$

We suppose that f_2 and f_3 are not constants. If f_1,f_2,f_3 are linearly independent, then by (40)–(42) and Lemma 6 we obtain

$$\begin{aligned}T(r,f_1)\leqslant{}&N_2\left(r,\tfrac{1}{f_1}\right)+N_2\left(r,\tfrac{1}{f_2}\right)+N_2\left(r,\tfrac{1}{f_3}\right)\\&+\overline{N}(r,f_1)+\overline{N}(r,f_2)+\overline{N}(r,f_3)+o(T(r))\\&\leqslant N_2\left(r,\tfrac{1}{f^{(k)}}\right)+2\overline{N}_L(r,g)+N_2\left(r,\tfrac{1}{g^{(k)}}\right)+2\overline{N}(r,f)\\&+\overline{N}_L(r,f)+2\log r+S(r,f)+S(r,g). \quad (43)\end{aligned}$$

By (41) and (43) we have

$$\begin{aligned}T(r,f^{(k)}) &\leqslant T(r,f_1)+\log r\\ &\leqslant N_2\left(r,\tfrac{1}{f^{(k)}}\right)+2\overline{N}_L(r,g)+N_2\left(r,\tfrac{1}{g^{(k)}}\right)\\ &\quad+2\overline{N}(r,f)+\overline{N}_L(r,f)+3\log r+S(r,f)+S(r,g)\\ &=N\left(r,\tfrac{1}{f^{(k)}}\right)-\left[N_{(3}\left(r,\tfrac{1}{f^{(k)}}\right)-2\overline{N}_{(3}\left(r,\tfrac{1}{f^{(k)}}\right)\right]+N\left(r,\tfrac{1}{g^{(k)}}\right)\\ &\quad-\left[N_{(3}\left(r,\tfrac{1}{g^{(k)}}\right)-2\overline{N}_{(3}\left(r,\tfrac{1}{g^{(k)}}\right)\right]+2\overline{N}_L(r,g)+2\overline{N}(r,f)\\ &\quad+\overline{N}_L(r,f)+3\log r+S(r,f)+S(r,g).\end{aligned} \tag{44}$$

Let z_0 be a zero of f with multiplicity l. Then z_0 is a zero of $f^{(k)}$ with multiplicity $l-k\geqslant 3$. Thus

$$N_{(3}\left(r,\tfrac{1}{f^{(k)}}\right)-2\overline{N}_{(3}\left(r,\tfrac{1}{f^{(k)}}\right)\geqslant(l-k-2)\overline{N}\left(r,\tfrac{1}{f}\right). \tag{45}$$

Similarly,

$$N_{(3}\left(r,\tfrac{1}{g^{(k)}}\right)-2\overline{N}_{(3}\left(r,\tfrac{1}{g^{(k)}}\right)\geqslant(l-k-2)\overline{N}\left(r,\tfrac{1}{g}\right). \tag{46}$$

It follows from Lemma 2 and Nevanlinna's first fundamental theorem that

$$\begin{aligned}m\left(r,\tfrac{1}{f}\right)&\leqslant m\left(r,\tfrac{1}{f^{(k)}}\right)+S(r,f)\\ &=T(r,f^{(k)})-N\left(r,\tfrac{1}{f^{(k)}}\right)+S(r,f),\\ T(r,f)&\leqslant T\left(r,f^{(k)}\right)-N\left(r,\tfrac{1}{f^{(k)}}\right)\\ &\quad+N\left(r,\tfrac{1}{f}\right)+S(r,f).\end{aligned} \tag{47}$$

By (44)–(47), Lemma 4 and f and g share ∞ IM we obtain

$$\begin{aligned}T(r,f)&\leqslant \tfrac{k+2}{n}N\left(r,\tfrac{1}{f}\right)+\tfrac{k+2}{n}N\left(r,\tfrac{1}{g}\right)+2\overline{N}_L(r,g)\\ &\quad+(k+2)\overline{N}(r,f)+\overline{N}_L(r,f)+3\log r\\ &\quad+S(r,f)+S(r,g).\end{aligned} \tag{48}$$

Similarly,

$$\begin{aligned}T(r,g)&\leqslant \tfrac{k+2}{n}N\left(r,\tfrac{1}{g}\right)+\tfrac{k+2}{n}N\left(r,\tfrac{1}{f}\right)\\ &\quad+2\overline{N}_L(r,f)+(k+2)\overline{N}(r,g)\\ &\quad+\overline{N}_L(r,g)+3\log r+S(r,f)+S(r,g).\end{aligned} \tag{49}$$

Noting that

$$\overline{N}_L(r,f)+\overline{N}_L(r,g)\leqslant\overline{N}(r,f)=\overline{N}(r,g). \tag{50}$$

By (48)–(50) we have

$$\begin{aligned}(n-2k-4)[T(r,f)+T(r,g)]&\leqslant\left(k+\tfrac{7}{2}\right)\left[\overline{N}(r,f)+\overline{N}(r,g)\right]\\ &\quad+6\log r+S(r,f)+S(r,g).\end{aligned} \tag{51}$$

In the following, we consider two cases.

Case 1. f and g have poles. Since f and g share ∞ IM, then we have

$$\overline{N}(r,f)=\overline{N}(r,g)\geqslant\log r. \tag{52}$$

Set

$$F=\frac{f^{(k)}}{z},\quad G=\frac{g^{(k)}}{z}. \tag{53}$$

From $f^{(k)}$ and $g^{(k)}$ share z CM, we know that F and G share 1 CM almost.

By (52), (53) and Lemma 4 we obtain

$$\begin{aligned}T(r,F)&\leqslant T\left(r,f^{(k)}\right)+\log r\\ &\leqslant T(r,f)+k\overline{N}(r,f)+\log r+S(r,f)\\ &\leqslant(k+2)T(r,f)+S(r,f).\end{aligned} \tag{54}$$

It follows $S(r,F)=S(r,f)$. Similarly, $S(r,G)=S(r,g)$.

Set

$$\phi=\frac{F'}{F(F-1)}-\frac{G'}{G(G-1)}. \tag{55}$$

Now, we consider two subcases.

Case 1.1. $\phi\equiv 0$. By (55) we obtain

$$\frac{F-1}{F}\equiv C\frac{G-1}{G}, \tag{56}$$

where C is a nonzero constant.

In the following, we consider two subcase.

Case 1.1.1. $C=1$. By (56) we get $F\equiv G$. That is $f^{(k)}\equiv g^{(k)}$. Using the same argument as used in proof of Theorem 1, we obtain $f\equiv g$.

Case 1.1.2. $C\neq 1$. By (56) we have

$$\frac{1}{F}-\frac{C}{G}=1-C. \tag{57}$$

Since f and g share ∞ IM, we know that F and G share ∞ IM, then by (57) we obtain a contradiction.

Case 1.2. $\phi\not\equiv 0$. Let z_0 be a pole of f with multiplicity l_1. By f and g share ∞ IM, we know that z_0 is a pole of g with multiplicity l_2. Set $l=\min\{l_1,l_2\}$. By (55) we deduce that z_0 is a zero of ϕ with multiplicity $\geqslant l+k-1$. From Lemma 2 and Nevanlinna's first fundamental theorem, we get

$$\begin{aligned}\overline{N}(r,f)=\overline{N}(r,g)&\leqslant\tfrac{1}{l+k-1}N\left(r,\tfrac{1}{\phi}\right)\\ &\leqslant\tfrac{1}{l+k-1}N(r,\phi)+\tfrac{1}{n+k-1}m(r,\phi)+O(1)\\ &\leqslant\tfrac{1}{l+k-1}\left[\overline{N}\left(r,\tfrac{1}{F}\right)+\overline{N}\left(r,\tfrac{1}{G}\right)\right]\\ &\quad+S(r,f)+S(r,g).\end{aligned} \tag{58}$$

By Lemma 4 we have

$$\begin{aligned}\overline{N}\left(r,\tfrac{1}{F}\right) &= \overline{N}\left(r,\tfrac{z}{f^{(k)}}\right) = \overline{N}\left(r,\tfrac{1}{f^{(k)}}\right)\\ &= N\left(r,\tfrac{1}{f^{(k)}}\right) - \left[N\left(r,\tfrac{1}{f^{(k)}}\right) - \overline{N}\left(r,\tfrac{1}{f^{(k)}}\right)\right]\\ &\leqslant N\left(r,\tfrac{1}{f}\right) + k\overline{N}(r,f)\\ &\quad - \left[N\left(r,\tfrac{1}{f^{(k)}}\right) - \overline{N}\left(r,\tfrac{1}{f^{(k)}}\right)\right] + S(r,f)\\ &\leqslant (k+1)\overline{N}\left(r,\tfrac{1}{f}\right) + k\overline{N}(r,f) + S(r,f)\\ &\leqslant \tfrac{2k+1}{n}T(r,f) + S(r,f).\end{aligned} \tag{59}$$

Similarly,

$$\overline{N}\left(r,\tfrac{1}{G}\right) \leqslant \tfrac{2k+1}{n}T(r,g) + S(r,g). \tag{60}$$

By (58)–(60) we obtain

$$\begin{aligned}\overline{N}(r,f) = \overline{N}(r,g) &\leqslant \tfrac{2k+1}{n(l+k-1)}[T(r,f)+T(r,g)]\\ &\quad + S(r,f) + S(r,g).\end{aligned} \tag{61}$$

It follows from (51), (52) and (61) that

$$\begin{aligned}&(n-2k-4)[T(r,f)+T(r,g)]\\ &\qquad \leqslant \tfrac{(2k+13)(2k+1)}{n+k-1}[T(r,f)+T(r,g)]\\ &\qquad\quad + S(r,f) + S(r,g).\end{aligned} \tag{62}$$

Since $n > 3k+8$, then by (62) we obtain $T(r,f) + T(r,g) \leqslant S(r,f) + S(r,g)$, a contradiction.

Therefore, we deduce that f_1, f_2, f_3 are linearly dependent. Hence, there exist three constants c_1, c_2, c_3 with $(c_1, c_2, c_3) \neq (0,0,0)$ such that

$$c_1 f_1 + c_2 f_2 + c_3 f_3 \equiv 0. \tag{63}$$

If $c_1 = 0$, then by (63), we have $c_2 f_2 + c_3 f_3 \equiv 0$ and $c_3 \neq 0$. That is

$$g^{(k)}(z) \equiv \frac{c_2}{c_3} z.$$

Thus g is a polynomial, a contradiction. Hence, $c_1 \neq 0$.

If $c_2 \neq 0$, from (63) and $f_1 + f_2 + f_3 \equiv 1$, we obtain

$$\left(1 - \tfrac{c_2}{c_1}\right) f_2 + \left(1 - \tfrac{c_3}{c_1}\right) f_3 \equiv 1, \tag{64}$$

and $c_1 \neq c_2$, $c_1 \neq c_3$.

By (39), (41), and (64) we get

$$\left(1 - \tfrac{c_3}{c_1}\right) \tfrac{g^{(k)}}{z} + \tfrac{g^{(k)}-z}{f^{(k)}-z} \equiv 1 - \tfrac{c_2}{c_1}. \tag{65}$$

Since f and g share ∞ IM, and f and g have poles, then by (65), we get a contradiction.

Therefore, $c_2 = 0$, $c_3 \neq 0$. By (63) and $f_1 + f_2 + f_3 \equiv 1$ we have

$$\left(1 - \tfrac{c_1}{c_3}\right) f_1 + f_2 \equiv 1.$$

Similar to above discussion, we get a contradiction.

Hence we deduce that either f_2 or f_3 is a constant.

Next, we consider two subcases.

Case 1.2.1 $f_2 = C$, where C is a constant. It follows from (41) and $f_1 + f_2 + f_3 \equiv 1$ that

$$\tfrac{f^{(k)}}{z} - C\tfrac{g^{(k)}}{z} \equiv 1 - C. \tag{66}$$

If $C \neq 1$, then by (66), we obtain $(f - Cg)^{(k)} \equiv (1-C)z$. Hence, $f - Cg \equiv P$, where P is a polynomial with $\deg P = k+1$. Using the same argument as used in proof of Theorem 1, we get a contradiction.

Therefore $f^{(k)} \equiv g^{(k)}$. Using the same argument as used in proof of Theorem 1, we have $f \equiv g$.

Case 1.2.2. $f_3 = C$, where C is a constant. By (39), (41) and $f_1 + f_2 + f_3 \equiv 1$, we obtain

$$\tfrac{f^{(k)}}{z} + \tfrac{f^{(k)}-z}{g^{(k)}-z} = 1 - C. \tag{67}$$

Similar to above discussion, we get a contradiction.

Case 2. f and g are two entire functions. By Lemma 8 we deduce that either f and g are two transcendental entire functions or f and g are two polynomials.

In the following, we consider two subcases.

Case 2.1. f and g are two transcendental entire functions. By the arguments similar to the proof of Case 1, we get either $f^{(k)} \equiv g^{(k)}$ or $f^{(k)} g^{(k)} \equiv z^2$.

If $f^{(k)} \equiv g^{(k)}$, then using the same argument as used in the proof of Theorem 1, we have $f \equiv g$.

If $f^{(k)} g^{(k)} \equiv z^2$, then from $n > k+3$, we obtain $f \neq 0$, $g \neq 0$. By Lemma 13, we get

(1) $k = 1$, $f(z) = c_1 \mathrm{e}^{cz^2}$, $g(z) = c_2 \mathrm{e}^{-cz^2}$, where c_1, c_2 and c are nonzero constants satisfying $4c_1c_2c^2 = -1$, or $f \equiv g$;

(2) $k \geqslant 2$, $f \equiv g$.

Case 2.2. f and g are two polynomials. Since $f^{(k)}$ and $g^{(k)}$ share z CM, then

$$f^{(k)} - z \equiv c\left(g^{(k)} - z\right), \tag{68}$$

where c is a nonzero constant.

If $c \neq 1$, then by (68) we have $(f - cg)^{(k)} \equiv (1-c)z$. Hence, $f - cg \equiv P$, where P is a polynomial with $\deg P = k+1$. Using the same argument as used in proof of Theorem 1, we obtain a contradiction. Thus, $f^{(k)} \equiv g^{(k)}$. Using the same argument as used in proof of Theorem 1, we have $f \equiv g$.

This completes the proof of Theorem 3.

PROOF OF THEOREM 4

From $f^{(k)}$ and $g^{(k)}$ share z CM, f and g share ∞ CM, we obtain

$$\mathrm{e}^h = \frac{f^{(k)} - z}{g^{(k)} - z},$$

where h is a nonconstant entire function.

Using the same argument as used in the proof of Theorem 3, we can prove Theorem 4.

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