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Global attractors for non-Newtonian equations on BCS-BEC crossover

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ABSTRACT: This paper considers the global attractor problem for the non-Newtonian equations on BCS (Bardeen-Cooper-Schrieffer)-BEC (Bose-Einstein condensation) crossover. These non-Newtonian equations can be translated into the time-dependent Ginzburg-Landau equations. In order to establish the attractors, we first prove the existence and uniqueness theorem of weak solutions by the standard Faedo-Galerkin approximation method. Then, we establish some suitable prior estimates of the weak solutions by combining Gagliardo-Nirenberg inequality, Agmon's inequality and Gronwall inequality, etc. Finally, using the existence theorem of the global attractor, we prove that there exists a compact global attractor for the time dependent Ginzburg-Landau equations of BCS-BEC crossover on atomic fermi gases near the Feshbach resonance.

KEYWORDS: global attractor, non-Newtonian equations, BCS-BEC crossover, time dependent Ginzburg-Landau equations, gronwall inequality

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INTRODUCTION

The Ginzburg-Landau theory is a useful tool for studying superconductivity and has very fruitful results [1–5]. However, there are few results for the Fermion-Boson model given by mathematical analysis. Most of them are obtained by physicist [6–8, 15]. In 2006, Machida and Koyama [9] developed a time-dependent Ginzburg-Landau (TDGL) theory for the superfluid atomic Fermion-Boson model based on the functional integral formalism, which has the following form:

$$-\mathrm{i}du_t = \left(-\frac{dg^2+1}{U} + a\right)u + g[a + d(2\nu - 2\mu)]\varphi + \frac{c}{4m}\Delta u + \frac{g}{4m}(c-d)\Delta\varphi - b|u+g\varphi|^2(u+g\varphi), \quad (1)$$

$$\mathrm{i}\varphi_t = -\frac{g}{U}\varphi + (2\nu - 2\mu)\varphi - \frac{1}{4m}\Delta\varphi, \quad (2)$$

where the functions u and φ stand for the fermion pair field and the condensed boson field, respectively. The coupling coefficient d is generally complex with $d = d_r + \mathrm{i}d_i$, and the others coupling coefficients of (1)–(2) are all real numbers. Especially, μ is the chemical potential, and $U > 0$ denotes the BCS (Bardeen-Cooper-Schrieffer) coupling constant. Just as illustration in [9], the coefficient d is a critical feature of the system (1)–(2), which dominates the dynamics of the superfluid atomic Fermi gases. In other words, d is used to as purely imaginary in the BCS limit, and the conventional TDGL equation for u enjoys the dissipative mechanism. On the contrary, the real part of d dominates the dynamics and the imaginary part of d vanishes. Usually, d is a complex number in the BCS-BEC (Bose-Einstein condensation) crossover region.

In order to analyze such Ginzburg-Landau equations on BCS-BEC crossover, we need to rearrange the equations. Thus, let $u + g\varphi = w$, the equations (1)–(2) can be rewritten as:

$$dw_t - \left(a - \frac{1}{U}\right)\mathrm{i}w - \frac{\mathrm{i}g}{U}\varphi - \frac{\mathrm{i}c}{4m}\Delta w + \mathrm{i}b|w|^2 w = 0, \quad (3)$$

$$\varphi_t - \frac{\mathrm{i}g}{U}w + \frac{\mathrm{i}g^2}{U}\varphi + \mathrm{i}(2\nu - 2\mu)\varphi - \frac{\mathrm{i}}{4m}\Delta\varphi = 0. \quad (4)$$

Through this method, Chen and Guo et al [10–13] constructed the existence theory of the solutions to the Ginzburg-Landau equations (1)–(2) under different conditions.

In addition to the existence of solutions, the authors even obtained the existence theorem of the global attractor under certain conditions [19]. However, it is regret that they cannot obtain the same results in the general case. In order to solve this problem, Fang et al. want to modified the Ginzburg-Landau equations (3)–(4) by given some external force terms as following.

$$dw_t - \left(a - \frac{1}{U}\right)\mathrm{i}w - \frac{\mathrm{i}g}{U}\varphi - \frac{\mathrm{i}c}{4m}\Delta w + \mathrm{i}b|w|^2 w = f(x), \quad (5)$$

$$\varphi_t - \frac{\mathrm{i}g}{U}w + \frac{\mathrm{i}g^2}{U}\varphi + \mathrm{i}(2\nu - 2\mu)\varphi - \frac{\mathrm{i}}{4m}\Delta\varphi = h(x). \quad (6)$$

Furthermore, Fang and Jin found that if they want to get the attractors results, they should continue to modify the equations (5)–(6) by increasing a dump term with the dumpling parameter $\gamma > 0$:

$$dw_t - \left(a - \frac{1}{U}\right)\mathrm{i}w - \frac{\mathrm{i}g}{U}\varphi - \frac{\mathrm{i}c}{4m}\Delta w + \mathrm{i}b|w|^2 w = f(x), \quad (7)$$

$$\varphi_t + \gamma\varphi - \frac{\mathrm{i}g}{U}w + \frac{\mathrm{i}g^2}{U}\varphi + \mathrm{i}(2\nu - 2\mu)\varphi - \frac{\mathrm{i}}{4m}\Delta\varphi = h(x). \quad (8)$$

Together with giving some restrictions of the coupling coefficients, they obtained the existence of global attractor [14]. And then, Jiang, Wu, and Guo extended the results in [14] and found that the global attractors of the equations (7)–(8) can be obtained under the more larger region of the coupling coefficients [16]. For more information on the theory of attractors, please refer to [20–23] and their respective citations.

Inspired by the above results, we naturally have the question: Does the general TDGL equations on BCS-BEC crossover with the following form have the similar results? This is the main problem we would consider in this paper.

$$dw_t - \mathrm{i}(a - \frac{1}{U})w - \frac{\mathrm{i}g}{U}\varphi - \frac{\mathrm{i}c}{4m}\Delta w + \mathrm{i}b|w|^p w + \gamma g\varphi = f(x,t), \quad (9)$$

$$\varphi_t + \gamma\varphi - \frac{\mathrm{i}g}{U}w + \frac{\mathrm{i}g^2}{U}\varphi + \mathrm{i}(2\nu - 2\mu)\varphi - \frac{\mathrm{i}}{4m}\Delta\varphi = h(x,t), \quad (10)$$

$$\begin{aligned} w(x,0) &= u_0(x) + g\varphi_0(x) = w_0(x), \\ \varphi(x,0) &= \varphi_0(x), \qquad x \in \Omega, \end{aligned} \quad (11)$$

$$w|_{\partial\Omega} = (u + g\varphi)|_{\partial\Omega} = 0, \qquad \varphi|_{\partial\Omega} = 0. \quad (12)$$

In fact, compared to the above results, the problem we considered in this paper has the following features.

(i) There are no results regarding the existence of global attractors for weak solutions to the equations (9)–(12) with the index $p > 0$ before.

(ii) The external force terms $f(x,t)$ and $h(x,t)$ here are not only dependent on the space variable x but also dependent on the time variable t.

(iii) The higher-order nonlinear term implies the more difficulties to obtain the prior estimation.

In order to overcome these difficulties, first, we combine Gagliardo-Nirenberg inequality, Agmon's inequality and Gronwall inequality, with the monotonicity properties of P-Laplace operator to deal with the difficult comes from the index $p > 0$. Then, combining with the space where the external force terms $f(x,t)$ and $h(x,t)$ belonged and the properties of the norm, the difficulties caused by the external force terms are overcome. Finally, through the properties of P-Laplace operator and the techniques such as interpolation inequality, and so on, the difficulties caused by higher-order nonlinear terms are resolved. Meanwhile, we obtain the existence of global attractor for the problem (9)–(12).

Theorem 1 *Supposes that the conditions (a)–(c) as followings are satisfied.*

(a) $f(x,t)$, $h(x,t) \in H_0^2(Q_T)$ *and* $u_0(x)$, $\varphi_0(x) \in H_0^2(\Omega)$ *with* $Q_T = (0,T)\times\Omega$*;*

(b) $U > 0$, $b > 0$, $c > 0$, $m > 0$, $aU < 1$, $\gamma > 0$*;*

(c) $d = d_r + \mathrm{i}d_i$, *where* d_r, $d_i \in \mathbb{R}$ *and* $d_i > 0$, $|d| = \sqrt{d_r^2 + d_i^2}$.

Then, the global weak solutions to the problem (9)–(12) define a strongly continuous nonlinear semigroup $S(t)$ *on* $H_0^1 \times H_0^1$*. Moreover, the semigroup operator* $S(t)$ *has a compact connected global attractor*

$$A \subset (H^2 \cap H_0^1) \times (H^2 \cap H_0^1).$$

PRELIMINARIES

This section includes some lemmas and definitions which would be used in the proof of the main results. Throughout this paper, the constant $C > 0$ often refers to the different constants in different places.

Definition 1 ([17]) Assume that $A \subset H$ satisfies the following conditions.

(1) A is an invariant set, that is, $S(t)A = A$, $\forall t \geqslant 0$;

(2) There exists an open set $U \subset A$, $\forall x_0 \in U$ such that $\lim_{t\to\infty} \mathrm{dist}(S(t)x_0, A) \to 0$ holds.

Then A is called an attractor. Where the Hausdorff semi-distance is defined as

$$\mathrm{dist}(S(t)U, A) = \sup_{x_0 \in U} \inf_{y \in A} d(S(t)x_0, y).$$

And the maximum open set U that satisfies the condition (2) is called the attraction area of A.

Lemma 1 ***([18])*** *For bounded sets* $B_0 \in E$*, if there exists a* $t_0 \in [0,t]$ *such that* $t_0(B_0) > 0$*, and for any bounded set* $B \subset E$*, there is*

$$S(t)B \subset B_0, \quad \forall t \geqslant t_0.$$

Then, B_0 *is called the bounded absorption set in E.*

Lemma 2 ***(The existence theorem of global attractor [18])*** *Let E be a Banach space,* $\{S(t), t \geqslant 0\}$ *be a semigroup operator,* $S(t): E \to E$, $S(t)\cdot S(\tau) = S(t+\tau)$, $S(0) = I$*, where I is an identity operator, and let the semigroup operator* $S(t)$ *satisfy the following conditions:*

(1) The semigroup operator $S(t)$ *is uniformly bounded in E, that is, for all* $R \geqslant 0$*, there exists a constant* $C(R)$*, such that when* $\|\overrightarrow{u}\|_E \leqslant R$*, there have* $\|S(t)\overrightarrow{u}\|_E \leqslant C(R)$, $\forall t \in [0,\infty)$.

(2) There exists a bounded absorption set B_0 *in E, that is, for any bounded set* $B \subset E$*, there exists a T, such that* $S(t)B \subset B_0$ *when* $t \geqslant T$.

(3) When $t > 0$, $S(t)$ *is a full continuous operator.*

Then, the semigroup $S(t)$ *has a tight global attractor.*

PRIORI ESTIMATES

Theorem 2 ***(The existence theorem of weak solutions)*** *Under the conditions (a)–(c), for any* $(w_0, \varphi_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *and* $T > 0$*, the initial boundary value problem* (9)–(12) *exist a pair of global weak solutions* (w, φ) *in* $Q_T = \Omega \times [0, T]$ *such that*

$$w, \varphi \in C([0,T], H_0^1(\Omega)),$$
$$w_t \in L^2((0,T), L^2(\Omega)), \quad \varphi_t \in L^2((0,T), H^{-1}(\Omega)).$$

And for any complex-valued functions $\psi \in H_0^1(\Omega)$ *and* $\xi \in C^1(0, T]$ *with* $\xi(T) = 0$*, the following equalities hold.*

$$\int_0^T \Big[-d(w, \xi_t\psi) - \mathrm{i}\Big(a - \frac{1}{U}\Big)(w, \xi\psi) - \frac{\mathrm{i}g}{U}(\varphi, \xi\psi) + \frac{\mathrm{i}c}{4m}(\nabla w, \xi\nabla\psi) + \gamma g(\varphi, \xi\psi) + \mathrm{i}b(|w|^2 w, \xi\psi)\Big]\mathrm{d}t = \int_0^T (f, \xi\psi)\,\mathrm{d}t + (w_0, \xi(0)\psi),$$

and

$$\int_0^T \Big[-(\varphi, \xi_t\psi) - \frac{\mathrm{i}g}{U}(w, \xi\psi) + \mathrm{i}\Big(\frac{g^2}{U} + 2\nu - 2\mu\Big) + \frac{\mathrm{i}}{4m}(\nabla\varphi, \xi\nabla\psi) + \gamma(\varphi, \xi\psi)\Big]\mathrm{d}t = \int_0^T (h, \xi\psi)\,\mathrm{d}t + (\psi_0, \xi(0)\psi).$$

This theorem can be proved by the standard Faedo-Galerkin approximation method. The detailed proof would be given later.

Theorem 3 *(The uniqueness theorem of weak solutions) Assume that any* $(w_1, \varphi_1), (w_2, \varphi_2) \in H_0^1(Q_T) \times H_0^1(Q_T)$ *are the weak solutions to the problem* (9)–(12) *with the initial values* (w_{01}, φ_{01}) *and* (w_{02}, φ_{02})*, respectively. Then, for any* $T > 0$*, it holds* $\forall t \in [0, T]$*,*

$$\|w_1(x,t) - w_2(x,t)\|_{H^1}^2 + \|\varphi_1(x,t) - \varphi_2(x,t)\|_{H^1}^2 + \int_0^t \|w_{1\tau}(x,\tau) - w_{2\tau}(x,\tau)\|^2\,\mathrm{d}\tau \leqslant L_1 \mathrm{e}^{L_2 t}(\|w_{01}(x) - w_{02}(x)\|_{H^1}^2 + \|\varphi_{01}(x) - \varphi_{02}(x)\|_{H^1}^2)$$

with $\|F\|^2 = \int |F|^2 dx$ *and where* L_1, L_2*, are positive constants depending on* $\|w_{01}\|_{H^1}$, $\|\varphi_{01}\|_{H^1}$, $\|w_{02}\|_{H^1}$, $\|\varphi_{02}\|_{H^1}$, $|\Omega|$ *and the coefficients of the system* (9)–(10).

The proof of the theorem is given by substituting the two pairs of weak solutions into the equations in the existence theorem of weak solutions (Theorem 2), respectively and then estimated. The details would be seen in the later of the paper.

Theorem 4 *Under the conditions (a)–(c), for any initial data* $(w_0, \varphi_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$*, there exist positive constants* C_5, C_6, C_7, C_8, C_9 *depending only on the coefficients of* (9)–(12)*, such that* $\forall t \geqslant 0$*,*

$$\|w(x,t)\|_{H^1}^2 + \|\varphi(x,t)\|_{H^1}^2 + \|w(x,t)\|_{L^{p+2}}^{p+2} \leqslant \tfrac{C_8}{C_7}\mathrm{e}^{-C_5 t}\big(\|w_0(x)\|_{H^1}^2 + C_9\|w_0(x)\|_{L^{p+2}}^{p+2} + \|\varphi_0(x)\|_{H^1}^2\big) + \tfrac{C_6}{C_5 C_7}.$$

Proof: Making the inner product of the equation (9) with $\bar{w}$, and then taking the imaginary parts, we find that

$$d_r \mathrm{Im}\Big[\int w_t \cdot \bar{w}\,\mathrm{d}x\Big] + \frac{d_i}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \Big(\frac{1}{U} - a\Big)\|w\|^2 - \frac{g}{U}\mathrm{Re}\Big[\int \varphi \cdot \bar{w}\,\mathrm{d}x\Big] + \frac{c}{4m}\|\nabla w\|^2 + b\int |w|^{p+2}\,\mathrm{d}x + \gamma g\,\mathrm{Im}\Big[\int \varphi\cdot\bar{w}\,\mathrm{d}x\Big] = \mathrm{Im}\Big[\int f(x,t)\cdot\bar{w}\,\mathrm{d}x\Big] \quad (13)$$

i.e.,

$$\frac{d_i}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \Big(\frac{1}{U} - a\Big)\|w\|^2 + \frac{c}{4m}\|\nabla w\|^2 + b\int_\Omega |w|^{p+2}\,\mathrm{d}x = \frac{g}{U}\mathrm{Re}\Big[\int_\Omega \varphi\cdot\bar{w}\,\mathrm{d}x\Big] - d_r\mathrm{Im}\Big[\int_\Omega w_t\cdot\bar{w}\,\mathrm{d}x\Big] + \mathrm{Im}\Big[\int_\Omega f(x,t)\cdot\bar{w}\,\mathrm{d}x\Big] - \gamma g\,\mathrm{Im}\Big[\int_\Omega \varphi\cdot\bar{w}\,\mathrm{d}x\Big]. \quad (14)$$

By Young inequality, (14) can be translated to

$$\frac{d_i}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \Big(\frac{1}{U} - a\Big)\|w\|^2 + \frac{c}{4m}\|\nabla w\|^2 + b\int_\Omega |w|^{p+2}\,\mathrm{d}x \leqslant \Big(\frac{|g|}{2\epsilon U} + \frac{|d_r|}{2\epsilon} + \frac{1}{\epsilon}\Big)\|w\|^2 + \Big(\frac{|g|\epsilon}{2U} + \frac{\gamma^2 g^2\epsilon}{2}\Big)\|\varphi\|^2 + \frac{\epsilon|d_r|}{2}\|w_t\|^2 + \frac{\epsilon}{2}\|f(x,t)\|^2. \quad (15)$$

Choosing suitable $\epsilon > 0$ such that

$$\frac{|g|}{2\epsilon U} + \frac{|d_r|}{2\epsilon} + \frac{1}{\epsilon} = \frac{1}{2}(\frac{1}{U} - a),$$

and let

$$C_1 = \max\Big\{\frac{g\epsilon}{2U} + \frac{\gamma^2 g^2\epsilon}{2}, \frac{\epsilon|d_r|}{2}, \frac{\epsilon}{2}\Big\}.$$

The equation (15) can be simplified as

$$\frac{d_i}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + (\frac{1}{U} - a)\|w\|^2 + \frac{c}{4m}\|\nabla w\|^2 + b\int_\Omega |w|^{p+2}\,\mathrm{d}x \leqslant \frac{1}{2}(\frac{1}{U} - a)\|w\|^2 + C_1(\|\varphi\|^2 + \|w_t\|^2 + f(x,t)). \quad (16)$$

Furthermore, making the inner product of the equation (9) with $\bar{w_t}$, we have

$$(d_r+\mathrm{i}d_i)\int w_t\cdot\bar{w_t}\,\mathrm{d}x-\mathrm{i}\Big(a-\frac{1}{U}\Big)\int w\cdot\bar{w_t}\,\mathrm{d}x -\frac{\mathrm{i}g}{U}\int\varphi\cdot\bar{w_t}\,\mathrm{d}x-\frac{\mathrm{i}c}{4m}\int\Delta w\cdot\bar{w_t}\,\mathrm{d}x +\mathrm{i}b\int|w|^p w\cdot\bar{w_t}\,\mathrm{d}x+\gamma g\int\varphi\cdot\bar{w_t}\,\mathrm{d}x=\int f(x,t)\cdot\bar{w_t}\,\mathrm{d}x.$$

Similarly, taking the imaginary parts,

$$d_i\|w_t\|^2+\frac{1}{2}\Big(\frac{1}{U}-a\Big)\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2-\frac{g}{U}\mathrm{Re}\Big[\int\varphi\cdot\bar{w_t}\,\mathrm{d}x\Big] +\frac{c}{8m}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|^2+\frac{b}{p+2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|w|^{p+2}\,\mathrm{d}x +\gamma g\mathrm{Im}\Big[\int\varphi\cdot\bar{w_t}\,\mathrm{d}x\Big]=\mathrm{Im}\Big[\int f(x,t)\cdot\bar{w_t}\,\mathrm{d}x\Big]. \quad (17)$$

Then, rearranging the equation (17), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{1}{2}\Big(\frac{1}{U}-a\Big)\|w\|^2+\frac{c}{8m}\|\nabla w\|^2+\frac{b}{p+2}\int_\Omega|w|^{p+2}\,\mathrm{d}x\Big] +d_i\|w_t\|^2=\frac{g}{U}\mathrm{Re}\Big[\int\varphi\cdot\bar{w_t}\,\mathrm{d}x\Big] +\mathrm{Im}\Big[\int f(x,t)\cdot\bar{w_t}\,\mathrm{d}x\Big]-\gamma g\mathrm{Im}\Big[\int\varphi\cdot\bar{w_t}\,\mathrm{d}x\Big]. \quad (18)$$

By Young inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{1}{2}\Big(\frac{1}{U}-a\Big)\|w\|^2+\frac{c}{8m}\|\nabla w\|^2+\frac{b}{p+2}\int_\Omega|w|^{p+2}\,\mathrm{d}x\Big] +d_i\|w_t\|^2\leqslant\Big|\frac{g}{2\epsilon U}+\frac{1}{\epsilon}\Big|\|w_t\|^2+\Big|\frac{g\epsilon}{2U}+\frac{\epsilon\gamma^2g^2}{2}\Big|\|\varphi\|^2 +\frac{\epsilon}{2}\|f(x,t)\|^2. \quad (19)$$

Choosing suitable $\epsilon>0$ such that $\frac{g}{2\epsilon U}+\frac{1}{\epsilon}=\frac{d_i}{2}$, and let $C_2=\max\left\{\frac{g\epsilon}{2U}+\frac{\epsilon\gamma^2g^2}{2},\frac{\epsilon}{2}\right\}$, we can get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{1}{2}(\frac{1}{U}-a)\|w\|^2+\frac{c}{8m}\|\nabla w\|^2+\frac{b}{p+2}\int|w|^{p+2}\,\mathrm{d}x\Big] +d_i\|w_t\|^2\leqslant\frac{d_i}{2}\|w_t\|^2+C_2(\|\varphi\|^2+\|f(x,t)\|^2). \quad (20)$$

Now, we proceed to take the inner product of the equation (10) with $\bar{\varphi}$, and get

$$\int\varphi_t\cdot\bar{\varphi}\,\mathrm{d}x+\gamma\int\varphi\cdot\bar{\varphi}\,\mathrm{d}x-\frac{\mathrm{i}g}{U}\int w\cdot\bar{\varphi}\,\mathrm{d}x +\frac{\mathrm{i}g^2}{U}\int\varphi\cdot\bar{\varphi}\,\mathrm{d}x+\mathrm{i}(2\nu-2\mu)\int\varphi\cdot\bar{\varphi}\,\mathrm{d}x -\frac{\mathrm{i}}{4m}\int\Delta\varphi\cdot\bar{\varphi}\,\mathrm{d}x=\int h(x,t)\cdot\bar{\varphi}\,\mathrm{d}x.$$

Taking the real parts,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|^2+\gamma\|\varphi\|^2=-\frac{g}{U}\mathrm{Im}\Big[\int_\Omega w\cdot\bar{\varphi}\,\mathrm{d}x\Big] +\mathrm{Re}\Big[\int_\Omega h(x,t)\cdot\bar{\varphi}\,\mathrm{d}x\Big]. \quad (21)$$

By Young inequality,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|^2+\gamma\|\varphi\|^2\leqslant\Big|\frac{g}{U}\Big|\Big(\frac{1}{2\epsilon}\|w\|^2+\frac{\epsilon}{2}\|\varphi\|^2\Big) +\frac{1}{2\epsilon}\|h(x,t)\|^2+\frac{\epsilon}{2}\|\varphi\|^2.$$

Choosing $\epsilon>0$ small enough such that

$$\frac{|g|\epsilon}{2U}+\frac{\epsilon}{2}=\frac{\gamma}{2},\quad\text{and let}\quad C_3=\max\left\{\frac{|g|}{2\epsilon U},\frac{1}{2\epsilon}\right\},$$

we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|^2+\gamma\|\varphi\|^2 \leqslant\frac{\gamma}{2}\|\varphi\|^2+C_3(\|w\|^2+\|h(x,t)\|^2). \quad (22)$$

Further, making the inner product of the equation (10) with $-\Delta\bar{\varphi}$,

$$\int\varphi_t\cdot(-\Delta\bar{\varphi})\,\mathrm{d}x+\gamma\int\varphi\cdot(-\Delta\bar{\varphi})\,\mathrm{d}x-\frac{\mathrm{i}g}{U}\int w\cdot(-\Delta\bar{\varphi})\,\mathrm{d}x +\frac{\mathrm{i}g^2}{U}\int\varphi\cdot(-\Delta\bar{\varphi})\,\mathrm{d}x+\mathrm{i}(2\nu-2\mu)\int\varphi\cdot(-\Delta\bar{\varphi})\,\mathrm{d}x -\frac{\mathrm{i}}{4m}\int\Delta\varphi\cdot(-\Delta\bar{\varphi})\,\mathrm{d}x=\int h(x,t)\cdot(-\Delta\bar{\varphi})\,\mathrm{d}x.$$

And then taking the real parts,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\varphi\|^2+\gamma\|\nabla\varphi\|^2=-\frac{g}{U}\mathrm{Im}\Big[\int\nabla w\cdot\nabla\bar{\varphi}\,\mathrm{d}x\Big] +\mathrm{Re}\Big[\int\nabla h(x,t)\cdot\nabla\bar{\varphi}\,\mathrm{d}x\Big]. \quad (23)$$

By Young inequality, we can obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\varphi\|^2+\gamma\|\nabla\varphi\|^2\leqslant\frac{|g|}{U}\Big(\frac{1}{2\epsilon}\|\nabla w\|^2+\frac{\epsilon}{2}\|\nabla\varphi\|^2\Big) +\frac{1}{2\epsilon}\|\nabla h(x,t)\|^2+\frac{\epsilon}{2}\|\nabla\varphi\|^2 =(\frac{|g|\epsilon}{2U}+\frac{\epsilon}{2})\|\nabla\varphi\|^2+\frac{1}{2\epsilon}\|\nabla h(x,t)\|^2+\frac{|g|}{2\epsilon U}\|\nabla w\|^2.$$

Similarly, choosing $\epsilon>0$ small enough such that

$$\frac{\epsilon}{2}+\frac{|g|\epsilon}{2U}=\frac{\gamma}{2},\quad\text{and let}\quad C_4=\max\left\{\frac{1}{2\epsilon},\frac{|g|}{2\epsilon U}\right\},$$

we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\varphi\|^2+\gamma\|\nabla\varphi\|^2 \leqslant \frac{\gamma}{2}\|\nabla\varphi\|^2+C_4(\|\nabla h(x,t)\|^2+\|\nabla w\|^2). \quad (24)$$

Then, we let (16) multiply k_1, (22) multiply k_2, and (24) multiply k_3,

$$\frac{k_1 d_i}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2+k_1\left(\frac{1}{U}-a\right)\|w\|^2+\frac{k_1 c}{4m}\|\nabla w\|^2 +k_1 b\int_\Omega |w|^{p+2}\,\mathrm{d}x \leqslant \frac{k_1}{2}\left(\frac{1}{U}-a\right)\|w\|^2 +C_1k_1(\|\varphi\|^2+\|w_t\|^2+\|f(x,t)\|^2), \quad (25)$$

$$\frac{k_2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|^2+k_2\gamma\|\varphi\|^2 \leqslant \frac{k_2\gamma}{2}\|\varphi\|^2+C_3k_2(\|w\|^2+\|h(x,t)\|^2), \quad (26)$$

$$\frac{k_3}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\varphi\|^2+\gamma k_3\|\nabla\varphi\|^2 \leqslant \frac{k_3\gamma}{2}\|\nabla\varphi\|^2+k_3C_4(\|\nabla h(x,t)\|^2+\|\nabla w\|^2), \quad (27)$$

Combining (25)–(27) with (20),

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{1}{2}(k_1d_i+\frac{1}{U}-a)\|w\|^2+\frac{c}{8m}\|\nabla w\|^2 + \frac{b}{p+2}\int_\Omega |w|^{p+2}\,\mathrm{d}x+\frac{k_2}{2}\|\varphi\|^2+\frac{k_3}{2}\|\nabla\varphi\|^2\Big]\\ &+\Big[\frac{k_1}{2}(\frac{1}{U}-a)-C_3k_2\Big]\|w\|^2+\left(\frac{k_1c}{4m}-C_4k_3\right)\|\nabla w\|^2\\ &+k_1b\int_\Omega |w|^{p+2}\,\mathrm{d}x+\left(\frac{\gamma k_2}{2}-C_1k_1-C_2\right)\|\varphi\|^2+\frac{\gamma k_3}{2}\|\nabla\varphi\|^2\\ &+\left(\frac{d_i}{2}-C_1k_1\right)\|w_t\|^2\leqslant (C_1k_1+C_2)\|f(x,t)\|^2 +C_3k_2\|h(x,t)\|^2+C_4k_3\|\nabla h(x,t)\|^2. \quad (28)\end{aligned}$$

Choosing $k_1=\frac{d_i}{4C_1}$, $k_2=\frac{4C_2+d_i}{\gamma}$, $k_3=\frac{cd_i}{32mC_1C_4}$, then there exists a constant $C_5>0$, such that the following inequality holds,

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1(t)+C_5E_1(t)+\frac{d_i}{4}\|w_t\|^2\leqslant C_6, \quad (29)$$

where

$$E_1(t)=\frac{1}{2}(k_1d_i+\frac{1}{U}-a)\|w\|^2+\frac{c}{8m}\|\nabla w\|^2 +\frac{b}{4}\int_\Omega |w|^{p+2}\,\mathrm{d}x+\frac{k_2}{2}\|\varphi\|^2+\frac{k_3}{2}\|\nabla\varphi\|^2.$$

Grönwall inequality yields that

$$E_1(t)\leqslant \mathrm{e}^{-C_5t}E_1(0)+\frac{C_6}{C_5}, \quad \forall t\geqslant 0. \quad (30)$$

For

$$C_7=\min\left\{\frac{1}{2}(k_1d_i+\frac{1}{U}-a),\frac{c}{8m},\frac{b}{4},\frac{k_2}{2},\frac{k_3}{2}\right\},$$
$$C_8=\max\left\{\frac{1}{2}(k_1d_i+\frac{1}{U}-a),\frac{c}{8m},\frac{b}{4},\frac{k_2}{2},\frac{k_3}{2}\right\},$$

we get, $\forall t\geqslant 0$,

$$\|w(x,t)\|^2_{H^1}+\|\varphi(x,t)\|^2_{H^1}+\|w(x,t)\|^{p+2}_{L^{p+2}} \leqslant \frac{C_8}{C_7}\mathrm{e}^{-C_5t}\Big(\|w_0(x)\|^2_{H^1}+\|w_0(x)\|^{p+2}_{L^{p+2}} +\|\varphi_0(x)\|^2_{H^1}\Big)+\frac{C_6}{C_5C_7}. \quad (31)$$

The proof of Theorem 4 is completed. □

Theorem 5 *Under the conditions of Theorem 2, the following uniform estimate holds.*

$$\|w(t)\|_{H^2(\Omega)}\leqslant C(1+\frac{1}{\beta}), \qquad \forall\, t\geqslant\beta>0,$$
$$\|\varphi\|_{H^2(\Omega)}\leqslant C_{10},$$

where C and C_{10} are positive constants depending on $\|w_0\|_{H^1}$, $\|\varphi_0\|_{H^1}$, Ω, $f(x,t)$, $h(x,t)$ and independent of t.

Proof: For any $t\geqslant 0$ and $\beta>0$, we integrate the inequality (29) from t to $t+\tau$, then from (31), we can infer that

$$\int_t^{t+\tau}\Big(\|w(\tau)\|^2_{H^1}+\|w(\tau)\|^{p+2}_{L^{p+2}}+\|\varphi(\tau)\|^2_{H^1}+\|w_t(\tau)\|^2\Big)\,\mathrm{d}\tau \leqslant C. \quad (32)$$

Noting that the inner product of the equation (9) and $-\triangle\bar{w}$ is,

$$\begin{aligned}&(d_r+\mathrm{i}d_i)\int w_t\cdot(-\triangle\bar{w})\,\mathrm{d}x-\mathrm{i}\left(a-\frac{1}{U}\right)\int w\cdot(-\triangle\bar{w})\,\mathrm{d}x\\ &-\frac{\mathrm{i}g}{U}\int\varphi\cdot(-\triangle\bar{w})\,\mathrm{d}x-\frac{\mathrm{i}c}{4m}\int\triangle w\cdot(-\triangle\bar{w})\,\mathrm{d}x\\ &+\mathrm{i}b\int\nabla(|w|^pw)\cdot\nabla\bar{w}\,\mathrm{d}x+\gamma g\int\varphi\cdot(-\triangle\bar{w})\,\mathrm{d}x\\ &=\int f(x,t)\cdot(-\triangle\bar{w})\,\mathrm{d}x.\end{aligned}$$

Taking the imaginary parts, we have

$$-d_r\mathrm{Im}\left[\int w_t\cdot\triangle\bar{w}\,\mathrm{d}x\right]+\frac{d_i}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|^2+\left(\frac{1}{U}-a\right)\|\nabla w\|^2+\frac{g}{U}\mathrm{Re}\left[\int\varphi\cdot\triangle\bar{w}\,\mathrm{d}x\right]+\frac{c}{4m}\|\Delta w\|^2+bp\mathrm{Re}\left[\int|w|^p|\nabla w|^2\,\mathrm{d}x\right]+\frac{pb}{2}\mathrm{Re}[\int|w|^{p-2}\cdot w^2\cdot(\nabla\bar{w})^2\,\mathrm{d}x]+\gamma g\mathrm{Im}\left[\int\varphi\cdot(-\triangle\bar{w})\,\mathrm{d}x\right]=-\mathrm{Im}\left[\int f(x,t)\cdot\triangle\bar{w}\,\mathrm{d}x\right].$$

By Young's inequality, we can get that

$$\frac{d_i}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|^2+\left(\frac{1}{U}-a\right)\|\nabla w\|^2+\frac{c}{4m}\|\triangle w\|^2+bp\mathrm{Re}\left[\int|w|^p|\nabla w|^2\,\mathrm{d}x\right]=-\frac{g}{U}\mathrm{Re}\left[\int\varphi\cdot\triangle\bar{w}\,\mathrm{d}x\right]-\frac{pb}{2}\mathrm{Re}\left[\int_\Omega|w|^{p-2}w^2(\nabla\bar{w})^2\,\mathrm{d}x\right]+d_r\mathrm{Im}\left[\int w_t\cdot\triangle\bar{w}dx\right]+\gamma g\mathrm{Im}\left[\int\varphi\cdot\triangle\bar{w}\,\mathrm{d}x\right]-\mathrm{Im}\left[\int_\Omega f(x,t)\cdot\triangle\bar{w}\,\mathrm{d}x\right]\leqslant\frac{c}{8m}\|\triangle w\|^2+\frac{pb}{2}\int_\Omega|w|^p|\nabla w|^2\,\mathrm{d}x+C_9(\|w_t\|^2+\|\varphi\|^2+\|f(x,t)\|^2).$$

Integrating the above inequality from t to $t+\tau$, and combining with (32), we can deduce

$$\int_t^{t+\tau}\|w(\tau)\|_{H^2}^2\,\mathrm{d}\tau\leqslant C.\tag{33}$$

Similarly, we can find that the inner product of the equation (9) and $-\Delta\bar{w}_t$ is,

$$(d_r+\mathrm{i}d_i)\int w_t\cdot(-\Delta\bar{w}_t)\,\mathrm{d}x-\mathrm{i}\left(a-\frac{1}{U}\right)\int w\cdot(-\Delta\bar{w}_t)\,\mathrm{d}x-\frac{\mathrm{i}g}{U}\int\varphi\cdot(-\Delta\bar{w}_t)\,\mathrm{d}x-\frac{\mathrm{i}c}{4m}\int\Delta w\cdot(-\Delta\bar{w}_t)\,\mathrm{d}x+\mathrm{i}b\int|w|^pw\cdot(-\Delta\bar{w}_t)\,\mathrm{d}x+\gamma g\int\varphi\cdot(-\Delta\bar{w}_t)\,\mathrm{d}x=\int f(x,t)\cdot(-\Delta\bar{w}_t)\,\mathrm{d}x.$$

Taking the imaginary parts,

$$d_i\|\nabla w_t\|^2+\frac{\left(\frac{1}{U}-a\right)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|^2-\frac{g}{U}\mathrm{Re}\left[\int\varphi\cdot(-\Delta\bar{w}_t)\,\mathrm{d}x\right]+\frac{c}{8m}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta w\|^2+b\mathrm{Re}\left[\int(|w|^pw)\cdot(-\Delta\bar{w}_t)\,\mathrm{d}x\right]+\gamma g\mathrm{Im}\left[\int\nabla\varphi\cdot(\nabla\bar{w}_t)\,\mathrm{d}x\right]=\mathrm{Im}\left[\int f(x,t)\cdot(-\Delta\bar{w}_t)\,\mathrm{d}x\right].$$

Using Young's inequality,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\left(\frac{1}{U}-a\right)\|\nabla w\|^2+\frac{c}{8m}\|\Delta w\|^2\right]+d_i\|\nabla w_t\|^2=\frac{g}{U}\mathrm{Re}\left[\int_\Omega\nabla\varphi\cdot(\nabla\bar{w}_t)\,\mathrm{d}x\right]-b\mathrm{Re}\left[\int_\Omega\nabla(|w|^pw)\cdot(\nabla\bar{w}_t)\,\mathrm{d}x\right]+\gamma g\mathrm{Im}\left[\int\varphi\cdot(-\triangle\bar{w}_t)\,\mathrm{d}x\right]+\mathrm{Im}\left[\int f(x,t)\cdot(-\Delta\bar{w}_t)\,\mathrm{d}x\right]=\frac{g}{U}\mathrm{Re}\left[\int_\Omega\nabla\varphi\cdot\nabla\bar{w}_t\,\mathrm{d}x\right]-b\mathrm{Re}\left[\int_\Omega\left(\frac{p}{2}+1\right)|w|^p\nabla w\nabla\bar{w}_t+\frac{p}{2}|w|^{p-2}w^2\nabla\bar{w}\nabla\bar{w}_t\,\mathrm{d}x\right]+\gamma g\mathrm{Im}\left[\int\nabla\varphi\cdot(\nabla\bar{w}_t)\,\mathrm{d}x\right]+\mathrm{Im}\left[\int f(x,t)\cdot(-\Delta\bar{w}_t)\,\mathrm{d}x\right]\leqslant\frac{d_i}{2}\|\nabla w_t\|^2+C(\|\nabla\varphi\|^2+\|\nabla f(x,t)\|^2+\int_\Omega|w|^{2p}|\nabla w|^2\,\mathrm{d}x)\leqslant\frac{d_i}{2}\|\nabla w_t\|^2+C(\|\nabla\varphi\|^2+\|\nabla f(x,t)\|^2+\|w\|_{L^\infty}^{2p}\|w\|_{H^1}^2).\tag{34}$$

Further, using the following Agmon's inequality:

$$\|w\|_{L^\infty}^2\leqslant C(\Omega)\|\nabla w\|\|\triangle w\|,$$

we have,

$$\|w\|_{L^\infty}^{2p}\|w\|_{H^1}^2\leqslant C(\Omega)\|\nabla w\|^p\|\Delta w\|^p\|w\|_{H^1}^2\leqslant C(\Omega)\|w\|_{H^1}^{2+p}\|\Delta w\|^p.$$

Thus, for the inequality (34), we can find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\left(\frac{1}{U}-a\right)\|\nabla w\|^2+\frac{c}{8m}\|\Delta w\|^2\right]+d_i\|\nabla w_t\|^2\leqslant\frac{d_i}{2}\|\nabla w_t\|^2+C(\|\nabla\varphi\|^2+\|\nabla f(x,t)\|^2+\|w\|_{H^1}^{2+p}\|\Delta w\|^p).$$

Let

$$y(t)=\frac{1}{2}(\frac{1}{U}-a)\|\nabla w(x,t)\|^2+\frac{c}{8m}\|\triangle w(x,t)\|^2,$$
$$h_1(t)=\|w(x,t)\|_{H^1}^{2+p},$$
$$h_2(t)=\|\varphi(x,t)\|_{H^1}^2+\|f(x,t)\|_{H^1}^2.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t)\leqslant Ch_1(t)y(t)+Ch_2(t).$$

Applying Grönwall inequality, we can infer from (31) and (33) that, for any $\beta>0$,

$$y(t+\beta)\leqslant C(1+\frac{1}{\beta}),\qquad\forall t\geqslant 0.\tag{35}$$

Similarly, making the inner product of the equation

(10) with $\nabla^4\bar{\varphi}$,

$$\int \varphi_t \cdot \nabla^4\bar{\varphi}\,\mathrm{d}x$$
$$= -\gamma \int \varphi\cdot\nabla^4\bar{\varphi}\,\mathrm{d}x + \frac{\mathrm{i}g}{U}\int w\cdot\nabla^4\bar{\varphi}\,\mathrm{d}x - \frac{\mathrm{i}g^2}{U}\|\Delta\varphi\|^2$$
$$-\mathrm{i}(2\nu-2\mu)\|\Delta\varphi\|^2 + \frac{\mathrm{i}}{4m}\int\Delta\varphi\cdot\nabla^4\bar{\varphi}\,\mathrm{d}x + \int h(x,t)\cdot\nabla^4\bar{\varphi}\,\mathrm{d}x.$$

Taking the real parts, we can get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\varphi\|^2 = -\gamma\|\Delta\varphi\|^2 + \mathrm{Re}\left[\frac{\mathrm{i}g}{U}\int \Delta w\cdot\Delta\bar{\varphi}\,\mathrm{d}x\right] + \mathrm{Re}\left[\int h(x,t)\cdot\nabla^4\bar{\varphi}\,\mathrm{d}x\right].$$

Using Young's inequality, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\varphi\|^2 = \left(-\gamma + \frac{|g|}{2\epsilon|U|} + \frac{1}{2\epsilon}\right)\|\Delta\varphi\|^2 + \frac{|g|\epsilon}{2|U|}\|\Delta w\|^2 + \frac{\epsilon}{2}\|\Delta h(x,t)\|^2.$$

Combining the inequality (35) with Gronwall's inequality, we can get

$$\|\varphi\|^2_{H^2} \leqslant C_{10}.$$

The proof of Theorem 5 is completed. □

Now, we can use the above results to prove the existence and uniqueness of weak solutions.

Proof: In order to prove Theorem 2, we should integrate (28) from t to $t+1$, $\forall t \geqslant 0$,

$$\int_t^{t+1}\left(\|w(\tau)\|^2_{H^1} + \|w_t(\tau)\|^2 + \|\varphi(\tau)\|^2_{H^1}\right)\mathrm{d}\tau \leqslant C, \quad (36)$$

where C depends on $\|w_0\|_{H^1}$ and $\|\varphi_0\|_{H^1}$.

Further, by the elliptic estimate, we easily get, $\forall t \geqslant 0$,

$$\int_t^{t+1}(\|w(\tau)\|^2_{H^2} + \|\varphi_t(\tau)\|^2_{H^{-1}})d\tau \leqslant C. \quad (37)$$

According to the uniform bounded, prior estimates obtained in Theorem 2 and the above estimate, the existence of weak solutions to (9)–(12) can be obtained by the limit in the Faedo-Galerkin procedure. □

Besides, we proceed to introduce the proof of weak solutions uniqueness.

Proof: Now we proceed to prove Theorem 3. Let $w = w_1 - w_2$, $\varphi = \varphi_1 - \varphi_2$, then from the equations (9) and (10), we have

$$dw_t + \mathrm{i}\left(\frac{1}{U} - a\right)w - \frac{\mathrm{i}g}{U}\varphi - \frac{\mathrm{i}c}{4m}\Delta w + \mathrm{i}b(|w_1|^p w_1 - |w_2|^p w_2) + \gamma g\varphi = 0, \quad (38)$$

$$\varphi_t + \gamma\varphi - \frac{\mathrm{i}g}{U}w + \frac{ig^2}{U}\varphi - \mathrm{i}(2\nu-2\mu)\varphi - \frac{\mathrm{i}}{4m}\Delta\varphi = 0. \quad (39)$$

Making the inner product of the equation (38) with $\bar{w}_t$,

$$(d_r + \mathrm{i}d_i)\int w_t\cdot\bar{w}_t\,\mathrm{d}x + i\left(\frac{1}{U} - a\right)\int w\cdot\bar{w}_t\,\mathrm{d}x - \frac{ig}{U}\int\varphi\cdot\bar{w}_t\,\mathrm{d}x - \frac{ic}{4m}\int\Delta w\cdot\bar{w}_t\,\mathrm{d}x + ib\int(|w_1|^p w_1 - |w_2|^p w_2)\cdot\bar{w}_t\,\mathrm{d}x + \gamma g\int\varphi\cdot\bar{w}_t\,\mathrm{d}x = 0.$$

Taking the imaginary parts,

$$d_i\|w_t\|^2 + \frac{\left(\frac{1}{U} - a\right)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 - \frac{g}{U}\mathrm{Re}\left[\int\varphi\cdot\bar{w}_t\,\mathrm{d}x\right] + \frac{c}{8m}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|^2 + b\mathrm{Re}\left[\int(|w|^p w_1 - |w_2|^p w_2)\cdot\bar{w}_t\,\mathrm{d}x\right] + \gamma g\mathrm{Im}\left[\int\varphi\cdot\bar{w}_t\,\mathrm{d}x\right] = 0.$$

By Young's inequality, we can get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\left(\frac{1}{U} - a\right)\|w\|^2 + \frac{c}{8m}\|\nabla w\|^2\right] + d_i\|w_t\|^2 \leqslant \left(\frac{g^2}{2U^2\epsilon} + \frac{\gamma^2 g^2}{2\epsilon}\right)\|\varphi\|^2 + \epsilon\|w_t\|^2 - b\mathrm{Re}\left[\int(|w_1|^p w_1 - |w_2|^p w_2)\cdot\bar{w}_t\,\mathrm{d}x\right]. \quad (40)$$

We further define $C = \frac{g^2}{2U^2\epsilon} + \frac{\gamma^2 g^2}{2\epsilon}$, $\epsilon = \frac{d_i - 1}{2}$, and note that

$$||w_1|^p w_1 - |w_2|^p w_2| \leqslant (p+1)sup(|w_1|^p, |w_2|^p)|w_1 - w_2|.$$

Combining Hölders inequality and Agmon's inequality, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}(\frac{1}{U} - a)\|w\|^2 + \frac{c}{8m}\|\nabla w\|^2\right] + d_i\|w_t\|^2 \\
&\leqslant C\|\varphi\|^2 + \frac{d_i - 1}{2}\|w_t\|^2 + b(p+1)\int\sup(|w_1|^p, |w_2|^p)|w_1 - w_2|\cdot|w_t|\,\mathrm{d}x \\
&\leqslant C\|\varphi\|^2 + \frac{d_i - 1}{2}\|w_t\|^2 + Cb(p+1)(\|w_1\|_{H^2}, \|w_2\|_{H^2})\int|w_1 - w_2|\cdot|w_t|\,\mathrm{d}x \\
&\leqslant C\|\varphi\|^2 + \frac{d_i - 1}{2}\|w_t\|^2 + \frac{Cb^2(p+1)^2}{2}(\|w_1\|_{H^2}, \|w_2\|_{H^2})^2\|w\|^2 + \frac{1}{2}\|w_t\|^2 \\
&= C\|\varphi\|^2 + \frac{d_i}{2}\|w_t\|^2 + \frac{Cb^2(p+1)^2}{2}(\|w_1\|_{H^2}, \|w_2\|_{H^2})^2\|w\|^2. \quad (41)
\end{aligned}$$

Similarly, making the inner product of the equation (39) and φ, we have

$$\int \varphi_t \cdot \bar{\varphi}\,\mathrm{d}x + \gamma \int \varphi \cdot \bar{\varphi}\,\mathrm{d}x - \frac{\mathrm{i}g}{U}\int w \cdot \bar{\varphi}\,\mathrm{d}x + \frac{\mathrm{i}g^2}{U}\int \varphi \cdot \bar{\varphi}\,\mathrm{d}x - \mathrm{i}(2\nu - 2\mu)\int \varphi \cdot \bar{\varphi}\,\mathrm{d}x - \frac{\mathrm{i}}{4m}\int \Delta\varphi \cdot \bar{\varphi}\,\mathrm{d}x = 0.$$

Taking the real parts, we can get that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|^2 + \gamma\|\varphi\|^2 + \frac{g}{U}\mathrm{Im}\left[\int w \cdot \bar{\varphi}\,\mathrm{d}x\right] = 0. \quad (42)$$

Further taking the inner product of the equation (39) and $-\Delta\bar{\varphi}$,

$$\int \varphi_t \cdot (-\Delta\bar{\varphi})\,\mathrm{d}x + \gamma \int \varphi \cdot (-\Delta\bar{\varphi})\,\mathrm{d}x - \frac{\mathrm{i}g}{U}\int w \cdot (-\Delta\bar{\varphi})\,\mathrm{d}x + \frac{\mathrm{i}g^2}{U}\int \varphi \cdot (-\Delta\bar{\varphi})\,\mathrm{d}x - \mathrm{i}(2\nu - 2\mu)\int \varphi \cdot (-\Delta\bar{\varphi})\,\mathrm{d}x - \frac{\mathrm{i}}{4m}\int \Delta\varphi \cdot (-\Delta\bar{\varphi})\,\mathrm{d}x = 0.$$

Taking the real parts

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\varphi\|^2 + \gamma\|\nabla\varphi\|^2 + \frac{g}{U}\mathrm{Im}\left[\int w \cdot (-\Delta\bar{\varphi})\,\mathrm{d}x\right] = 0. \quad (43)$$

Combining (42) with (43),

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\varphi\|^2 + \|\nabla\varphi\|^2) + \gamma(\|\varphi\|^2 + \|\nabla\varphi\|^2) &= -\frac{g}{U}\mathrm{Im}\left[\int_\Omega (w \cdot \bar{\varphi} + \nabla w \cdot \nabla\bar{\varphi})\,\mathrm{d}x\right] \\ &\leqslant \frac{\gamma}{2}(\|\varphi\|^2 + \|\nabla\varphi\|^2) + C(\|w\|^2 + \|\nabla w\|^2).\end{aligned}$$

According to (41), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}(\frac{1}{U} - a)\|w\|^2 + \frac{c}{8m}\|\nabla w\|^2 + \|\varphi\|^2 + \|\nabla\varphi\|^2\right] \\ &+ d_i\|w_t\|^2 + \gamma(\|\varphi\|^2 + \|\nabla\varphi\|^2) \leqslant C\|\varphi\|^2 + \frac{d_i}{2}\|w_t\|^2 \\ &+ \frac{\gamma}{2}\|\nabla\varphi\|^2 + C(\|w\|^2 + \|\nabla w\|^2) + C(\|w_1\|_{H^2}, \|w_2\|_{H^2})^2\|w\|^2.\end{aligned}$$

From the results of Theorems 4 and 5 and standard Grönwall inequality, the uniqueness of weak solutions is proved. □

Moreover, we estimate $\|w\|_{L^{p+2}}^{p+2}$ and $\|w\|_{L^{2p+2}}^{2p+2}$, and obtain the following theorem.

Theorem 6 *Under the conditions (a)–(c), the following uniform estimate hold.*

$$\|w\|_{L^{p+2}}^{p+2} \leqslant C, \quad \|w\|_{L^{2p+2}}^{2p+2} \leqslant C,$$

here C is a positive constant and independent on t.

Proof: The equation (9) can be rewritten as

$$w_t = \frac{\mathrm{i}}{d}\left(a - \frac{1}{U}\right)w + \frac{\mathrm{i}g}{dU}\varphi + \frac{\mathrm{i}c}{4md}\Delta w - \frac{\mathrm{i}b}{d}|w|^p w - \frac{\gamma g}{d}\varphi + \frac{1}{d}f(x,t). \quad (44)$$

Then, we making the inner product of this equation and $|w|^p\bar{w}$,

$$\begin{aligned}&\int w_t \cdot |w|^p\bar{w}\,\mathrm{d}x \\ &= \left(\frac{\mathrm{i}a}{d} - \frac{\mathrm{i}}{dU}\right)\int w \cdot |w|^p\bar{w}\,\mathrm{d}x + \frac{\mathrm{i}g}{dU}\int \varphi \cdot |w|^p\bar{w}\,\mathrm{d}x \\ &+ \frac{\mathrm{i}c}{4md}\int \Delta w \cdot |w|^p\bar{w}\,\mathrm{d}x - \frac{\mathrm{i}b}{d}\int |w|^p w \cdot |w|^p\bar{w}\,\mathrm{d}x \\ &- \frac{\gamma g}{d}\int \varphi \cdot |w|^p\bar{w}\,\mathrm{d}x + \frac{1}{d}\int f(x,t) \cdot |w|^p\bar{w}\,\mathrm{d}x.\end{aligned}$$

Taking the real parts,

$$\begin{aligned}&\frac{1}{p+2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^{p+2}}^{p+2} \\ &= \mathrm{Re}\left[\frac{\mathrm{i}c}{4md}\int \Delta\omega \cdot |w|^p\bar{w}\,\mathrm{d}x\right] + \frac{d_i}{|d|^2}(a - \frac{1}{U})\|w\|_{L^{p+2}}^{p+2} \\ &+ \mathrm{Re}\left[\frac{\mathrm{i}g}{dU}\int \varphi \cdot |w|^p\bar{w}\,\mathrm{d}x\right] - \frac{bd_i}{|d|^2}\|w\|_{L^{2p+2}}^{2p+2} \\ &- \frac{\gamma g}{d}\mathrm{Re}\left[\int \varphi \cdot |w|^p\bar{w}\,\mathrm{d}x\right] + \mathrm{Re}\left[\frac{1}{d}\int f(x,t) \cdot |w|^p\bar{w}\,\mathrm{d}x\right].\end{aligned}$$

By Young inequality, we can get

$$\begin{aligned}&\frac{1}{p+2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^{p+2}}^{p+2} \\ &\leqslant \frac{c^2}{32\epsilon m^2 d^2}\|\Delta w\|^2 + \frac{\epsilon}{2}\|w\|_{L^{2p+2}}^{2p+2} + \frac{d_i}{|d|^2}(a - \frac{1}{U})\|w\|_{L^{p+2}}^{p+2} \\ &+ \frac{g^2}{2\epsilon|d|^2U^2}\|\varphi\|^2 + \frac{\epsilon}{2}\|w\|_{L^{2p+2}}^{2p+2} - \frac{bd_i}{|d|^2}\|w\|_{L^{2p+2}}^{2p+2} + \frac{\gamma^2g^2}{2\epsilon d^2}\|\varphi\|^2 \\ &+ \frac{\epsilon}{2}\|w\|_{L^{2p+2}}^{2p+2} + \frac{1}{2\epsilon d^2}\|f(x,t)\|^2 + \frac{\epsilon}{2}\|w\|_{L^{2p+2}}^{2p+2} \\ &\leqslant \frac{d_i}{|d|^2}\left(a - \frac{1}{U}\right)\|w\|_{L^{p+2}}^{p+2} + \left(2\epsilon - \frac{bd_i}{|d|^2}\right)\|w\|_{L^{2p+2}}^{2p+2} \\ &+ C(\|\Delta w\|^2 + \|\varphi\|^2 + \|f(x,t)\|^2),\end{aligned}$$

where $C = \max\left\{\frac{c^2}{32\epsilon m^2 d^2}, \frac{g^2}{2\epsilon|d|^2U^2} + \frac{\gamma^2g^2}{2\epsilon d^2}, \frac{1}{2\epsilon d^2}\right\}$.

Choosing $\epsilon > 0$ small enough such that $2\epsilon = \frac{bd_i}{|d|^2}$ and then by Grönwall inequality and the results of Theorems 4 and 5, we have

$$\|w\|_{L^{p+2}}^{p+2} \leqslant C.$$

By Gagliardo-Nirenberg inequality

$$\|f\|_{L^p} \leqslant C_G\|f\|_{H^k}^{\theta}\|f\|_{L^Q}^{1-\theta}$$

with
$$\frac{1}{P}=\theta(\frac{1}{2}-k)+(1-\theta)\frac{1}{Q},$$
we obtain the inequality

$$\|w\|_{L^{2p+2}}\leqslant C_G\|w\|_{H^1}^{\frac{p}{(p+1)(p+4)}}\|w\|_{L^{p+2}}^{\frac{(p+2)^2}{(p+1)(p+4)}}$$

for $P=2p+2$, $k=1$, $\theta=\frac{p}{(p+1)(p+4)}$, and $Q=p+2$. Thus,

$$\begin{aligned}\|w\|_{L^{2p+2}}^{2p+1}&\leqslant C_G^{2p+2}\|w\|_{H^1}^{\frac{2p}{p+4}}\|w\|_{L^{p+2}}^{\frac{2(p+2)^2}{p+4}}\\&\leqslant C_G^{2p+2}\|w\|_{H^1}^2+C_G^{2p+2}\|w\|_{L^{p+2}}^{\frac{(p+2)^2}{2}}.\end{aligned}$$

Combining the results of Theorem 4 with the above inequality, we have

$$\|w\|_{L^{2p+2}}^{2p+2}\leqslant C.$$

Now, we finish the proof of Theorem 6. □

Theorem 7 *Under the conditions of Theorem 4, the following uniform estimates hold*

$$\|w_t\|^2\leqslant C,\quad \|\varphi_t\|^2\leqslant C,$$

where C is a positive constant and independent on t.

Proof: Making the inner product of the equation (44) and $\bar{w}_t$, and taking the real parts,

$$\begin{aligned}\|w_t\|^2&=\mathrm{Re}\left[\frac{\mathrm{i}}{d}(a-\frac{1}{U})\int w\cdot w_t\,\mathrm{d}x\right]+\mathrm{Re}\left[\frac{\mathrm{i}g}{dU}\int\varphi\cdot\bar{w}_t\,\mathrm{d}x\right]\\&+\mathrm{Re}\left[\frac{\mathrm{i}c}{4md}\int\Delta w\cdot\bar{w}_t\,\mathrm{d}x\right]-\mathrm{Re}\left[\frac{\mathrm{i}b}{d}\int|w|^pw\cdot\bar{w}_t\,\mathrm{d}x\right]\\&-\mathrm{Re}\left[\frac{\gamma g}{d}\int\varphi\cdot\bar{w}_t\,\mathrm{d}x\right]+\mathrm{Re}\left[\frac{1}{d}\int f(x,t)\cdot\bar{w}_t\,\mathrm{d}x\right]\\&\leqslant\frac{1}{|d|}\left|a-\frac{1}{U}\right|\int|w|\cdot|\bar{w}_t|\,\mathrm{d}x+\frac{|g|}{|d|U}\int|\varphi||\bar{w}_t|\,\mathrm{d}x\\&+\frac{c}{4m|d|}\int|\Delta w|\cdot|w_t|\,\mathrm{d}x+\frac{b}{|d|}\int|w|^{p+1}|\bar{w}_t|\,\mathrm{d}x\\&+|\frac{\gamma g}{d}|\int|\varphi||\bar{w}_t|\,\mathrm{d}x+\frac{1}{|d|}\int|f(x,t)||\bar{w}_t|\,\mathrm{d}x.\end{aligned}$$

By Young's inequality, we have

$$\begin{aligned}\|w_t\|^2-3\epsilon\|w_t\|^2&\leqslant\frac{c^2}{32m^2|d|^2\epsilon}\|\Delta w\|^2+\frac{(a-\frac{1}{U})^2}{2|d|^2\epsilon}\|w\|^2\\&+\frac{g^2}{2|d|^2U^2\epsilon}\|\varphi\|^2+\frac{b^2}{2|d|^2\epsilon}\|w\|_{2p+2}^{2p+2}\\&+\frac{\gamma^2g^2}{2\epsilon d^2}\|\varphi\|^2+\frac{1}{2|d|^2\epsilon}\|f(x,t)\|^2.\end{aligned}$$

Choosing $0<\epsilon<\frac{1}{3}$, and using the results of Theorems 4–6, yields

$$\|w_t\|\leqslant C,$$

where constant $C>0$ independent on t.

Similarly, taking the inner product of the equation (10) and $\|\varphi_t\|^2$, we have

$$\begin{aligned}\int\varphi_t\cdot\bar{\varphi}_t\,\mathrm{d}x&=-\gamma\int\varphi\cdot\bar{\varphi}_t\,\mathrm{d}x+\frac{\mathrm{i}g}{U}\int w\cdot\bar{\varphi}_t\,\mathrm{d}x\\&-\frac{\mathrm{i}g^2}{U}\int\varphi\cdot\bar{\varphi}_t\,\mathrm{d}x-\mathrm{i}(2\nu-2\mu)\int\varphi\cdot\bar{\varphi}_t\,\mathrm{d}x\\&+\frac{\mathrm{i}}{4m}\int\Delta\varphi\cdot\bar{\varphi}_t\,\mathrm{d}x+\int h(x,t)\cdot\bar{\varphi}_t\,\mathrm{d}x.\end{aligned}$$

Taking the real parts and using Young's inequality

$$\begin{aligned}\|\varphi_t\|^2&\leqslant\left|\gamma+\frac{g^2}{U}+2\nu-2\mu\right|\int|\varphi||\bar{\varphi}_t|\,\mathrm{d}x+\left|\frac{g}{U}\right|\int|w||\bar{\varphi}_t|\,\mathrm{d}x\\&+\frac{1}{4m}\int|\Delta\varphi||\bar{\varphi}_t|\,\mathrm{d}x+\int|h(x,t)|\cdot|\bar{\varphi}_t|\,\mathrm{d}x\\&\leqslant\frac{g^2}{2U^2\epsilon}\|w\|^2+\frac{\epsilon}{2}\|\varphi_t\|^2+\frac{1}{2\epsilon}\Big(\gamma+\frac{g^2}{U}+2\nu-2\mu\Big)^2\|\varphi\|^2\\&+\frac{\epsilon}{2}\|\varphi_t\|^2+\frac{1}{32m^2\epsilon}\|\Delta\varphi\|^2+\frac{\epsilon}{2}\|\varphi_t\|^2\\&+\frac{1}{2\epsilon}\|h(x,t)\|^2+\frac{\epsilon}{2}\|\varphi_t\|^2.\end{aligned}$$

By arranging the above equation, we get

$$\begin{aligned}(1-2\epsilon)\|\varphi_t\|^2&\leqslant\frac{g^2}{2U^2\epsilon}\|w\|^2+\frac{1}{2\epsilon}\Big(\gamma+2\nu-2\mu+\frac{g^2}{U}\Big)^2\|\varphi\|^2\\&+\frac{1}{32m^2\epsilon}\|\Delta\varphi\|^2+\frac{1}{2\epsilon}\|h(x,t)\|^2.\end{aligned}$$

Choosing $\epsilon=\frac{1}{4}$, and using the estimates of Theorems 4 and 5, then

$$\|\varphi_t\|^2\leqslant C,$$

where constant $C>0$ is independent on t. The proof of Theorem 7 is completed. □

THE EXISTENCE OF GLOBAL ATTRACTOR

This section we would use the existence theorem of global attractor (Lemma 2) to prove the main result of this paper. Thus, we should verify the three conditions of Lemma 2 in turn.

Proof: Now we would prove the results of Theorem 1. For the convenience of readers, we state the abstract results on the existence of global attractors as Lemma 2. And then verify the conditions of it one by one. First, with the help of Theorems 3–5, we can find that the global weak solution (w,φ) to problem (9)–(12) is unique, and the weak solutions can generate a strongly continuous nonlinear semigroup $S(t)$ acting on $H_0^1(Q)\times H_0^1(Q)$, such that $S(t)(w_0,\varphi_0)=(w(t),\varphi(t))$. Moreover, for these theorems, we find that the operator $S(t)$ is a uniformly bounded operator. This is just the first condition in Lemma 2.

Second, for the results of Theorems 4 and 5, there exists a positive constant R_0 such that the ball $B_0=\left\{(w,\varphi)\in H_0^1(Q)\times H_0^1(Q)\ :\ \|w\|_{H^1}^2+\|\varphi\|_{H^1}^2\leqslant R_0\right\}$ is a bounded absorbing set for the dynamical system $S(t)$ which comes from the problem (9)–(12). Thus, for any bounded set $B \subset H_0^1(Q) \times H_0^1(Q)$, there exists a t_0 such that $S(t)B \subset B_0$ for every $t \geqslant t_0$. The ball B_0 in $H_0^1 \times H_0^1$ of radius $(R_0)^{\frac{1}{2}}$ will contains any bounded set of B, uniformly in time, where $R_0 = \frac{C_6}{C_5 C_7}$. Noting that

$$\tilde{B_0} = \bigcup_{t \geqslant 0} S(t)B \subset B_0.$$

$\tilde{B_0}$ is an absorbing set of $S(t)$, for $t \geqslant 0$. Then, we verify the second condition of Lemma 2.

Third, the results of Theorems 6 and 7 mean that $S(t)$ is a full continuous operator when $t > 0$. And noting that the continuous embedding $H^2 \hookrightarrow H^1$ is compact. The third condition of Lemma 2 is verified. Thus, by Lemma 2, there exists a global attractor of the equations (9)–(12). The proof of Theorem 1 is completed. □

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