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Unicity of meromorphic functions whose lower order is finite and noninteger

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ABSTRACT: In this paper, we study unicity of meromorphic functions whose lower order is finite and noninteger and mainly prove: Let f and g be two nonconstant meromorphic functions, let $n \geqslant 6$ be an integer, $S=\{z \mid \frac{(n-1)(n-2)}{4}z^n - \frac{n(n-2)}{2}z^{n-1}+\frac{n(n-1)}{4}z^{n-2}-1=0\}$. If f and g share S, ∞ CM, and the lower order of f is finite and noninteger, then $f \equiv g$. This answers a question posed by Gross for meromorphic functions whose lower order is finite and noninteger.

KEYWORDS: meromorphic function, shared set, lower order, small function

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INTRODUCTION AND MAIN RESULTS

In this paper, a meromorphic function means meromorphic in the complex plane. We use the following standard notations in value distribution theory, such as $T(r,f)$, $N(r,f)$, $m(r,f), \dots$, see [1–3].

We denote by $S(r,f)$ any quantity satisfying $S(r,f)=o(T(r,f))$, $r \to \infty$, $r \notin E$, where E is a set of finite linear measure. A meromorphic function α is said to be a small function of f if it satisfies $T(r,\alpha)=S(r,f)$.

Let f be a nonconstant meromorphic function, let k be a positive integer, and let α be a small function of f, we denote the counting function for the zeros of $f-\alpha$ with multiplicities $\leqslant k$ (ignoring multiplicity) as $N_{k)}\left(r,\frac{1}{f-\alpha}\right)\left(\overline{N}_{k)}\left(r,\frac{1}{f-\alpha}\right)\right)$, and denote the set of zeros of $f-\alpha$ with multiplicities $\leqslant k$ (ignoring multiplicity) as $E_{k)}(\alpha,f)\left(\overline{E}_{k)}(\alpha,f)\right)$.

We define the order $\lambda(f)$ of f and the lower order $\mu(f)$ of f by

$$\lambda(f)=\overline{\lim_{r\to\infty}}\frac{\log^+ T(r,f)}{\log r},$$

$$\mu(f)=\underline{\lim_{r\to\infty}}\frac{\log^+ T(r,f)}{\log r}.$$

$$\delta(\alpha,f)=1-\overline{\lim_{r\to\infty}}\frac{N\left(r,\frac{1}{f-\alpha}\right)}{T(r,f)},$$

$$\Theta(\alpha,f)=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}\left(r,\frac{1}{f-\alpha}\right)}{T(r,f)}.$$

Let α be a small function of both f and g. If $f-\alpha$ and $g-\alpha$ have the same zeros counting multiplicities (ignoring multiplicity), then we call that f and g share α CM(IM).

Let $S=\{a_1,a_2,\dots,a_n\}$ be a set of finite complex numbers. If $\prod_{i=1}^n(f-a_i)$ and $\prod_{i=1}^n(g-a_i)$ have the same zeros counting multiplicities (ignoring multiplicity), then we call that f and g share the set S CM(IM).

In 1926, Nevanlinna (see [3]) proved the famous five-value theorem.

Theorem A *Let f and g be two nonconstant meromorphic functions, and let a_i $(i=1,2,\dots,5)$ be five distinct values (one may be ∞). If f and g share a_i $(i=1,2,\dots,5)$ IM, then $f\equiv g$.*

In 1977, Gross [4] proposed the following question. Whether there exist two (even one) finite sets S_j $(j=1,2)$ such that any two nonconstant entire functions f and g share two sets S_j $(j=1,2)$ CM can imply $f \equiv g$?

In 1994, Yi [5] gave an affirmative answer to the question and proved

Theorem B *Let f and g be two nonconstant entire functions, let $n \geqslant 5$ be an integer, $S=\{z \mid z^n-1=0\}$, and let a be a nonzero constant with $a^{2n}\neq 1$. If f and g share S, a CM, then $f\equiv g$.*

In 1996–1998, Fang and Xu [6], Yi [7] proved

Theorem C *Let f and g be two nonconstant entire functions, and let $S=\{z \mid z^3-z^2-1=0\}$. If f and g share S, 0 CM, then $f\equiv g$.*

In 1995, Yi [8] proved

Theorem D *Let f and g be two nonconstant entire functions, let n and m be two positive integers such that n and m have no common factor with $n\geqslant 2m+5$, and let $S=\{z \mid z^n+az^{n-m}+b=0\}$, where a and b are two nonzero constants such that the algebraic equation*

$z^n + az^{n-m} + b = 0$ has no multiple roots. If f and g share S CM, then $f \equiv g$.

In 1998, Frank and Reinders [9] proved

Theorem E *Let f and g be two nonconstant meromorphic functions, let $n \geqslant 11$ be an integer, and let $S = \{z \mid \frac{(n-1)(n-2)}{2}z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2}z^{n-2} - c = 0\}$, where $\frac{(n-1)(n-2)}{2}z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2}z^{n-2} - c = 0$ has no multiple roots and $c(\neq 0,1)$ is a complex number. If f and g share S CM, then $f \equiv g$.*

In 1926, Nevanlinna (see [3]) proved

Theorem F *Let f and g be two nonconstant meromorphic functions, and let a_1, a_2, a_3 be three distinct complex numbers. If f and g share a_1, a_2, a_3 CM, and the lower order of f is finite and noninteger, then $f \equiv g$.*

There are several papers (see [10–15]) dealing with the problems of unique range sets of meromorphic function whose order is finite and noninteger.

In this paper, we study unicity of meromorphic functions whose lower order is finite and noninteger, we prove the following results.

Theorem 1 *Let f and g be two nonconstant meromorphic functions, let $n \geqslant 6$ be an integer, and let $S = \{z \mid \frac{(n-1)(n-2)}{4}z^n - \frac{n(n-2)}{2}z^{n-1} + \frac{n(n-1)}{4}z^{n-2} - 1 = 0\}$. If f and g share S, ∞ CM, and the lower order of f is finite and noninteger, then $f \equiv g$.*

Theorem 2 *Let f and g be two nonconstant meromorphic functions having finitely many poles, let $n \geqslant 5$ be an integer, and let $S = \left\{z \mid z^n - z^{n-1} - 1 = 0\right\}$. If f and g share S CM, and the lower order of f is finite and noninteger, then $f \equiv g$.*

In 2000, Li and Qiao [16] improved Theorem A and proved

Theorem G *Let f and g be two nonconstant meromorphic functions, and let α_i ($i = 1, 2, \dots, 5$) be five distinct small functions of both f and g (one may be ∞). If f and g share α_i ($i = 1, 2, \dots, 5$) IM, then $f \equiv g$.*

Yi (see [3]) proved

Theorem H *Let f and g be two nonconstant entire functions, let a_1, a_2 be two distinct finite nonzero complex numbers such that $\overline{E}_{1)}(a_i, f) = \overline{E}_{1)}(a_i, g)$ ($i = 1, 2$) and $\max\{\Theta(0, f), \delta(a_1, f), \delta(a_2, f)\} > 0$. If f and g share 0 CM and the lower order of f is finite and noninteger, then $f \equiv g$.*

Theorem I *Let f and g be two nonconstant entire functions, let $k_1(\geqslant 1)$, $k_2(\geqslant 2)$ be two integers, and let a_1, a_2 be two distinct finite nonzero complex numbers such that $\overline{E}_{k_i)}(a_i, f) = \overline{E}_{k_i)}(a_i, g)$ ($i = 1, 2$). If f and g share 0 CM, and the lower order of f is finite and noninteger, then $f \equiv g$.*

By above four theorems, we naturally pose the following question.

Question Whether Theorems H and I are valid or not if 0, a_1, a_2 are replaced by three distinct small functions of both f and g?

In this paper, we give a positive answer to this question.

Theorem 3 *Let f and g be two nonconstant entire functions, and let α_1, α_2, α_3 be three distinct small functions of both f and g such that $\overline{E}_{1)}(\alpha_i, f) = \overline{E}_{1)}(\alpha_i, g)$ ($i = 1, 2$) and $\max\{\delta(\alpha_1, f), \delta(\alpha_2, f), \Theta(\alpha_3, f)\} > 0$. If f and g share α_3 CM, and the lower order of f is finite and noninteger, then $f \equiv g$.*

Theorem 4 *Let f and g be two nonconstant entire functions, let k_1, k_2 be two positive integers with $k_1 + k_2 \geqslant 3$, and let α_1, α_2, α_3 be three distinct small functions of both f and g such that $\overline{E}_{k_i)}(\alpha_i, f) = \overline{E}_{k_i)}(\alpha_i, g)$ ($i = 1, 2$). If f and g share α_3 CM, and the lower order of f is finite and noninteger, then $f \equiv g$.*

By the proof of Theorem in [17, p. 293], it follows the following result.

Theorem J *Let f be a nonconstant entire function. If the lower order of f is finite and noninteger, then f assumes every finite value infinitely often.*

In this paper we obtain the following result.

Theorem 5 *Let f be a nonconstant entire function whose lower order is finite and noninteger, and let α ($\not\equiv \infty$) be an entire small function of f. Then $f - \alpha$ has infinitely many zeros.*

LEMMAS

For the proof of our results, we need the following lemmas.

Lemma 1 ([18]) *Let f be a nonconstant meromorphic function, let n be a positive integer, and let $P(f) = a_0 f^n + a_1 f^{n-1} + \cdots + a_n$ ($a_0 \neq 0$), where a_i ($i = 0, 1, 2, \dots, n$) are constants. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 2 ([3, 19]) *Let $T_1(r)$ and $T_2(r)$ be two nonnegative, nondecreasing real functions defined in $r > r_0 > 0$. If $T_1(r) = O(T_2(r))$, $r \to \infty$, $r \notin E$, where E is a set of finite measure, then*

$$\overline{\lim_{r\to\infty}} \frac{\log^+ T_1(r)}{\log r} \leqslant \overline{\lim_{r\to\infty}} \frac{\log^+ T_2(r)}{\log r},$$
$$\underline{\lim_{r\to\infty}} \frac{\log^+ T_1(r)}{\log r} \leqslant \underline{\lim_{r\to\infty}} \frac{\log^+ T_2(r)}{\log r}.$$

Lemma 3 ([3]) *Let f be a nonconstant meromorphic function. If $f \neq 0, \infty$, then $f = \mathrm{e}^h$, where h is an entire function.*

Lemma 4 ([3]) *Let $f = \mathrm{e}^h$, where h is a nonconstant entire function. Then $\lambda(f) = \mu(f)$ and $\mu(f)$ is an integer or infinite.*

Lemma 5 ([20]) *Let f and g be two nonconstant meromorphic functions and let $n(\geqslant 6)$ be an integer. If*

$$\left[\tfrac{(n-1)(n-2)}{2}f^2 - n(n-2)f + \tfrac{n(n-1)}{2}\right]f^{n-2} \equiv \left[\tfrac{(n-1)(n-2)}{2}g^2 - n(n-2)g + \tfrac{n(n-1)}{2}\right]g^{n-2},$$

then $f \equiv g$.

Lemma 6 ([3]) *Let f be a nonconstant meromorphic function, and let $a_1, a_2, \ldots, a_q$ be $q(\geqslant 3)$ distinct values in the extended complex plane. Then*

$$(q-2)T(r,f) \leqslant \sum_{i=1}^{q} \overline{N}\left(r, \tfrac{1}{f-a_i}\right) + S(r,f).$$

Lemma 7 ([21]) *Let f be a nonconstant meromorphic function, and let $\alpha_1, \alpha_2, \ldots, \alpha_q$ be $q(\geqslant 3)$ distinct small functions of f. Then, for any $\varepsilon > 0$, we have*

$$(q-2)T(r,f) \leqslant \sum_{i=1}^{q} \overline{N}\left(r, \tfrac{1}{f-\alpha_i}\right) + \varepsilon T(r,f) + S(r,f).$$

It follows from [22–24] that the following result.

Lemma 8 *Let $n \geqslant 4$ be an integer, and let h and g be two nonconstant meromorphic functions satisfying*

$$g(h^n - 1) - (h^{n-1} - 1) \equiv 0, \tag{1}$$

where g have finitely many poles. Then g is a rational function.

Proof: From (1) and h is not constant, we obtain

$$g = \frac{(h-\eta)(h-\eta^2)\cdots(h-\eta^{n-2})}{(h-\nu)(h-\nu^2)\cdots(h-\nu^{n-1})},$$

where $\eta = \cos\frac{2\pi}{n-1} + \mathrm{i}\sin\frac{2\pi}{n-1}$ and $\nu = \cos\frac{2\pi}{n} + \mathrm{i}\sin\frac{2\pi}{n}$.

Obviously, $\eta, \eta^2, \ldots, \eta^{n-2}, \nu, \nu^2, \cdots, \nu^{n-1}$ are distinct. Thus, by Lemma 6, we have

$$\begin{aligned}(n-3)\,T(r,h) &\leqslant \sum_{i=1}^{n-1}\overline{N}\left(r, \frac{1}{h-\nu^i}\right) + S(r,h)\\ &\leqslant \overline{N}(r,g) + S(r,h)\\ &= O(\log r) + S(r,h).\end{aligned}$$

By $n \geqslant 4$, we know that h is a rational function. It follows from (1) that g is a rational function. □

PROOF OF Theorem 1

Proof: Set

$$F = \frac{(n-1)(n-2)}{4}f^n - \frac{n(n-2)}{2}f^{n-1} + \frac{n(n-1)}{4}f^{n-2},$$
$$G = \frac{(n-1)(n-2)}{4}g^n - \frac{n(n-2)}{2}g^{n-1} + \frac{n(n-1)}{4}g^{n-2}.$$

Since f and g share S, ∞ CM, we know that F and G share $1, \infty$ CM. By Lemma 3, we have

$$\frac{F-1}{G-1} = \mathrm{e}^h, \tag{2}$$

where h is an entire function.

By F and G share 1 CM, Lemma 1, Lemma 6 and Nevanlinna's first fundamental theorem, we have

$$\begin{aligned}nT(r,f) &= T(r,F) + S(r,f)\\
&\leqslant \overline{N}(r,F) + \overline{N}\left(r,\tfrac{1}{F}\right) + \overline{N}\left(r,\tfrac{1}{F-1}\right) + S(r,f)\\
&\leqslant \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f^{n-2}\left(\frac{(n-1)(n-2)}{4}f^2 - \frac{n(n-2)}{2}f + \frac{n(n-1)}{4}\right)}\right)\\
&\quad + \overline{N}\left(r, \frac{1}{\frac{(n-1)(n-2)}{4}g^n - \frac{n(n-2)}{2}g^{n-1} + \frac{n(n-1)}{4}g^{n-2} - 1}\right) + S(r,f)\\
&\leqslant \overline{N}(r,f) + \overline{N}\left(r,\tfrac{1}{f}\right) + \overline{N}\left(r, \frac{1}{\frac{(n-1)(n-2)}{4}f^2 - \frac{n(n-2)}{2}f + \frac{n(n-1)}{4}}\right)\\
&\quad + T\left(r, \frac{1}{\frac{(n-1)(n-2)}{4}g^n - \frac{n(n-2)}{2}g^{n-1} + \frac{n(n-1)}{4}g^{n-2} - 1}\right) + S(r,f)\\
&\leqslant 4T(r,f) + nT(r,g) + S(r,f).\end{aligned}$$

It follows from $n \geqslant 5$ that

$$T(r,f) = O(T(r,g)).$$

Similarly,

$$T(r,g) = O(T(r,f)). \tag{3}$$

By Nevanlinna's first fundamental theorem, Lemma 1, (2) and (3), we have

$$\begin{aligned}T\left(r,\mathrm{e}^h\right) &= T\left(r, \frac{\frac{(n-1)(n-2)}{4}f^n - \frac{n(n-2)}{2}f^{n-1} + \frac{n(n-1)}{4}f^{n-2} - 1}{\frac{(n-1)(n-2)}{4}g^n - \frac{n(n-2)}{2}g^{n-1} + \frac{n(n-1)}{4}g^{n-2} - 1}\right)\\
&\leqslant nT(r,f) + nT(r,g) + S(r,f)\\
&\leqslant O(T(r,f)) + S(r,f) + S(r,g).\end{aligned} \tag{4}$$

From Lemma 2, Lemma 4 and (4), we obtain $\lambda\left(\mathrm{e}^h\right) = \mu\left(\mathrm{e}^h\right) \leqslant \mu(f)$. Noting that $\mu(f)$ is finite and noninteger and $\lambda\left(\mathrm{e}^h\right)\left(= \mu\left(\mathrm{e}^h\right)\right)$ is an integer, we have $\lambda\left(\mathrm{e}^h\right) < \mu(f)$. Hence,

$$T\left(r,\mathrm{e}^h\right) = S(r,f). \tag{5}$$

It follows from (2) that

$$F = \mathrm{e}^h G - \mathrm{e}^h + 1. \tag{6}$$

By Nevanlinna's first fundamental theorem, (2) and (5), we get

$$\begin{aligned} nT(r,f) &= T(r,F)+S(r,f)\\ &= T(r,F-1)+S(r,f)\\ &= T\left(r,(G-1)e^{h}\right)+S(r,f)\\ &\leqslant T(r,G-1)+T\left(r,e^{h}\right)+S(r,f)\\ &\leqslant T(r,G)+S(r,f)=nT(r,g)+S(r,f). \end{aligned}$$

Hence, we have

$$T(r,f)\leqslant T(r,g)+S(r,f). \tag{7}$$

Similarly,

$$T(r,g)\leqslant T(r,f)+S(r,g). \tag{8}$$

Now, we consider two cases.

Case 1: $e^h\equiv 1$. By (2), we have $\frac{(n-1)(n-2)}{2}f^n-n(n-2)f^{n-1}+\frac{n(n-1)}{2}f^{n-2}\equiv\frac{(n-1)(n-2)}{2}g^n-n(n-2)g^{n-1}+\frac{n(n-1)}{2}g^{n-2}$. It follows from $n\geqslant 6$ and Lemma 5 that $f\equiv g$.

Case 2: $e^h\not\equiv 1$. In this case, we consider two subcases.

Case 2.1: $e^h\equiv\frac{1}{2}$. From (6), we obtain

$$F\equiv\tfrac{1}{2}G+\tfrac{1}{2}. \tag{9}$$

By (7), (9), Lemma 1 and Lemma 7 ($q=4,\varepsilon=\frac{1}{7}$), we get

$$\begin{aligned} 2nT(r,g) &= 2T(r,G)+S(r,g)\\ &\leqslant \overline{N}(r,G)+\overline{N}\left(r,\tfrac{1}{G}\right)+\overline{N}\left(r,\tfrac{1}{G+1}\right)\\ &\quad+\overline{N}\left(r,\tfrac{1}{G-\frac{1}{2}}\right)+\tfrac{1}{7}T(r,G)+S(r,g)\\ &\leqslant \overline{N}(r,G)+\overline{N}\left(r,\tfrac{1}{G}\right)+\overline{N}\left(r,\tfrac{1}{F}\right)\\ &\quad+\overline{N}\left(r,\tfrac{1}{G-\frac{1}{2}}\right)+\tfrac{1}{7}T(r,G)+S(r,g)\\ &\leqslant \overline{N}(r,g)+\overline{N}\left(r,\tfrac{1}{g}\right)+2T(r,g)\\ &\quad+\overline{N}\left(r,\tfrac{1}{f}\right)+2T(r,f)+\overline{N}\left(r,\tfrac{1}{g-1}\right)\\ &\quad+(n-3)T(r,g)+\tfrac{n}{7}T(r,g)+S(r,g)\\ &\leqslant (n+\tfrac{n}{7}+5)T(r,g)+S(r,g). \end{aligned}$$

It follows from $n\geqslant 6$ that $T(r,g)\leqslant S(r,g)$, a contradiction.

Case 2.2: $e^h\not\equiv\frac{1}{2}$. By (6), (8), Lemma 1 and Lemma 7 ($q=4,\varepsilon=\frac{1}{7}$), we get

$$\begin{aligned} 2nT(r,f) &= 2T(r,F)+S(r,f)\\ &\leqslant \overline{N}(r,F)+\overline{N}\left(r,\tfrac{1}{F}\right)+\overline{N}\left(r,\tfrac{1}{F+e^h-1}\right)\\ &\quad+\overline{N}\left(r,\tfrac{1}{F-\frac{1}{2}}\right)+\tfrac{1}{7}T(r,F)+S(r,f)\\ &\leqslant \overline{N}(r,F)+\overline{N}\left(r,\tfrac{1}{F}\right)+\overline{N}\left(r,\tfrac{1}{G}\right)\\ &\quad+\overline{N}\left(r,\tfrac{1}{F-\frac{1}{2}}\right)+\tfrac{1}{7}T(r,F)+S(r,f)\\ &\leqslant \overline{N}(r,f)+\overline{N}\left(r,\tfrac{1}{f}\right)+2T(r,f)\\ &\quad+\overline{N}\left(r,\tfrac{1}{g}\right)+2T(r,g)+\overline{N}\left(r,\tfrac{1}{f-1}\right)\\ &\quad+(n-3)T(r,f)+\tfrac{n}{7}T(r,f)+S(r,f)\\ &\leqslant (n+\tfrac{n}{7}+5)T(r,f)+S(r,f). \end{aligned}$$

It follows from $n\geqslant 6$ that $T(r,f)\leqslant S(r,f)$, a contradiction.

This completes the proof of Theorem 1. □

PROOF OF Theorem 2

Proof: Set

$$F=f^n-f^{n-1},\quad G=g^n-g^{n-1}. \tag{10}$$

Since f and g have finitely many poles, we know that

$$\begin{aligned} N(r,f)=O(\log r),\quad N(r,F)=O(\log r),\\ N(r,g)=O(\log r),\quad N(r,G)=O(\log r). \end{aligned}$$

By (10) and Lemma 1, we have

$$\begin{aligned} T(r,F)=nT(r,f)+S(r,f),\\ T(r,G)=nT(r,g)+S(r,g). \end{aligned}$$

Since f and g share S CM, we know that F and G share 1 CM. By Lemma 3, we obtain

$$\frac{F-1}{G-1}=Q\,e^h, \tag{11}$$

where Q is a rational function and h is an entire function.

By f and g share S CM, Lemma 1, Lemma 6 and Nevanlinna's first fundamental theorem, we have

$$\begin{aligned} nT(r,f) &= T\left(r,f^n-f^{n-1}\right)+S\left(r,f\right)\\ &\leqslant \overline{N}\left(r,f^n-f^{n-1}\right)+\overline{N}\left(r,\tfrac{1}{f^n-f^{n-1}}\right)\\ &\quad+\overline{N}\left(r,\tfrac{1}{f^n-f^{n-1}-1}\right)+S(r,f)\\ &\leqslant \overline{N}\left(r,f\right)+\overline{N}\left(r,\tfrac{1}{f^{n-1}(f-1)}\right)\\ &\quad+\overline{N}\left(r,\tfrac{1}{g^n-g^{n-1}-1}\right)+S(r,f)\\ &\leqslant \overline{N}\left(r,f\right)+\overline{N}\left(r,\tfrac{1}{f}\right)+\overline{N}\left(r,\tfrac{1}{f-1}\right)\\ &\quad+T(r,g^n-g^{n-1}-1)+S(r,f)\\ &\leqslant 2T(r,f)+nT(r,g)+O(\log r)+S(r,f). \end{aligned}$$

It follows from $n \geqslant 3$ that

$$T(r,f)=O(T(r,g)).$$

Similarly,

$$T(r,g)=O(T(r,f)). \tag{12}$$

From Nevanlinna's first fundamental theorem, (11) and (12), we have

$$\begin{aligned}T\left(r,Q\,\mathrm{e}^h\right)&=T\left(r,\frac{f^n-f^{n-1}-1}{g^n-g^{n-1}-1}\right)\\&\leqslant T\left(r,f^n-f^{n-1}-1\right)+T\left(r,g^n-g^{n-1}-1\right)+S(r,f)\\&\leqslant O(T(r,f))+S(r,f)+S(r,g).\end{aligned} \tag{13}$$

By Lemma 2, Lemma 4 and (13), we obtain $\lambda\left(\mathrm{e}^h\right)=\mu\left(\mathrm{e}^h\right)\leqslant\mu(f)$. Noting that $\mu(f)$ is finite and noninteger and $\lambda\left(\mathrm{e}^h\right)\left(=\mu\left(\mathrm{e}^h\right)\right)$ is an integer, we have $\lambda\left(\mathrm{e}^h\right)<\mu(f)$. Hence,

$$T\left(r,Qe^h\right)=S(r,f). \tag{14}$$

It follows from Nevanlinna's first fundamental theorem, (11) and (14) that

$$\begin{aligned}nT(r,f)&=T(r,F)+S(r,f)\\&=T(r,F-1)+S(r,f)\\&=T\left(r,(G-1)Q\,\mathrm{e}^h\right)+S(r,f)\\&\leqslant T(r,G-1)+T\left(r,Q\,\mathrm{e}^h\right)+S(r,f)\\&\leqslant T(r,G)+S(r,f)=nT(r,g)+S(r,f).\end{aligned}$$

Hence, we have

$$T(r,f)\leqslant T(r,g)+S(r,f).$$

Similarly,

$$T(r,g)\leqslant T(r,f)+S(r,g). \tag{15}$$

Now, we consider two cases.

Case 1: $Q\,\mathrm{e}^h\equiv 1$. By (11), we have

$$(f^n-g^n)-\left(f^{n-1}-g^{n-1}\right)\equiv 0.$$

Set $h=\frac{f}{g}$. Then we have

$$g(h^n-1)-(h^{n-1}-1)\equiv 0. \tag{16}$$

Next, we consider two subcases.

Case 1.1: h is constant. By (16) and g is a nonconstant meromorphic function, we deduce that $h^i-1\equiv 0$ ($i=n,n-1$). Thus $h\equiv 1$, that is $f\equiv g$.

Case 1.2: h is not constant. It follows Lemma 8 that $\mu(g)=0$, a contradiction.

Case 2: $Q\,\mathrm{e}^h\not\equiv 1$. From (11), we get

$$f^n-f^{n-1}=Q\,\mathrm{e}^h(g^n-g^{n-1})+1-Q\,\mathrm{e}^h. \tag{17}$$

By (15), (17), Lemma 6 and Nevanlinna's first fundamental theorem and Lemma 1, we get

$$\begin{aligned}nT(r,f)&=T\left(r,f^n-f^{n-1}\right)+S(r,f)\\&\leqslant\overline{N}\left(r,f^n-f^{n-1}\right)+\overline{N}\left(r,\tfrac{1}{f^n-f^{n-1}}\right)\\&\quad+\overline{N}\left(r,\tfrac{1}{f^n-f^{n-1}+Q\,\mathrm{e}^h-1}\right)+S(r,f)\\&\leqslant\overline{N}(r,f)+\overline{N}\left(r,\tfrac{1}{f^{n-1}(f-1)}\right)\\&\quad+\overline{N}\left(r,\tfrac{1}{g^{n-1}(g-1)}\right)+S(r,f)\\&\leqslant\overline{N}(r,f)+\overline{N}\left(r,\tfrac{1}{f}\right)+\overline{N}\left(r,\tfrac{1}{f-1}\right)\\&\quad+\overline{N}\left(r,\tfrac{1}{g}\right)+\overline{N}\left(r,\tfrac{1}{g-1}\right)+S(r,f)\\&\leqslant 2T(r,f)+2T(r,g)+O(\log r)+S(r,f)\\&\leqslant 4T(r,f)+S(r,f).\end{aligned}$$

It follows from $n\geqslant 5$ that $T(r,f)\leqslant S(r,f)$, a contradiction.

This completes the proof of Theorem 2. □

PROOF OF Theorem 3

Proof: Since f and g share α_3 CM, by Lemma 3, we have

$$\frac{f-\alpha_3}{g-\alpha_3}=\mathrm{e}^h, \tag{18}$$

where h is a polynomial.

From Lemma 7 ($q=4$) and Nevanlinna's first fundamental theorem, we obtain

$$\begin{aligned}2T(r,f)&\leqslant\overline{N}(r,f)+\overline{N}\left(r,\tfrac{1}{f-\alpha_1}\right)+\overline{N}\left(r,\tfrac{1}{f-\alpha_2}\right)\\&\quad+\overline{N}\left(r,\tfrac{1}{f-\alpha_3}\right)+\varepsilon T(r,f)+S(r,f)\\&\leqslant\tfrac{1}{2}\overline{N}_{1)}\left(r,\tfrac{1}{f-\alpha_1}\right)+\tfrac{1}{2}N\left(r,\tfrac{1}{f-\alpha_1}\right)\\&\quad+\tfrac{1}{2}\overline{N}_{1)}\left(r,\tfrac{1}{f-\alpha_2}\right)+\tfrac{1}{2}N\left(r,\tfrac{1}{f-\alpha_2}\right)\\&\quad+\overline{N}\left(r,\tfrac{1}{f-\alpha_3}\right)+\varepsilon T(r,f)+S(r,f).\end{aligned} \tag{19}$$

Since $\overline{E}_{1)}(\alpha_i,f)=\overline{E}_{1)}(\alpha_i,g)$ ($i=1,2$), by f and g share α_3 CM, (19), we have

$$\begin{aligned}(1-\varepsilon)T(r,f)&\leqslant\tfrac{1}{2}\overline{N}_{1)}\left(r,\tfrac{1}{g-\alpha_1}\right)\\&\quad+\tfrac{1}{2}\overline{N}_{1)}\left(r,\tfrac{1}{g-\alpha_2}\right)+\overline{N}\left(r,\tfrac{1}{g-\alpha_3}\right)+S(r,f)\\&\leqslant 2T(r,g)+S(r,f).\end{aligned}$$

That is $T(r,f)\leqslant O(T(r,g))+S(r,f)$. It follows from Lemma 2 that $\mu(f)\leqslant\mu(g)$.

Similarly,

$$T(r,g)\leqslant O(T(r,f))+S(r,g), \tag{20}$$

and it follows from Lemma 2 that $\mu(g) \leqslant \mu(f)$. Hence, $\mu(g) = \mu(f)$.

By (18), (20) and Nevanlinna's first fundamental theorem, we have

$$\begin{aligned} T\left(r, \mathrm{e}^h\right) &= T\left(r, \tfrac{f-\alpha_3}{g-\alpha_3}\right) \\ &\leqslant T(r,f) + T(r,g) + S(r,f) \\ &\leqslant O(T(r,f)) + S(r,f). \end{aligned} \tag{21}$$

By Lemma 2, Lemma 4 and (21), we obtain $\lambda\left(\mathrm{e}^h\right) = \mu\left(\mathrm{e}^h\right) \leqslant \mu(f)$. Noting that $\mu(f)$ is finite and noninteger and $\lambda\left(\mathrm{e}^h\right)\left(= \mu\left(\mathrm{e}^h\right)\right)$ is an integer, we have $\lambda\left(\mathrm{e}^h\right) < \mu(f)$. Hence, e^h is a small function of both f and g.

Now, we consider two cases.

Case 1: $\mathrm{e}^h \equiv 1$. In this case, by (18), we have $f \equiv g$.

Case 2: $\mathrm{e}^h \not\equiv 1$. By $\overline{E}_{1)}(a_i, f) = \overline{E}_{1)}(a_i, g)$ $(i = 1, 2)$, (18) and Nevanlinna's first fundamental theorem, we have

$$\begin{aligned} \overline{N}_{1)}\left(r, \tfrac{1}{f-\alpha_1}\right) &\leqslant N\left(r, \tfrac{1}{\mathrm{e}^h-1}\right) \\ &\leqslant T\left(r, \mathrm{e}^h\right) + O(1) \leqslant S(r,f). \end{aligned} \tag{22}$$

Similarly,

$$\overline{N}_{1)}\left(r, \tfrac{1}{f-\alpha_2}\right) \leqslant S(r,f). \tag{23}$$

From (19), (22) and (23), we get

$$\begin{aligned} 2T(r,f) &\leqslant \tfrac{1}{2}\overline{N}_{1)}\left(r, \tfrac{1}{f-\alpha_1}\right) + \tfrac{1}{2}N\left(r, \tfrac{1}{f-\alpha_1}\right) \\ &\quad + \tfrac{1}{2}\overline{N}_{1)}\left(r, \tfrac{1}{f-\alpha_2}\right) + \tfrac{1}{2}N\left(r, \tfrac{1}{f-\alpha_2}\right) \\ &\quad + \overline{N}\left(r, \tfrac{1}{f-\alpha_3}\right) + \varepsilon T(r,f) + S(r,f) \qquad (24) \\ &\leqslant \tfrac{1}{2}N\left(r, \tfrac{1}{f-\alpha_1}\right) + \tfrac{1}{2}N\left(r, \tfrac{1}{f-\alpha_2}\right) \\ &\quad + \overline{N}\left(r, \tfrac{1}{f-\alpha_3}\right) + \varepsilon T(r,f) + S(r,f). \end{aligned}$$

Then we have

$$\begin{aligned} (2-\varepsilon)T(r,f) &\leqslant \tfrac{1}{2}N\left(r, \tfrac{1}{f-\alpha_1}\right) \\ &\quad + \tfrac{1}{2}N\left(r, \tfrac{1}{f-\alpha_2}\right) + \overline{N}\left(r, \tfrac{1}{f-\alpha_3}\right) + S(r,f), \end{aligned}$$

$$\begin{aligned} \underline{\lim}_{r\to\infty}(2-\varepsilon) &\leqslant \underline{\lim}_{r\to\infty}\left[\frac{N\left(r, \frac{1}{f-\alpha_1}\right)}{2T(r,f)}\right. \\ &\quad \left. + \frac{N\left(r, \frac{1}{f-\alpha_2}\right)}{2T(r,f)} + \frac{\overline{N}\left(r, \frac{1}{f-\alpha_3}\right)}{T(r,f)} + \frac{S(r,f)}{T(r,f)}\right], \end{aligned}$$

$$\begin{aligned} 2-\varepsilon &\leqslant \overline{\lim}_{r\to\infty}\left[\frac{N\left(r, \frac{1}{f-\alpha_1}\right)}{2T(r,f)} + \frac{N\left(r, \frac{1}{f-\alpha_2}\right)}{2T(r,f)}\right. \\ &\quad \left. + \frac{\overline{N}\left(r, \frac{1}{f-\alpha_3}\right)}{T(r,f)}\right] + \underline{\lim}_{r\to\infty}\frac{S(r,f)}{T(r,f)} \end{aligned}$$

$$\begin{aligned} 2-\varepsilon &\leqslant \frac{1}{2}(1-\delta(\alpha,f)) + \frac{1}{2}(1-\delta(\alpha_2,f)) \\ &\quad + (1-\Theta(\alpha_3,f)). \end{aligned}$$

Let $\varepsilon \to 0$, it follows that $\frac{1}{2}\delta(\alpha_1,f) + \frac{1}{2}\delta(\alpha_2,f) + \Theta(\alpha_3,f) \leqslant 0$, a contradiction.

This completes the proof of Theorem 3. □

PROOF OF Theorem 4

Proof: By Lemma 7 $\left(q = 4, \varepsilon = \frac{1}{7}\right)$, it is easy to prove Theorem 4 by imitating the proof of Theorem 3 and replacing (24) with the following formula.

$$\begin{aligned} 2T(r,f) &\leqslant \overline{N}(r,f) + \overline{N}\left(r, \tfrac{1}{f-\alpha_1}\right) + \overline{N}\left(r, \tfrac{1}{f-\alpha_2}\right) \\ &\quad + \overline{N}\left(r, \tfrac{1}{f-\alpha_3}\right) + \tfrac{1}{7}T(r,f) + S(r,f) \\ &\leqslant \tfrac{k_1}{k_1+1}\overline{N}_{k_1)}\left(r, \tfrac{1}{f-\alpha_1}\right) + \tfrac{1}{k_1+1}N\left(r, \tfrac{1}{f-\alpha_1}\right) \\ &\quad + \tfrac{k_2}{k_2+1}\overline{N}_{k_2)}\left(r, \tfrac{1}{f-\alpha_2}\right) + \tfrac{1}{k_2+1}N\left(r, \tfrac{1}{f-\alpha_2}\right) \\ &\quad + \overline{N}\left(r, \tfrac{1}{f-\alpha_3}\right) + \tfrac{1}{7}T(r,f) + S(r,f) \\ &\leqslant \tfrac{1}{k_1+1}N\left(r, \tfrac{1}{f-\alpha_1}\right) + \tfrac{1}{k_2+1}N\left(r, \tfrac{1}{f-\alpha_2}\right) \\ &\quad + \overline{N}\left(r, \tfrac{1}{f-\alpha_3}\right) + \tfrac{1}{7}T(r,f) + S(r,f) \\ &\leqslant \left(\tfrac{1}{k_1+1} + \tfrac{1}{k_2+1} + 1 + \tfrac{1}{7}\right)T(r,f) + S(r,f). \end{aligned}$$

It follows that $T(r,f) \leqslant S(r,f)$, a contradiction.

This completes the proof of Theorem 4. □

PROOF OF Theorem 5

Proof: We prove Theorem 5 by contradiction. Suppose that there exists an entire small function $\alpha(\not\equiv \infty)$ of f such that $f - \alpha$ has finitely many zeros.

Obviously,

$$f - \alpha = Q\,\mathrm{e}^h, \tag{25}$$

where Q is a polynomial and h is an entire function.

It follows from (25) that $\mu(f) = \mu(f-\alpha) = \mu\left(\mathrm{e}^h\right)$. Noting that $\mu(f)$ is finite and noninteger and $\mu\left(\mathrm{e}^h\right)$ is an integer, a contradiction.

This completes the proof of Theorem 5. □

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REFERENCES

1. Hayman WK (1964) *Meromorphic Functions*, Clarendon Press, Oxford.
2. Yang L (1993) *Value Distribution Theory*, Springer-Verlag, Berlin.
3. Yang CC, Yi HX (2003) *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers Group, Dordrecht.
4. Gross F (1977) *Factorization of Meromorphic Functions and Some Open Problems, Complex Analysis*, Springer, Berlin-New York.
5. Yi HX (1994) Uniqueness of meromorphic functions and question of Gross. *Sci China Ser A* **37**, 802–813.

6. Fang ML, Xu WS (1996) On the uniqueness of entire functions. *Bull Malaysian Math Soc* **19**, 29–37.
7. Yi HX (1998) On a question of Gross concerning uniqueness of entire functions. *Bull Aust Math Soc* **57**, 343–349.
8. Yi HX (1995) A question of Gross and the uniqueness of entire functions. *Nagoya Math J* **138**, 169–177.
9. Frank G, Reinders M (1998) A unique range set for meromorphic functions with 11 elements. *Complex Variables Theory Appl* **37**, 185–193.
10. Chen JF (2017) Uniqueness of meromorphic functions sharing two finite sets. *Open Math* **15**, 1244–1250.
11. Chen JF (2021) Meromorphic functions sharing unique range sets with one or two elements. *Georgian Math J* **28**, 349–354.
12. Chen JF, Cai XH (2019) Uniqueness results of meromorphic functions with finite and noninteger order. *Houston J Math* **45**, 115–127.
13. Sahoo P, Sarkar A (2020) On the uniqueness of meromorphic functions sharing two sets. *Bol Soc Mat Mex* **26**, 417–423.
14. Chakraborty B, Pal AK, Saha S, Kamila J (2022) Unique range sets of meromorphic functions of non-integer finite order. *Asian Eur J Math* **15**, 2250076.
15. Banerjee A, Kundu A (2022) A note on uniqueness of meromorphic functions sharing two singleton sets related to a question of Chen. *Novi Sad J Math* **52**, 41–68.
16. Li YH, Qiao JY (2000) The uniqueness of meromorphic functions concerning small functions. *Sci China Ser A* **43**, 581–590.
17. Conway JB (1973) *Functions of One Complex Variable*, Springer-Verlag.
18. Yang CC (1972) On deficiencies of differential polynomials II. *Math Z* **125**, 107–112.
19. Doeringer W (1982) Exceptional values of differential polynomials. *Pacific J Math* **98**, 55–62.
20. Fang ML, Guo H (1997) On meromorphic functions sharing two values. *Analysis* **17**, 355–366.
21. Yamanoi K (2004) The second main theorem for small functions and related problems. *Acta Math* **192**, 225–294.
22. Yi HX (1995) On a question of Gross. *Sci China Ser A* **38**, 8–16.
23. Mues E, Reinders M (1995) Meromorphic functions sharing one value and unique range sets. *Kodai Math J* **18**, 515–522.
24. Li P, Yang CC (1996) On the unique range set of meromorphic functions. *Proc Amer Math Soc* **124**, 177–185.