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Further multi-term refinements of Young's type inequalities and its applications

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ABSTRACT: We obtain some refined Young's type inequalities in this paper. And the results are given as follows. Let $a,\ b > 0$, $0 \leqslant \nu \leqslant \tau \leqslant 1$, then for any positive integer N: if $\frac{m}{2^N} < \nu \leqslant \tau \leqslant \frac{m+1}{2^N}$ for $m \in \{0, 1, 2, \cdots, 2^N - 1\}$, we have

$$a\nabla_\nu b \geqslant K^{2^N\nu - m}\left(\sqrt[2^N]{\tfrac{b}{a}}, 2\right) a\sharp_\nu b + \sum_{l=0}^{N-1} r_l(\nu) \sum_{k=1}^{2^l}\left(\sqrt{a^{1-\frac{k-1}{2^l}} b^{\frac{k-1}{2^l}}} - \sqrt{a^{1-\frac{k}{2^l}} b^{\frac{k}{2^l}}}\right)^2 \mathscr{X}_{\left(\frac{k-1}{2^l}, \frac{k}{2^l}\right)}(\nu)$$
$$+ \frac{2^N\nu - m}{2^N\tau - m}\left(\left(m+1-2^N\tau\right) a^{1-\frac{\lfloor 2^N\tau\rfloor}{2^N}} b^{\frac{\lfloor 2^N\tau\rfloor}{2^N}} + \left(2^N\tau - m\right) a^{1-\frac{\lfloor 2^N\tau\rfloor+1}{2^N}} b^{\frac{\lfloor 2^N\tau\rfloor+1}{2^N}} - K^{2^N\tau - m}\left(\sqrt[2^N]{\tfrac{b}{a}}, 2\right) a\sharp_\tau b\right),$$

where $r_0(\nu) = \min\{\nu, 1-\nu\}$, $r_l(\nu) = \min\{2r_{l-1}(\nu), 1 - 2r_{l-1}(\nu)\}$, and $K(h, 2) = \frac{(h+1)^2}{4h}$ with $h = \frac{b}{a}$. We also get some applications of Young's type inequalities.

KEYWORDS: Young's inequalities, Kantorovich constant, operator inequalities, norms

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INTRODUCTION

The classical Young inequality states that

$$(1-\nu)a + \nu b \geqslant a^{1-\nu} b^{\nu}, \tag{1}$$

where $a, b > 0$ and $0 \leqslant \nu \leqslant 1$. This inequality, though very simple, has attracted researchers working in operator theory due to its applications in this field. Refining this inequality and its reverse by finding intermediate terms by adding some positive quantities has taken the attention of numerous researches.

The constant $K(h, 2) = (h+1)^2/4h$, $(h > 0)$ is called the Kantorovich constant and satisfies the following properties:

(i) $K(1, 2) = 1$
(ii) $K(h, 2) = K(1/h, 2)$ for $h > 0$
(iii) $K(h, 2)$ is monotone increasing and monotone decreasing on the intervals $[1, \infty]$ and $(0, 1]$, respectively.

Zuo et al [1] refined the inequality with the Kantorovich constant in the following form:

$$(1-\nu)a + \nu b \geqslant K(h, 2)^{r_0(\nu)} a^{1-\nu} b^{\nu} \tag{2}$$

for positive real numbers a, b, ν and $0 \leqslant \nu \leqslant 1$, where $r_0(\nu) = \min\{\nu, 1-\nu\}$ and $h = b/a$.

Liao et al [2] shown the reverse inequality (2),

$$(1-\nu)a + \nu b \leqslant K(h, 2)^{R(\nu)} a^{1-\nu} b^{\nu}, \tag{3}$$

where $R(\nu) = \max\{\nu, 1-\nu\}$.

In 2016, Choi [3] showed a multi-term refinement of Young's inequality as follows.

Theorem 1 *Let a and b be two positive numbers and $0 \leqslant \nu \leqslant 1$. Then we have*

$$(1-\nu)a + \nu b \geqslant K^{r_N(\nu)}\left(\sqrt[2^{N-1}]{\tfrac{b}{a}}, 2\right) a^{1-\nu} b^{\nu}$$
$$+ \sum_{l=0}^{N-1} r_l(\nu) \sum_{k=1}^{2^l}\left(\sqrt{a^{1-\frac{k-1}{2^l}} b^{\frac{k-1}{2^l}}} - \sqrt{a^{1-\frac{k}{2^l}} b^{\frac{k}{2^l}}}\right)^2 \mathscr{X}_{\left(\frac{k-1}{2^l}, \frac{k}{2^l}\right)}(\nu), \tag{4}$$

where $r_l(\nu) = \min\{2r_{l-1}(\nu), 1 - 2r_{l-1}(\nu)\}$ and $\mathscr{X}_I$ is the characteristic function defined by

$$\mathscr{X}_I(x) = \begin{cases} 1, & \text{if } x \in I, \\ 0, & \text{if } x \notin I. \end{cases}$$

Alzer et al [4] obtained the interesting refinement of Young's inequality.

Theorem 2 *Let a, $b > 0$ and let λ, ν and τ be real numbers with $\lambda \geqslant 1$ and $0 \leqslant \nu \leqslant \tau \leqslant 1$. Then*

$$\left(\frac{\nu}{\tau}\right)^{\lambda} \leqslant \frac{(a\nabla_\nu b)^{\lambda} - (a\sharp_\nu b)^{\lambda}}{(a\nabla_\tau b)^{\lambda} - (a\sharp_\tau b)^{\lambda}} \leqslant \left(\frac{1-\nu}{1-\tau}\right)^{\lambda},$$

where $a\nabla_\nu b = (1-\nu)a + \nu b$, $a\nabla_\tau b = (1-\tau)a + \tau b$, $a\sharp_\nu b = a^{1-\nu} b^{\nu}$ and $a\sharp_\tau b = a^{1-\tau} b^{\tau}$.

The Young's inequality and its reverse are important in functional analysis, matrix theory, operator theory, electrical networks, etc. Many scholars had done much research in this topic. We refer the readers to the recent papers [5–8].

In recent work, Yang and Wang [9] gave the following Young type inequality.

Theorem 3 *Let $1/2 \leqslant \nu \leqslant \tau \leqslant 1$ and a, b are real positive numbers. Then*

$$\frac{K^{\nu}(h,2)a\sharp_{\nu}b-a\nabla_{\nu}b}{K^{\tau}(h,2)a\sharp_{\tau}b-a\nabla_{\tau}b}\leqslant\frac{\nu}{\tau}, \tag{5}$$

where $h=b/a$, $a\nabla_{\nu}b=(1-\nu)a+\nu b$, $a\nabla_{\tau}b=(1-\tau)a+\tau b$, $a\sharp_{\nu}b=a^{1-\nu}b^{\nu}$ and $a\sharp_{\tau}b=a^{1-\tau}b^{\tau}$.

Similar to Theorem 3, we also have the following results.

Theorem 4 *Let $0 \leqslant \nu \leqslant \tau \leqslant 1/2$ and a, b are real positive numbers. Then*

$$\frac{a\nabla_{\nu}b-K^{\nu}(h,2)a\sharp_{\nu}b}{a\nabla_{\tau}b-K^{\tau}(h,2)a\sharp_{\tau}b}\geqslant\frac{\nu}{\tau}, \tag{6}$$

where $h=b/a$, $a\nabla_{\nu}b=(1-\nu)a+\nu b$, $a\nabla_{\tau}b=(1-\tau)a+\tau b$, $a\sharp_{\nu}b=a^{1-\nu}b^{\nu}$ and $a\sharp_{\tau}b=a^{1-\tau}b^{\tau}$.

Proof: Let

$$f(t)=\frac{1-t+tc-K^{t}(c,2)c^{t}}{t}=\frac{1-t+tc-(\frac{1+c}{2})^{2t}}{t}$$

where $c\in(0,\infty)$, $t\in(0,1/2]$. Then we can get

$$f'(t)=\frac{h(c)}{t^2}$$

where

$$h(c)=\Big(c-1-2\big(\tfrac{1+c}{2}\big)^{2t}\ln\tfrac{1+c}{2}\Big)t-\Big(1-t+tc-\big(\tfrac{1+c}{2}\big)^{2t}\Big)$$

and

$$\begin{aligned}h'(c)&=\Big(1-2t\big(\tfrac{1+c}{2}\big)^{2t-1}\ln\tfrac{1+c}{2}-\big(\tfrac{1+c}{2}\big)^{2t-1}\Big)t-\Big(t-t\big(\tfrac{1+c}{2}\big)^{2t-1}\Big)\\&=-2t^2\big(\tfrac{1+c}{2}\big)^{2t-1}\ln\tfrac{1+c}{2}.\end{aligned}$$

It means that $c\in(0,1]$, $h'(c)\geqslant0$, so $h(c)$ is increasing on $(0,1]$; $c\in[1,\infty)$, $h'(c)\leqslant0$, so $h(c)$ is decreasing on $[1,\infty)$. Therefore, $h(c)\leqslant h(1)=0$, $f'(t)\leqslant0$ $(c>0)$. When $c>0$, $f(t)$ is decreasing on $(0,1/2]$, $f(t)\geqslant f(1/2)=0$. Taking $c=b/a$, we can get inequality (6). □

We use inequality (2) to get that the denominator and numerator of inequality (6) are greater than zero. Similarly, we use inequality (3) to get that the denominator and numerator of inequality (5) are also greater than zero.

So the two inequalities (5) and (6) can be sorted out into one inequality as follows:

$$a\nabla_{\nu}b\geqslant K^{\nu}(h,2)a\sharp_{\nu}b+\frac{\nu}{\tau}\Big(a\nabla_{\tau}b-K^{\tau}(h,2)a\sharp_{\tau}b\Big), \tag{7}$$

where $0\leqslant\nu\leqslant\tau\leqslant1$.

A MULTI-TERM REFINEMENT OF YOUNG'S TYPE INEQUALITY

In this section, we mainly present the direct refinements of the Young's inequality (7).

Theorem 5 *Let a, $b\geqslant0$ and $0\leqslant\nu\leqslant\tau\leqslant1$.*
(i) *If $0<\nu\leqslant\tau\leqslant\frac{1}{2}$, then*

$$\begin{aligned}a\nabla_{\nu}b\geqslant{}&K^{2\nu}\Big(\sqrt{\tfrac{b}{a}},2\Big)a\sharp_{\nu}b+\nu\Big(\sqrt{a}-\sqrt{b}\Big)^2\\&+\frac{\nu}{\tau}\Big((1-2\tau)a+2\tau\sqrt{ab}-K^{2\tau}(\sqrt{\tfrac{b}{a}},2)a\sharp_{\tau}b\Big).\end{aligned} \tag{8}$$

(ii) *If $\frac{1}{2}<\nu\leqslant\tau\leqslant1$, then*

$$\begin{aligned}a\nabla_{\nu}b\geqslant{}&K^{2\nu-1}\Big(\sqrt{\tfrac{b}{a}},2\Big)a\sharp_{\nu}b+(1-\nu)\Big(\sqrt{a}-\sqrt{b}\Big)^2\\&+\frac{2\nu-1}{2\tau-1}\Big((2-2\tau)\sqrt{ab}+(2\tau-1)b\\&-K^{2\tau-1}\Big(\sqrt{\tfrac{b}{a}},2\Big)a\sharp_{\tau}b\Big).\end{aligned} \tag{9}$$

Proof: (i) When $0<\nu\leqslant\tau\leqslant1/2$, by simple calculation and inequality (7), then we have

$$\begin{aligned}(1-\nu)a+\nu b-\nu(\sqrt{a}-\sqrt{b})^2&=a\nabla_{2\nu}\sqrt{ab}\\&\geqslant K^{2\nu}(\tfrac{\sqrt{ab}}{a},2)a^{1-2\nu}(\sqrt{ab})^{2\nu}\\&\quad+\frac{2\nu}{2\tau}\Big(a\nabla_{2\tau}\sqrt{ab}-K^{2\tau}(\tfrac{\sqrt{ab}}{a},2)a^{1-2\tau}(\sqrt{ab})^{2\tau}\Big)\\&=K^{2\nu}(\sqrt{\tfrac{b}{a}},2)a\sharp_{\nu}b\\&\quad+\frac{\nu}{\tau}\Big((1-2\tau)a+(2\tau)\sqrt{ab}-K^{2\tau}(\sqrt{\tfrac{b}{a}},2)a\sharp_{\tau}b\Big).\end{aligned}$$

So inequality (8) holds.

(ii) When $1/2<\nu\leqslant\tau\leqslant1$, by simple calculation and inequality (7), then we have

$$\begin{aligned}(1-\nu)a+\nu b-(1-\nu)(\sqrt{a}-\sqrt{b})^2&=\sqrt{ab}\nabla_{2\nu-1}b\\&\geqslant K^{2\nu-1}(\tfrac{b}{\sqrt{ab}},2)b^{2\nu-1}(\sqrt{ab})^{2-2\nu}+\tfrac{2\nu-1}{2\tau-1}\Big(\sqrt{ab}\nabla_{2\tau-1}b\\&\quad-K^{2\tau-1}(\tfrac{b}{\sqrt{ab}},2)b^{2\tau-1}(\sqrt{ab})^{2-2\tau}\Big)\\&=K^{2\nu-1}(\sqrt{\tfrac{b}{a}},2)a\sharp_{\nu}b+\tfrac{2\nu-1}{2\tau-1}\Big((2-2\tau)\sqrt{ab}\\&\quad+(2\tau-1)b-K^{2\tau-1}(\sqrt{\tfrac{b}{a}},2)a\sharp_{\tau}b\Big).\end{aligned}$$

So inequality (9) holds. □

By analogy with this approach, we get a more general generalization.

Theorem 6 *Let a, $b>0$, $0\leqslant\nu\leqslant\tau\leqslant1$, then for any positive integer N: if $m/2^N<\nu\leqslant\tau\leqslant(m+1)/2^N$ for*

$m \in \{0,1,2,\dots,2^N-1\}$, *we have*

$$
\begin{aligned}
a\nabla_\nu b \geqslant{}& K^{2^N\nu-m}\Big(\sqrt[2^N]{\tfrac{b}{a}},2\Big)a\sharp_\nu b\\
&+\sum_{l=0}^{N-1} r_l(\nu)\sum_{k=1}^{2^l}\Big(\sqrt{a^{1-\frac{k-1}{2^l}}\,b^{\frac{k-1}{2^l}}}-\sqrt{a^{1-\frac{k}{2^l}}\,b^{\frac{k}{2^l}}}\Big)^2\mathscr{X}_{\left(\frac{k-1}{2^l},\frac{k}{2^l}\right)}(\nu)\\
&+\frac{2^N\nu-m}{2^N\tau-m}\Big((m+1-2^N\tau)a^{1-\frac{\lfloor 2^N\tau\rfloor}{2^N}}\,b^{\frac{\lfloor 2^N\tau\rfloor}{2^N}}\\
&+(2^N\tau-m)a^{1-\frac{\lfloor 2^N\tau\rfloor+1}{2^N}}\,b^{\frac{\lfloor 2^N\tau\rfloor+1}{2^N}}\\
&-K^{2^N\tau-m}\Big(\sqrt[2^N]{\tfrac{b}{a}},2\Big)a\sharp_\tau b\Big).
\end{aligned}
\tag{10}
$$

where $r_0(\nu)=\min\{\nu,1-\nu\}$, $r_l(\nu)=\min\{2r_{l-1}(\nu),1-2r_{l-1}(\nu)\}$, $\lfloor x\rfloor$ *is the greatest integer less than or equal to* x.

Proof: To complete the proof, we need the following three steps.

Step 1: When $N=1$, inequality (10) can be written as inequality (8) or inequality (9).

Step 2: Let inequality (10) hold for some $N>1$.

If $m/2^{N+1}<\nu\leqslant\tau\leqslant(m+1)/2^{N+1}$ for some $m\in\{0,1,2,\dots,2^N-1\}$, then $m/2^N<2\nu\leqslant 2\tau\leqslant(m+1)/2^N$, we have

$$
\begin{aligned}
&(1-\nu)a+\nu b-\nu(\sqrt{a}-\sqrt{b})^2\\
&=a\nabla_{2\nu}\sqrt{ab}\\
&\geqslant K^{2^N(2\nu)-m}\Big(\sqrt[2^N]{\tfrac{\sqrt{ab}}{a}},2\Big)a^{1-2\nu}(\sqrt{ab})^{2\nu}\\
&\quad+\sum_{l=0}^{N-1} r_l(2\nu)\sum_{k=1}^{2^l}\Big(\sqrt{a^{1-\frac{k-1}{2^l}}(\sqrt{ab})^{\frac{k-1}{2^l}}}\\
&\quad-\sqrt{a^{1-\frac{k}{2^l}}(\sqrt{ab})^{\frac{k}{2^l}}}\Big)^2\mathscr{X}_{\left(\frac{k-1}{2^l},\frac{k}{2^l}\right)}(2\nu)\\
&\quad+\tfrac{2^N(2\nu)-m}{2^N(2\tau)-m}\Big((m+1-2^N(2\tau))a^{1-\frac{\lfloor 2^N(2\tau)\rfloor}{2^N}}(\sqrt{ab})^{\frac{\lfloor 2^N(2\tau)\rfloor}{2^N}}\\
&\quad+(2^N(2\tau)-m)a^{1-\frac{\lfloor 2^N(2\tau)\rfloor+1}{2^N}}(\sqrt{ab})^{\frac{\lfloor 2^N(2\tau)\rfloor+1}{2^N}}\\
&\quad-K^{2^N(2\tau)-m}\Big(\sqrt[2^N]{\tfrac{\sqrt{ab}}{a}},2\Big)a^{1-2\tau}(\sqrt{ab})^{2\tau}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&=K^{2^{N+1}\nu-m}\Big(\sqrt[2^{N+1}]{\tfrac{b}{a}},2\Big)a\sharp_\nu b+\sum_{l=0}^{N-1} r_{l+1}(\nu)\\
&\times\sum_{k=1}^{2^{l+1}}\Big(\sqrt{a^{1-\frac{k-1}{2^{l+1}}}\,b^{\frac{k-1}{2^{l+1}}}}-\sqrt{a^{1-\frac{k}{2^{l+1}}}\,b^{\frac{k}{2^{l+1}}}}\Big)^2\mathscr{X}_{\left(\frac{k-1}{2^{l+1}},\frac{k}{2^{l+1}}\right)}(\nu)\\
&\quad+\tfrac{2^{N+1}\nu-m}{2^{N+1}\tau-m}\Big((m+1-2^{N+1}\tau)a^{1-\frac{\lfloor 2^{N+1}\tau\rfloor}{2^{N+1}}}\,b^{\frac{\lfloor 2^{N+1}\tau\rfloor}{2^{N+1}}}\\
&\quad+(2^{N+1}\tau-m)a^{1-\frac{\lfloor 2^{N+1}\tau\rfloor+1}{2^{N+1}}}\,b^{\frac{\lfloor 2^{N+1}\tau\rfloor+1}{2^{N+1}}}\\
&\quad-K^{2^{N+1}\tau-m}\Big(\sqrt[2^{N+1}]{\tfrac{b}{a}},2\Big)a\sharp_\tau b\Big)
\end{aligned}
$$

$$
\begin{aligned}
&=K^{2^{N+1}\nu-m}\Big(\sqrt[2^{N+1}]{\tfrac{b}{a}},2\Big)a\sharp_\nu b+\sum_{l=1}^{N} r_l(\nu)\\
&\quad\times\sum_{k=1}^{2^l}\Big(\sqrt{a^{1-\frac{k-1}{2^l}}\,b^{\frac{k-1}{2^l}}}-\sqrt{a^{1-\frac{k}{2^l}}\,b^{\frac{k}{2^l}}}\Big)^2\mathscr{X}_{\left(\frac{k-1}{2^l},\frac{k}{2^l}\right)}(\nu)\\
&\quad+\tfrac{2^{N+1}\nu-m}{2^{N+1}\tau-m}\Big((m+1-2^{N+1}\tau)a^{1-\frac{\lfloor 2^{N+1}\tau\rfloor}{2^{N+1}}}\,b^{\frac{\lfloor 2^{N+1}\tau\rfloor}{2^{N+1}}}\\
&\quad+(2^{N+1}\tau-m)a^{1-\frac{\lfloor 2^{N+1}\tau\rfloor+1}{2^{N+1}}}\,b^{\frac{\lfloor 2^{N+1}\tau\rfloor+1}{2^{N+1}}}\\
&\quad-K^{2^{N+1}\tau-m}\Big(\sqrt[2^{N+1}]{\tfrac{b}{a}},2\Big)a\sharp_\tau b\Big).
\end{aligned}
$$

Step 3: If $m/2^{N+1}<\nu\leqslant\tau\leqslant(m+1)/2^{N+1}$ for some $m\in\{2^N,2^N+1,\dots,2^{N+1}-1\}$, then $\left(m-2^N\right)/2^N<2\nu-1\leqslant 2\tau-1\leqslant\left(m+1-2^N\right)/2^N$.

Let $m_1=m-2^N$, then $m_1\in\{0,1,2,\dots,2^N-1\}$ and $m_1/2^N<2\nu-1\leqslant 2\tau-1\leqslant(m_1+1)/2^N$, at the same time $k_1=k-2^l$ by inequality (10):

$$
\begin{aligned}
&(1-\nu)a+\nu b-(1-\nu)(\sqrt{a}-\sqrt{b})^2=\sqrt{ab}\nabla_{2\nu-1}b\\
&\geqslant K^{2^N(2\nu-1)-m_1}\Big(\sqrt[2^N]{\tfrac{b}{\sqrt{ab}}},2\Big)(\sqrt{ab})^{2-2\nu}b^{2\nu-1}\\
&\quad+\sum_{l=0}^{N-1} r_l(2\nu-1)\sum_{k_1=1}^{2^l}\Big(\sqrt{(\sqrt{ab})^{1-\frac{k_1-1}{2^l}}\,b^{\frac{k_1-1}{2^l}}}\\
&\quad-\sqrt{(\sqrt{ab})^{1-\frac{k_1}{2^l}}\,b^{\frac{k_1}{2^l}}}\Big)^2\mathscr{X}_{\left(\frac{k_1-1}{2^l},\frac{k_1}{2^l}\right)}(2\nu-1)\\
&\quad+\tfrac{2^N(2\nu-1)-m_1}{2^N(2\tau-1)-m_1}\Big((2^N(2\tau-1)-m_1)\\
&\quad\times(\sqrt{ab})^{1-\frac{\lfloor 2^N(2\tau-1)\rfloor+1}{2^N}}\,b^{\frac{\lfloor 2^N(2\tau-1)\rfloor+1}{2^N}}\\
&\quad+(m_1+1-2^N(2\tau-1))(\sqrt{ab})^{1-\frac{\lfloor 2^N(2\tau-1)\rfloor}{2^N}}\,b^{\frac{\lfloor 2^N(2\tau-1)\rfloor}{2^N}}\\
&\quad-K^{2^N(2\tau-1)-m_1}\Big(\sqrt[2^N]{\tfrac{b}{\sqrt{ab}}},2\Big)(\sqrt{ab})^{2-2\tau}b^{2\tau-1}\Big)\\
&=K^{2^{N+1}\nu-2^N-m_1}\Big(\sqrt[2^{N+1}]{\tfrac{b}{a}},2\Big)a\sharp_\nu b\\
&\quad+\sum_{l=0}^{N-1} r_{l+1}(\nu)\sum_{k=1}^{2^{l+1}}\Big(\sqrt{a^{1-\frac{2^l+k_1-1}{2^{l+1}}}\,b^{\frac{2^l+k_1-1}{2^{l+1}}}}\\
&\quad-\sqrt{a^{1-\frac{2^l+k_1}{2^{l+1}}}\,b^{\frac{2^l+k_1}{2^{l+1}}}}\Big)^2\mathscr{X}_{\left(\frac{k_1+2^l-1}{2^{l+1}},\frac{2^l+k_1}{2^{l+1}}\right)}(\nu)\\
&\quad+\tfrac{2^{N+1}\nu-2^N-m_1}{2^{N+1}\tau-2^N-m_1}\Big((2^{N+1}\tau-2^N-m_1)a^{1-\frac{\lfloor 2^{N+1}\tau\rfloor}{2^{N+1}}}\,b^{\frac{\lfloor 2^{N+1}\tau\rfloor}{2^{N+1}}}\\
&\quad+(2^N+m_1+1-2^{N+1}\tau)a^{1-\frac{\lfloor 2^{N+1}\tau\rfloor+1}{2^{N+1}}}\,b^{\frac{\lfloor 2^{N+1}\tau\rfloor+1}{2^{N+1}}}\\
&\quad-K^{2^{N+1}\tau-2^N-m_1}\Big(\sqrt[2^{N+1}]{\tfrac{b}{a}},2\Big)a\sharp_\tau b\Big)\\
&=K^{2^{N+1}\nu-m}\Big(\sqrt[2^{N+1}]{\tfrac{b}{a}},2\Big)a\sharp_\nu b+\sum_{l=1}^{N} r_l(\nu)\\
&\times\sum_{k=1}^{2^l}\Big(\sqrt{a^{1-\frac{k-1}{2^l}}\,b^{\frac{k-1}{2^l}}}-\sqrt{a^{1-\frac{k}{2^l}}\,b^{\frac{k}{2^l}}}\Big)^2\mathscr{X}_{\left(\frac{k-1}{2^l},\frac{k}{2^l}\right)}(\nu)\\
&+\tfrac{2^{N+1}\nu-m}{2^{N+1}\tau-m}\Big((m+1-2^{N+1}\tau)a^{1-\frac{\lfloor 2^{N+1}\tau\rfloor}{2^{N+1}}}\,b^{\frac{\lfloor 2^{N+1}\tau\rfloor}{2^{N+1}}}\\
&+(2^{N+1}\tau-m)a^{1-\frac{\lfloor 2^{N+1}\tau\rfloor+1}{2^{N+1}}}\,b^{\frac{\lfloor 2^{N+1}\tau\rfloor+1}{2^{N+1}}}-K^{2^{N+1}\tau-m}\Big(\sqrt[2^{N+1}]{\tfrac{b}{a}},2\Big)a\sharp_\tau b\Big).
\end{aligned}
$$

So we have

$$a\nabla_\nu b \geqslant K^{2^{N+1}\nu-m}\Big(\sqrt[2^{N+1}]{\tfrac{b}{a}},2\Big)a\sharp_\nu b + \sum_{l=0}^{N} r_l(\nu)\sum_{k=1}^{2^l}\Big(\sqrt{a^{1-\frac{k-1}{2^l}}b^{\frac{k-1}{2^l}}} - \sqrt{a^{1-\frac{k}{2^l}}b^{\frac{k}{2^l}}}\Big)^2\mathscr{X}_{\big(\frac{k-1}{2^l},\frac{k}{2^l}\big)}(\nu) + \tfrac{2^{N+1}\nu-m}{2^{N+1}\tau-m}\Big((m+1-2^{N+1}\tau)a^{1-\frac{\lfloor 2^{N+1}\tau\rfloor}{2^{N+1}}}b^{\frac{\lfloor 2^{N+1}\tau\rfloor}{2^{N+1}}} + (2^{N+1}\tau-m)a^{1-\frac{\lfloor 2^{N+1}\tau\rfloor+1}{2^{N+1}}}b^{\frac{\lfloor 2^{N+1}\tau\rfloor+1}{2^{N+1}}} - K^{2^{N+1}\tau-m}\Big(\sqrt[2^{N+1}]{\tfrac{b}{a}},2\Big)a\sharp_\tau b\Big),$$

where $\lfloor 2^{N+1}\tau - 2^N\rfloor = -\lfloor 2^N - 2^{N+1}\tau\rfloor - 1 = \lfloor 2^{N+1}\tau\rfloor - 2^N$, i.e. $N+1$, the inequality (10) holds.

So, combining the above steps, we use the inductive hypothesis method to complete the results. □

Remark 1 Replacing a and b by their square in inequality (10), we get some results as follows.
If $m/2^N < \nu \leqslant \tau \leqslant (m+1)/2^N$ for $m \in \{0,1,2,\dots,2^N-1\}$, we have

$$a^2\nabla_\nu b^2 \geqslant K^{2^N\nu-m}\Big(\sqrt[2^{N-1}]{\tfrac{b}{a}},2\Big)(a\sharp_\nu b)^2 + \sum_{l=0}^{N-1} r_l(\nu)\sum_{k=1}^{2^l}(a^{1-\frac{k-1}{2^l}}b^{\frac{k-1}{2^l}} - a^{1-\frac{k}{2^l}}b^{\frac{k}{2^l}})^2\mathscr{X}_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu) + \tfrac{2^N\nu-m}{2^N\tau-m}\Big((m+1-2^N\tau)(a^{1-\frac{\lfloor 2^N\tau\rfloor}{2^N}}b^{\frac{\lfloor 2^N\tau\rfloor}{2^N}})^2 + (2^N\tau-m)(a^{1-\frac{\lfloor 2^N\tau\rfloor+1}{2^N}}b^{\frac{\lfloor 2^N\tau\rfloor+1}{2^N}})^2 - K^{2^N\tau-m}\Big(\sqrt[2^{N-1}]{\tfrac{b}{a}},2\Big)(a\sharp_\tau b)^2\Big).$$

On the other hand, we get

$$a^2\nabla_\nu b^2 - r_0(\nu)(a-b)^2 = (a\nabla_\nu b)^2 - r_0^2(\nu)(a-b)^2.$$

Therefore, by the above two relations, we obtain the following corollary.

Corollary 1 *Let a, $b>0$, $0 \leqslant \nu \leqslant \tau \leqslant 1$. Then, for any positive integer N: if $m/2^N < \nu \leqslant \tau \leqslant (m+1)/2^N$, for $m \in \{0,1,2,\dots,2^N-1\}$, we have*

$$(a\nabla_\nu b)^2 \geqslant K^{2^N\nu-m}\Big(\sqrt[2^{N-1}]{\tfrac{b}{a}},2\Big)(a\sharp_\nu b)^2 + r_0^2(\nu)(a-b)^2 + \sum_{l=1}^{N-1} r_l(\nu)\sum_{k=1}^{2^l}(a^{1-\frac{k-1}{2^l}}b^{\frac{k-1}{2^l}} - a^{1-\frac{k}{2^l}}b^{\frac{k}{2^l}})^2\mathscr{X}_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu) + \tfrac{2^N\nu-m}{2^N\tau-m}\Big((m+1-2^N\tau)(a^{1-\frac{\lfloor 2^N\tau\rfloor}{2^N}}b^{\frac{\lfloor 2^N\tau\rfloor}{2^N}})^2 + (2^N\tau-m)(a^{1-\frac{\lfloor 2^N\tau\rfloor+1}{2^N}}b^{\frac{\lfloor 2^N\tau\rfloor+1}{2^N}})^2 - K^{2^N\tau-m}\Big(\sqrt[2^{N-1}]{\tfrac{b}{a}},2\Big)(a\sharp_\tau b)^2\Big), \quad (11)$$

where $r_0(\nu) = \min\{\nu, 1-\nu\}$, $r_l(\nu) = \min\{2r_{l-1}(\nu), 1-2r_{l-1}(\nu)\}$, $\lfloor x\rfloor$ is the greatest integer less than or equal to x.

APPLICATIONS

In this section, we mainly give an operator inequality and Hilbert-Schmidt norm for the improved Young inequality.

Before giving the main result of this part, we need to recall certain useful knowledge.

Let $\mathbb{B}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators acting on a complex (separable) Hilbert space $(\mathscr{H},\langle\cdot,\cdot\rangle)$ and I be its identity. An operator $A \in \mathbb{B}(\mathscr{H})$ is said to be positive semi-definite (denote by $A \geqslant 0$) if $\langle Ax,x\rangle \geqslant 0$ for all vectors $x \in \mathscr{H}$. The set of all positive operators is denoted by $\mathbb{B}(\mathscr{H})^+$. If $\langle Ax,x\rangle > 0$ for all nonzero vectors $x \in \mathscr{H}$, A is said to be positive (denotes $A > 0$). The set of all invertible operators in $\mathbb{B}(\mathscr{H})^+$ is denoted by $\mathbb{B}(\mathscr{H})^{++}$. For self-adjoint operators A, $B \in \mathbb{B}(\mathscr{H})$.

For positive invertible operators A, $B \in \mathbb{B}(\mathscr{H})$, the weighted operator arithmetic mean and geometric mean of A and B defined, respectively, by

$$A\nabla_\nu B = (1-\nu)A + \nu B,$$
$$A\sharp_\nu B = A^{\frac12}(A^{-\frac12}BA^{-\frac12})^\nu A^{\frac12},$$

where $\nu \in [0,1]$. When $\nu = 1/2$, $A\nabla_{1/2}B$ and $A\sharp_{1/2}B$ are called, respectively, operator arithmetic mean and operator geometric mean, which are denoted by $A\nabla B$ and $A\sharp B$.

Let $\mathscr{M}_n(\mathbb{C})$ denotes the space of all $n\times n$ complex matrices and $\mathscr{M}_n^+(\mathbb{C})$ denotes the space of all $n\times n$ positive semi-definite matrices in $\mathscr{M}_n(\mathbb{C})$. A norm $|||\cdot|||$ is called unitarily invariant norm if $|||UAV||| = |||A|||$ for all $A \in \mathscr{M}_n(\mathbb{C})$ and for all unitary matrices U, $V \in \mathscr{M}_n(\mathbb{C})$. For $A = [a_{ij}] \in \mathscr{M}_n(\mathbb{C})$, the Hilbert-Schmidt norm of A is defined by

$$\|A\|_2 = \sqrt{\sum_{i=1}^n s_i^2(A)} = \sqrt{\sum_{i,j=1}^n |a_{ij}|^2},$$

where $s_1(A) \geqslant s_2(A) \geqslant \cdots \geqslant s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive matrix $|A| = (A^*A)^{1/2}$, arranged in decreasing order and repeated according to multiplicity. The Hilbert-Schmidt norm is unitarily invariant.

Lemma 1 ([10]) *Let $A \in \mathbb{B}(\mathscr{H})$ be self-adjoint. If f and g are both continuous functions with $f(t) \geqslant g(t)$ for $t \in Sp(A)$ (where the sign $Sp(A)$ denotes the spectrum of operator A), then $f(A) \geqslant g(A)$.*

The following theorem presents the operator version of Theorem 6.

Theorem 7 *Let A, $B \in \mathbb{B}(\mathscr{H})^{++}$ and $0 \leqslant \nu \leqslant \tau \leqslant 1$. If all positive numbers q, q' and Q, Q' satisfy either of the following conditions: $0 < q'I \leqslant A \leqslant qI < QI \leqslant B \leqslant Q'I$, $0 < q'I \leqslant B \leqslant qI < QI \leqslant A \leqslant Q'I$. Then for all positive* *integer N, we have: if $m/2^N < \nu \leqslant \tau \leqslant (m+1)/2^N$, for $m \in \{0,1,2,\dots,2^N-1\}$*

$$A\nabla_\nu B \geqslant K^{2^N\nu-m}(\sqrt[2^N]{h},2)A\sharp_\nu B + \sum_{l=0}^{N-1} r_l(\nu)\sum_{k=1}^{2^l}(A\sharp_{\frac{k-1}{2^l}}B + A\sharp_{\frac{k}{2^l}}B - 2A\sharp_{\frac{2k-1}{2^{l+1}}}B)\mathscr{X}_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu) + \frac{2^N\nu-m}{2^N\tau-m}\Big((m+1-2^N\tau)A\sharp_{\frac{\lfloor 2^N\tau\rfloor}{2^N}}B + (2^N\tau-m)A\sharp_{\frac{\lfloor 2^N\tau\rfloor+1}{2^N}}B - K^{2^N\tau-m}(\sqrt[2^N]{h'},2)A\sharp_\tau B\Big), \quad (12)$$

where $h = Q/q$, $h' = Q'/q'$.

Proof: Taking $a=1$, $b=t$ in inequality (10),

$$(1-\nu)+\nu b \geqslant K^{2^N\nu-m}(\sqrt[2^N]{t},2)t^\nu + \sum_{l=0}^{N-1} r_l(\nu)\sum_{k=1}^{2^l}\Big(\sqrt{t^{\frac{k-1}{2^l}}}-\sqrt{t^{\frac{k}{2^l}}}\Big)^2\mathscr{X}_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu) + \frac{2^N\nu-m}{2^N\tau-m}\Big((m+1-2^N\tau)t^{\frac{\lfloor 2^N\tau\rfloor}{2^N}} + (2^N\tau-m)t^{\frac{\lfloor 2^N\tau\rfloor+1}{2^N}} - K^{2^N\tau-m}(\sqrt[2^N]{t},2)t^\tau\Big).$$

For $X = A^{-\frac12}BA^{-\frac12}$, we get

$$(1-\nu)I+\nu X \geqslant K_{\min}^{2^N\nu-m}(\sqrt[2^N]{t},2)X^\nu + \sum_{l=0}^{N-1} r_l(\nu)\sum_{k=1}^{2^l}\Big(\sqrt{X^{\frac{k-1}{2^l}}}-\sqrt{X^{\frac{k}{2^l}}}\Big)^2\mathscr{X}_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu) + \frac{2^N\nu-m}{2^N\tau-m}\Big((m+1-2^N\tau)X^{\frac{\lfloor 2^N\tau\rfloor}{2^N}} + (2^N\tau-m)X^{\frac{\lfloor 2^N\tau\rfloor+1}{2^N}} - K_{\max}^{2^N\tau-m}(\sqrt[2^N]{t},2)X^\tau\Big).$$

(i) $I \leqslant hI = (Q/q)I \leqslant X \leqslant Q'/q' = h'I$ and the $Sp(X) \subseteq [h,h'] \subseteq [1,\infty)$. Then by Lemma 1 and Kantorovich constant $K(h,2) = (h+1)^2/4h$ is an increasing function for $h > 1$, so we have

$$K_{\min}^{2^N\nu-m}(\sqrt[2^N]{t},2) = K^{2^N\nu-m}(\sqrt[2^N]{h},2),$$
$$K_{\max}^{2^N\tau-m}(\sqrt[2^N]{t},2) = K^{2^N\tau-m}(\sqrt[2^N]{h'},2).$$

(ii) $0 < (1/h')I \leqslant X \leqslant (1/h)I \leqslant I$. Since the Kantorovich constant $K(h,2) = (h+1)^2/4h$ is an decreasing function for $0 < h < 1$, so we have

$$K_{\min}^{2^N\nu-m}(\sqrt[2^N]{t},2) = K^{2^N\nu-m}\Big(\sqrt[2^N]{\tfrac{1}{h}},2\Big),$$
$$K_{\max}^{2^N\tau-m}(\sqrt[2^N]{t},2) = K^{2^N\tau-m}\Big(\sqrt[2^N]{\tfrac{1}{h'}},2\Big).$$

Using $K(1/h,2) = K(h,2)$ for $h > 0$, we get

$$(1-\nu)I+\nu X \geqslant K^{2^N\nu-m}(\sqrt[2^N]{h},2)X^\nu + \sum_{l=0}^{N-1} r_l(\nu)\sum_{k=1}^{2^l}\Big(\sqrt{X^{\frac{k-1}{2^l}}}-\sqrt{X^{\frac{k}{2^l}}}\Big)^2\mathscr{X}_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu) + \frac{2^N\nu-m}{2^N\tau-m}\Big((m+1-2^N\tau)X^{\frac{\lfloor 2^N\tau\rfloor}{2^N}} + (2^N\tau-m)X^{\frac{\lfloor 2^N\tau\rfloor+1}{2^N}} - K^{2^N\tau-m}(\sqrt[2^N]{h'},2)X^\tau\Big).$$

Multiplying both sides by $A^{1/2}$ to the above inequality, we can deduce

$$A\nabla_\nu B \geqslant K^{2^N\nu-m}(\sqrt[2^N]{h},2)A\sharp_\nu B + \sum_{l=0}^{N-1} r_l(\nu)\sum_{k=1}^{2^l}(A\sharp_{\frac{k-1}{2^l}}B + A\sharp_{\frac{k}{2^l}}B - 2A\sharp_{\frac{2k-1}{2^{l+1}}}B)\mathscr{X}_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu) + \frac{2^N\nu-m}{2^N\tau-m}\Big((m+1-2^N\tau)A\sharp_{\frac{\lfloor 2^N\tau\rfloor}{2^N}}B + (2^N\tau-m)A\sharp_{\frac{\lfloor 2^N\tau\rfloor+1}{2^N}}B - K^{2^N\tau-m}(\sqrt[2^N]{h'},2)A\sharp_\tau B\Big).$$

The proof is completed. □

The following theorem presents the Hilbert-Schmidt norm version of Corollary 1.

Theorem 8 *Suppose A, B, $X \in \mathscr{M}_n(\mathbb{C})$ such that A and B are two positive definite matrices and satisfy $0 < qI \leqslant A$, $B \leqslant QI$, where I represents the identity matrix and q, $Q \in \mathbb{R}$. For any positive integer N and $0 \leqslant \nu \leqslant \tau \leqslant 1$, we have: if $m/2^N < \nu \leqslant \tau \leqslant (m+1)/2^N$ for $m \in \{0,1,2,\dots,2^N-1\}$,*

$$\|(1-\nu)AX+\nu XB\|_2^2 \geqslant K_{\min}^{2^N\nu-m}(\sqrt[2^{N-1}]{t_{ij}},2)\|A^{1-\nu}XB^\nu\|_2^2 + r_0^2(\nu)\|AX-XB\|_2^2 + \sum_{l=1}^{N-1} r_l(\nu)\sum_{k=1}^{2^l}\Big(\|A^{1-\frac{k-1}{2^l}}XB^{\frac{k-1}{2^l}}\|_2^2 + \|A^{1-\frac{k}{2^l}}XB^{\frac{k}{2^l}}\|_2^2 - 2\|A^{1-\frac{2k-1}{2^{l+1}}}XB^{\frac{2k-1}{2^{l+1}}}\|_2^2\Big)\mathscr{X}_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu) + \frac{2^N\nu-m}{2^N\tau-m}\Big((m+1-2^N\tau)\|A^{1-\frac{\lfloor 2^N\tau\rfloor}{2^N}}XB^{\frac{\lfloor 2^N\tau\rfloor}{2^N}}\|_2^2 + (2^N\tau-m)\|A^{1-\frac{\lfloor 2^N\tau\rfloor+1}{2^N}}XB^{\frac{\lfloor 2^N\tau\rfloor+1}{2^N}}\|_2^2 - K_{\max}^{2^N\tau-m}(\sqrt[2^{N-1}]{t_{ij}},2)\|A^{1-\tau}XB^\tau\|_2^2\Big), \quad (13)$$

where $q/Q = 1/h \leqslant t_{ij} = \lambda_i/\mu_j \leqslant h = Q/q$.

Proof: Since A and B are positive definite, it follows by the spectral theorem that there exist unitary matrices U, $V \in \mathscr{M}_n(\mathbb{C})$ such that

$$A = U\Lambda_1U^*, \qquad B = V\Lambda_2V^*,$$

where $\Lambda_1 = \mathrm{diag}(\lambda_1,\lambda_2,\dots,\lambda_n)$, $\Lambda_2 = \mathrm{diag}(\mu_1,\mu_2,\dots,\mu_n)$, λ_i, $\mu_i \geqslant 0$, $i = 1,2,\dots,n$.

Let $Y = U^*XV = [y_{ij}]$, $i, j = 1, 2, \ldots, n$, then

$$\begin{aligned}(1-\nu)AX+\nu XB &= U((1-\nu)\Lambda_1 Y + \nu Y\Lambda_2)V^* \\ &= U[((1-\nu)\lambda_i+\nu\mu_j)y_{ij}]V^*.\\ AX-XB &= U[(\lambda_i-\mu_j)y_{ij}]V^*,\\ A^{1-\nu}XB^{\nu} &= U[\lambda_i^{1-\nu}\mu_j^{\nu} y_{ij}]V^*.\end{aligned}$$

Next, we use the inequality (11) in Corollary 1, we can get

$$\begin{aligned}
&\|(1-\nu)AX+\nu XB\|_2^2 = \sum_{i,j=1}^{n}((1-\nu)\lambda_i+\nu\mu_j)^2|y_{ij}|^2\\
&\geqslant \sum_{i,j=1}^{n}\Big(K^{2^N\nu-m}\big(\sqrt[2^{N-1}]{\tfrac{\mu_j}{\lambda_i}},2\big)(\lambda_i^{1-\nu}\mu_j^{\nu})^2+r_0^2(\nu)(\lambda_i-\mu_j)^2\\
&\quad+\sum_{l=1}^{N-1}r_l(\nu)\sum_{k=1}^{2^l}(\lambda_i^{1-\frac{k-1}{2^l}}\mu_j^{\frac{k-1}{2^l}}-\lambda_i^{1-\frac{k}{2^l}}\mu_j^{\frac{k}{2^l}})^2\mathscr{X}_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu)\\
&\quad+\frac{2^N\nu-m}{2^N\tau-m}\Big((m+1-2^N\tau)(\lambda_i^{1-\frac{\lfloor 2^N\tau\rfloor}{2^N}}\mu_j^{\frac{\lfloor 2^N\tau\rfloor}{2^N}})^2\\
&\quad+(2^N\tau-m)(\lambda_i^{1-\frac{\lfloor 2^N\tau\rfloor+1}{2^N}}\mu_j^{\frac{\lfloor 2^N\tau\rfloor+1}{2^N}})^2\\
&\quad-K^{2^N\tau-m}\big(\sqrt[2^{N-1}]{\tfrac{\mu_j}{\lambda_i}},2\big)(\lambda_i^{1-\tau}\mu_j^{\tau})^2\Big)\Big)|y_{ij}|^2
\end{aligned}$$

$$\begin{aligned}
&\geqslant K_{\min}^{2^N\nu-m}\big(\sqrt[2^{N-1}]{t_{ij}},2\big)\sum_{i,j=1}^{n}(\lambda_i^{1-\nu}\mu_j^{\nu})^2|y_{ij}|^2\\
&\qquad+r_0^2(\nu)\sum_{i,j=1}^{n}(\lambda_i-\mu_j)^2|y_{ij}|^2\\
&+\sum_{l=1}^{N-1}r_l(\nu)\sum_{k=1}^{2^l}\sum_{i,j=1}^{n}(\lambda_i^{1-\frac{k-1}{2^l}}\mu_j^{\frac{k-1}{2^l}}-\lambda_i^{1-\frac{k}{2^l}}\mu_j^{\frac{k}{2^l}})^2\mathscr{X}_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu)|y_{ij}|^2\\
&\quad+\tfrac{2^N\nu-m}{2^N\tau-m}\Big((m+1-2^N\tau)\sum_{i,j=1}^{n}(\lambda_i^{1-\frac{\lfloor 2^N\tau\rfloor}{2^N}}\mu_j^{\frac{\lfloor 2^N\tau\rfloor}{2^N}})^2|y_{ij}|^2\\
&\qquad+(2^N\tau-m)\sum_{i,j=1}^{n}(\lambda_i^{1-\frac{\lfloor 2^N\tau\rfloor+1}{2^N}}\mu_j^{\frac{\lfloor 2^N\tau\rfloor+1}{2^N}})^2|y_{ij}|^2\\
&\qquad-K_{\max}^{2^N\tau-m}\big(\sqrt[2^{N-1}]{t_{ij}},2\big)(\lambda_i^{1-\tau}\mu_j^{\tau})^2|y_{ij}|^2\Big).
\end{aligned}$$

So we have

$$\begin{aligned}
&\|(1-\nu)AX+\nu XB\|_2^2\geqslant K_{\min}^{2^N\nu-m}\big(\sqrt[2^{N-1}]{t_{ij}},2\big)\|A^{1-\nu}XB^{\nu}\|_2^2\\
&+r_0^2(\nu)\|AX-XB\|_2^2+\sum_{l=1}^{N-1}r_l(\nu)\sum_{k=1}^{2^l}\Big(\|A^{1-\frac{k-1}{2^l}}XB^{\frac{k-1}{2^l}}\|_2^2\\
&+\|A^{1-\frac{k}{2^l}}XB^{\frac{k}{2^l}}\|_2^2-2\|A^{1-\frac{2k-1}{2^{l+1}}}XB^{\frac{2k-1}{2^{l+1}}}\|_2^2\Big)\mathscr{X}_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu)\\
&+\tfrac{2^N\nu-m}{2^N\tau-m}\Big((m+1-2^N\tau)\|A^{1-\frac{\lfloor 2^N\tau\rfloor}{2^N}}XB^{\frac{\lfloor 2^N\tau\rfloor}{2^N}}\|_2^2\\
&+(2^N\tau-m)\|A^{1-\frac{\lfloor 2^N\tau\rfloor+1}{2^N}}XB^{\frac{\lfloor 2^N\tau\rfloor+1}{2^N}}\|_2^2\\
&-K_{\max}^{2^N\tau-m}\big(\sqrt[2^{N-1}]{t_{ij}},2\big)\|A^{1-\tau}XB^{\tau}\|_2^2\Big).
\end{aligned}$$

The proof is established. □

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