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An alternative functional equation related to the quadratic equation

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ABSTRACT: Given a rational number $\alpha \neq 2$, we establish a criterion for the existence of the general solution of an alternative quadratic functional equation of the form

$$f(xy)+f(xy^{-1})=2f(x)+2f(y) \quad \text{or} \quad f(xy)+f(xy^{-1})=\alpha f(x)+2f(y),$$

where f is a mapping from an abelian group $(G,\cdot)$ to a uniquely divisible abelian group $(H,+)$. We also find the general solution in the cases when G is a 6-divisible abelian group and G is a cyclic group.

KEYWORDS: alternative quadratic, quadratic functional equation, quadratic function

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INTRODUCTION

One of the most interesting problem of functional equations is solving the alternative functional equation. For instance, the alternative Cauchy functional equation

$$\big(f(x+y)-af(x)-bf(y)\big)\big(f(x+y)-f(x)-f(y)\big)=0$$

related to the classical Cauchy functional equation

$$f(x+y)=f(x)+f(y)$$

has been studied by Kannappan et al [1]. Forti [2] extended the above work by finding the general solution in a more general setting of the form

$$\big(cf(x+y)-af(x)-bf(y)-d\big)\big(f(x+y)-f(x)-f(y)\big)=0,$$

and it also extended the work of Ger [3] as well as that of Forti and Paganoni [4, 5]. Skof [6] proposed the four alternative functional equations as follows:

$$\begin{aligned}|f(x+y)| &= |2f(x)+2f(y)-f(x-y)|,\\ |f(x-y)| &= |2f(x)+2f(y)-f(x+y)|,\\ |2f(y)| &= |f(x+y)+f(x-y)-2f(x)|,\\ \text{and } |2f(x)| &= |f(x+y)+f(x-y)-2f(y)|\end{aligned}$$

and proved that each of above functional equations is equivalent to the classical quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{1}$$

where f is a function from a real linear space to the set of real numbers. The quadratic functional equation (1) can be stated as

$$f(xy)+f(xy^{-1})=2f(x)+2f(y) \tag{2}$$

when the domain of f is a group. Nakmahachalasint [7] showed that an alternative quadratic functional equation of the form

$$f(xy)+f(xy^{-1})=\pm\big(2f(x)+2f(y)\big)$$

is equivalent to the quadratic functional equation (2), where f is a function from a 2-divisible group to a uniquely abelian group. Forti [8] studied the solution of an alternative functional equation of the form

$$f(xy)+f(xy^{-1})-2f(x)-2f(y)\in\{0,1\},$$

where f is a function from a group to the set of real numbers. Tipyan [9] proved that an alternative quadratic functional equation of the form

$$f(xy)+f(xy^{-1})=2f(x)\pm 2f(y)$$

is equivalent to the quadratic functional equation (2), where f is a function from a 2-divisible abelian group to a linear space.

In this paper, given a rational number $\alpha \neq 2$, we establish a criterion for the existence of the general solution of an alternative quadratic functional equation of the form

$$\begin{aligned}&f(xy)+f(xy^{-1})=2f(x)+2f(y) \quad \text{or}\\ &f(xy)+f(xy^{-1})=\alpha f(x)+2f(y)\end{aligned} \tag{3}$$

where f is a mapping from an abelian group to a uniquely divisible abelian group. We also find the general solution in the cases when G is a 6-divisible abelian group and G is a cyclic group.

AUXILIARY LEMMAS

Throughout the paper, we let $(G,\cdot)$ be an abelian group and H be a uniquely divisible abelian group. Given a rational number $\alpha\neq 2$ and a function $f:G\to H$. For every pair of $x,y\in G$, we denote the statement

$$\mathscr{P}f_y^{(\alpha)}(x)=\Big(f(xy)+f(xy^{-1})=2f(x)+2f(y)\\ \text{or } f(xy)+f(xy^{-1})=\alpha f(x)+2f(y)\Big).$$

The set of all solutions to the statement $\mathscr{P}f_y^{(\alpha)}(x)$ will be denoted by $\mathscr{A}_{(G,H)}^{(\alpha)}$, i.e.,

$$\mathscr{A}_{(G,H)}^{(\alpha)}=\{f:G\to H\mid \mathscr{P}f_y^{(\alpha)}(x)\ \text{ for all } x,y\in G\}.$$

In this section, we will prove some auxiliary lemmas that will lead to the proof of main results.

Lemma 1 *Let $f\in\mathscr{A}_{(G,H)}^{(\alpha)}$. If $f(e)\neq 0$, then $\alpha=0$ and f is constant.*

Proof: Assume that $f(e)\neq 0$. By the alternatives in $\mathscr{P}f_e^{(\alpha)}(e)$, we obtain that $\alpha=0$. By $f(e)\neq 0$ and the alternatives in $\mathscr{P}f_e^{(\alpha)}(x)$, we get $f(x)=f(e)$ for all $x\in G$. Hence, we have the desired result. □

Lemma 2 *Let $f\in\mathscr{A}_{(G,H)}^{(\alpha)}$. Then $f(x^{-1})=f(x)$ for all $x\in G$, i.e., f is even.*

Proof: If $f(e)\neq 0$, then by Lemma 1, f is constant. If $f(e)=0$, then the alternatives in $\mathscr{P}f_x^{(\alpha)}(e)$ gives $f(x^{-1})=f(x)$ for any $x\in G$. Therefore, we get the desired result. □

Lemma 3 *Let $f\in\mathscr{A}_{(G,H)}^{(\alpha)}$ and $f(e)=0$. If $\alpha\notin\{-2,-1\}$, then $f(x^2)=4f(x)$ for all $x\in G$.*

Proof: It should be noted that f is even by Lemma 2. We will prove this lemma by contradiction. Assume that $\alpha\notin\{-2,-1\}$ and there exists $a\in G$ such that

$$f(a^2)\neq 4f(a). \tag{4}$$

The alternatives in $\mathscr{P}f_a^{(\alpha)}(a)$ gives

$$f(a^2)=(2+\alpha)f(a). \tag{5}$$

It should be noted that if $f(a)=0$, then $f(a^2)=0$, a contradiction to (4). By (5) and the alternatives in $\mathscr{P}f_{a^2}^{(\alpha)}(a)$, we have

$$f(a^3)\in\{(5+2\alpha)f(a),(3+3\alpha)f(a)\}. \tag{6}$$

By (5) and the alternatives in $\mathscr{P}f_a^{(\alpha)}(a^2)$, we get

$$f(a^3)\in\{(5+2\alpha)f(a),(1+2\alpha+\alpha^2)f(a)\}. \tag{7}$$

Next, we consider two possible cases of $f(a^3)$ as follows. Suppose $f(a^3)\neq(5+2\alpha)f(a)$. By (6) and (7), we obtain $f(a)=0$, a contradiction. Suppose $f(a^3)=(5+2\alpha)f(a)$. By (5), the alternatives in $\mathscr{P}f_{a^3}^{(\alpha)}(a)$ and $\mathscr{P}f_a^{(\alpha)}(a^3)$ gives

$$f(a^4)\in\{(10+3\alpha)f(a),(8+4\alpha)f(a)\} \text{ and}\\ f(a^4)\in\{(10+3\alpha)f(a),(4\alpha+2\alpha^2)f(a)\}.$$

Hence we conclude that

$$f(a^4)=(10+3\alpha)f(a) \text{ or } \alpha=-\frac{5}{2}. \tag{8}$$

By (5), the alternatives in $\mathscr{P}f_{a^2}^{(\alpha)}(a^2)$, we have

$$f(a^4)\in\{(8+4\alpha)f(a),(4+4\alpha+\alpha^2)f(a)\}. \tag{9}$$

Suppose $\alpha\neq-5/2$. By (8) and (9), we get $\alpha=-3$. Hence, $f(a^2)=f(a^3)=-f(a)$ and $f(a^4)=f(a)$. Thus, the alternatives in $\mathscr{P}f_a^{(-3)}(a^4)$ gives $f(a^5)\in\{5f(a),0\}$, while the alternatives in $\mathscr{P}f_{a^2}^{(-3)}(a^3)$ gives $f(a^5)\in\{-4f(a),f(a)\}$. Hence, $f(a)=0$, a contradiction. Suppose $\alpha=-5/2$. We have $f(a^3)=0$. Thus, (5) becomes $f(a^2)=-\frac{1}{2}f(a)$ and (9) becomes

$$f(a^4)\in\Big\{-2f(a),\frac{1}{4}f(a)\Big\}. \tag{10}$$

By the alternatives in $\mathscr{P}f_a^{(-3)}(a^3)$, we get

$$f(a^4)=\frac{5}{2}f(a). \tag{11}$$

By (10) and (11), we have $f(a)=0$, a contradiction. □

Lemma 4 *Let $f\in\mathscr{A}_{(G,H)}^{(\alpha)}$ and $x\in G$. If $f(x^2)=4f(x)$, then $f(x^n)=n^2f(x)$ for all integers n.*

Proof: It should be noted that f is even by Lemma 2. Assume that $f(x^2)=4f(x)$. If $f(e)\neq 0$, then by Lemma 1, we obtain that f is constant, i.e., $f(x)=f(e)$. Hence, $f(x)=f(x^2)=4f(x)$ and then $f(x)=0$, a contradiction. We must have $f(e)=0$. Suppose $f(x^3)\neq 9f(x)$. The alternatives in $\mathscr{P}f_{x^2}^{(\alpha)}(x)$ gives

$$f(x^3)=(7+\alpha)f(x), \tag{12}$$

while the alternatives in $\mathscr{P}f_x^{(\alpha)}(x^2)$ gives

$$f(x^3)=(1+4\alpha)f(x). \tag{13}$$

By (12) and (13), we conclude $f(x^3) = f(x) = 0$, a contradiction to $f(x^3) \neq 9f(x)$. Now we obtain that

$$f(x^n) = n^2 f(x) \tag{14}$$

holds for $n = 1, 2, 3$. Let $k \geqslant 3$ be an integer. Assume that (14) holds for $n = 1, 2, \ldots, k$. We have $f(x^{k-1}) = (k-1)^2 f(x)$ and $f(x^k) = k^2 f(x)$. Suppose $f(x^{k+1}) \neq (k+1)^2 f(x)$. The alternatives in $\mathscr{P} f_{x^k}^{(\alpha)}(x)$ gives

$$f(x^{k+1}) = (\alpha + 2k^2 - (k-1)^2) f(x),$$

while the alternatives in $\mathscr{P} f_x^{(\alpha)}(x^k)$ gives

$$f(x^{k+1}) = (\alpha k^2 + 2 - (k-1)^2) f(x).$$

Hence, $f(x^{k+1}) = f(x) = 0$, a contradiction to $f(x^{k+1}) \neq (k+1)^2 f(x)$. Therefore, (14) must hold for $n = k+1$. By induction and the evenness of f, we conclude that (14) holds for all integers n as desired. □

Lemma 5 *Let $f \in \mathscr{A}_{(G,H)}^{(\alpha)}$. If $f(e) = 0$ and there exists $a \in G$ such that $f(a^2) \neq 4f(a)$, then $f(a) \neq 0$ and*

(i) $\alpha = -1$ *and*

$$f(a^n) = \begin{cases} 0 & \text{if } 3 \mid n, \\ f(a) & \text{otherwise} \end{cases}$$

for all integers n, or

(ii) $\alpha = -2$ *and*

$$f(a^n) = \begin{cases} 0 & \text{if } n \text{ is even}, \\ f(a) & \text{otherwise} \end{cases}$$

for all integers n.

Proof: It is should be noted that f is even by Lemma 2. Assume that $f(e) = 0$ and there exists $a \in G$ such that $f(a^2) \neq 4f(a)$. The alternatives in $\mathscr{P} f_a^{(\alpha)}(a)$ gives $f(a^2) = (2+\alpha) f(a)$. It should be noted that if $f(a) = 0$, then $f(a^2) = 0$, a contradiction. By the assumptions, Lemma 3 gives $\alpha \in \{-1, -2\}$.

Case (i). Suppose $\alpha = -1$. We get $f(a^2) = f(a)$. The alternatives in $\mathscr{P} f_a^{(-1)}(a^2)$ gives

$$f(a^3) \in \{3f(a), 0\}. \tag{15}$$

Suppose $f(a^3) \neq 0$ and so $f(a^3) = 3f(a)$. The alternatives in $\mathscr{P} f_a^{(-1)}(a^3)$ gives

$$f(a^4) \in \{7f(a), -2f(a)\},$$

while the alternatives in $\mathscr{P} f_{a^2}^{(-1)}(a^2)$ gives

$$f(a^4) \in \{4f(a), f(a)\}.$$

We get $f(a) = 0$, a contradiction. We must have $f(a^3) = 0$. Observe that the alternatives in $\mathscr{P} f_{a^3}^{(-1)}(a^3)$ gives $f(a^6) = 0$, and the alternatives in $\mathscr{P} f_a^{(-1)}(a^3)$ and $\mathscr{P} f_{a^2}^{(-1)}(a^3)$ gives $f(a^4) = f(a^5) = f(a)$. Then, the alternatives in $\mathscr{P} f_{a^6}^{(-1)}(a^3)$ gives $f(a^9) = 0$, and the alternatives in $\mathscr{P} f_{a^4}^{(-1)}(a^3)$ and $\mathscr{P} f_{a^5}^{(-1)}(a^3)$ gives $f(a^7) = f(a^8) = f(a)$. For each a positive integer n, we use the alternatives in $\mathscr{P} f_{a^{3n}}^{(-1)}(a^3)$ to prove $f(a^{3n+3}) = 0$ and we use the alternatives in $\mathscr{P} f_{a^{3n-2}}^{(-1)}(a^3)$ and $\mathscr{P} f_{a^{3n-1}}^{(-1)}(a^3)$ to prove $f(a^{3n+1}) = f(a^{3n+2}) = f(a)$. By induction and the evenness of f, we get the desired result.

Case (ii). Suppose $\alpha = -2$. We get $f(a^2) = 0$. Observe that the alternatives in $\mathscr{P} f_{a^2}^{(-2)}(a^2)$ gives $f(a^4) = 0$ and the alternatives in $\mathscr{P} f_a^{(-2)}(a^2)$ gives $f(a^3) = f(a)$. Then the alternatives in $\mathscr{P} f_{a^4}^{(-2)}(a^2)$ gives $f(a^6) = 0$ and the alternatives in $\mathscr{P} f_{a^3}^{(-2)}(a^2)$ gives $f(a^5) = f(a)$. For each a positive integer n, we use the alternatives in $\mathscr{P} f_{a^{2n}}^{(-2)}(a^2)$ to prove $f(a^{2n+2}) = 0$ and we use alternatives in $\mathscr{P} f_{a^{2n-1}}^{(-2)}(a^2)$ to prove $f(a^{2n+1}) = f(a)$. By induction and the evenness of f, we get the desired result. □

Lemma 6 *Let $f \in \mathscr{A}_{(G,H)}^{(\alpha)}$. If $f(x^2) = 4f(x)$ for all $x \in G$, then f is quadratic.*

Proof: It should be noted that f is even by Lemma 2. We will prove this lemma by contradiction. Assume that

$$f(x^2) = 4f(x) \tag{16}$$

for all $x \in G$ but f is not quadratic. Setting $x = e$ in (16), we get $f(e) = 0$. Thus, there exist $x, y \in G$ such that

$$f(xy) + f(xy^{-1}) \neq 2f(x) + 2f(y). \tag{17}$$

By the alternatives in $\mathscr{P} f_y^{(\alpha)}(x)$, we obtain

$$f(xy) + f(xy^{-1}) = \alpha f(x) + 2f(y). \tag{18}$$

By (16), (17) and the alternatives in $\mathscr{P} f_{xy^{-1}}^{(\alpha)}(xy)$, we get

$$4f(x) + 4f(y) = \alpha f(xy) + 2f(xy^{-1}). \tag{19}$$

Let $k := f(x)$. Eliminating $f(xy^{-1})$ and $f(y)$ from (18) and (19), we have $f(xy) = -2k$. Similarly, by replacing y by y^{-1} in the process of (19), we get $f(xy^{-1}) = -2k$. Hence, (19) reduces to

$$2f(y) = (-4 - \alpha)k. \tag{20}$$

By (16), we have $f(y^2) = 4f(y)$ and then by (20), we get

$$f(y^2) = (-8 - 2\alpha)k. \tag{21}$$

If $k = 0$, then by (18) and (20), we obtain $2f(y) = 0$ and $f(xy) + f(xy^{-1}) = 0$, a contradiction to (17).

Suppose $k \neq 0$. By (20), the alternatives in $\mathscr{P} f_y^{(\alpha)}(xy)$ gives

$$f(xy^2) \in \{(-9-\alpha)k, (-5-3\alpha)k\} \quad (22)$$

and the alternatives in $\mathscr{P} f_y^{(\alpha)}(xy^{-1})$ gives

$$f(xy^{-2}) \in \{(-9-\alpha)k, (-5-3\alpha)k\}. \quad (23)$$

Next, we consider the possible cases of (22) and (23) as follows.

Case (i). Assume that $f(xy^2) = f(xy^{-2}) = (-9-\alpha)k$ or $f(xy^2) = f(xy^{-2}) = (-5-3\alpha)k$. By (21) and the alternatives in $\mathscr{P} f_{y^2}^{(\alpha)}(x)$, we conclude that $k = 0$, a contradiction.

Case (ii). Assume that $f(xy^2) = (-9-\alpha)k$ and $f(xy^{-2}) = (-5-3\alpha)k$. By $f(y^2) = 4f(y)$, Lemma 4 gives

$$f(y^3) = 9f(y). \quad (24)$$

By (20) and (24), we get

$$f(y^3) = \Big(-18 - \frac{9}{2}\alpha\Big)k. \quad (25)$$

By (20), the alternatives in $\mathscr{P} f_y^{(\alpha)}(xy^2)$ gives

$$f(xy^3) \in \{\, (-20-3\alpha)k, (-2-10\alpha-\alpha^2)k\}, \quad (26)$$

and the alternatives in $\mathscr{P} f_y^{(\alpha)}(xy^{-2})$ gives

$$f(xy^{-3}) \in \{\, (-12-7\alpha)k, (-2-6\alpha-3\alpha^2)k\}. \quad (27)$$

We next consider four possible cases of $f(xy^3)$ and $f(xy^{-3})$ as follows.

(a) Suppose $f(xy^3) = (-20-3\alpha)k$ and $f(xy^{-3}) = (-12-7\alpha)k$. By (25) and the alternatives in $\mathscr{P} f_{y^3}^{(\alpha)}(x)$, we obtain that $k = 0$, a contradiction.

(b) Suppose $f(xy^3) = (-20-3\alpha)k$ and $f(xy^{-3}) = (-2-6\alpha-3\alpha^2)k$. By (25), the alternatives in $\mathscr{P} f_{y^3}^{(\alpha)}(x)$ gives $\alpha \in \Big\{-2, -7/3\Big\}$, while the alternatives in $\mathscr{P} f_x^{(\alpha)}(y^3)$ gives $\alpha \in \{-2, -8\}$. Hence, $\alpha = -2$ and then

$$f(y^2) = -4k \text{ and } f(xy^3) = -14k. \quad (28)$$

By $f(xy) = f(xy^{-1}) = -2k$ and (28), the alternatives in $\mathscr{P} f_{y^2}^{(\alpha)}(xy)$ gives $k = 0$, a contradiction.

(c) Suppose $f(xy^3) = (-2-10\alpha-\alpha^2)k$ and $f(xy^{-3}) = (-12-7\alpha)k$. By (25), the alternatives in $\mathscr{P} f_{y^3}^{(\alpha)}(x)$ gives $\alpha \in \{-10, -11\}$, while the alternatives in $\mathscr{P} f_x^{(\alpha)}(y^3)$ gives $\alpha \in \Big\{-10, -16/7\Big\}$. Hence, $\alpha = -10$ and then

$$f(y^2) = 12k \text{ and } f(xy^3) = -2k. \quad (29)$$

By $f(xy) = f(xy^{-1}) = -2k$ and (29), the alternatives in $\mathscr{P} f_{y^2}^{(\alpha)}(xy)$ gives $k = 0$, a contradiction.

(d) Suppose $f(xy^3) = (-2-10\alpha-\alpha^2)k$ and $f(xy^{-3}) = (-2-6\alpha-3\alpha^2)k$. By (25), the alternatives in $\mathscr{P} f_{y^3}^{(\alpha)}(x)$ gives $\alpha \in \Big\{-15/4, -4\Big\}$, while the alternatives in $\mathscr{P} f_x^{(\alpha)}(y^3)$ gives $\alpha \in \Big\{-15/4, -6\Big\}$. Hence, $\alpha = -15/4$, and then

$$f(y^2) = -\frac{1}{2}k \text{ and } f(xy^3) = \frac{343}{16}k. \quad (30)$$

By $f(xy) = f(xy^{-1}) = -2k$ and (30), the alternatives in $\mathscr{P} f_{y^2}^{(\alpha)}(xy)$ gives $k = 0$, a contradiction.

Case (iii). If $f(xy^2) = (-5-3\alpha)k$ and $f(xy^{-2}) = (-9-\alpha)k$, then the proof is as in Case (ii) after replacing y by y^{-1}. □

MAIN RESULTS AND EXAMPLES

In this section, we shall use the lemmas in the previous section to obtain the main theorems. Moreover, we will give the general solution of the alternative quadratic functional equation (3) on a cyclic group. We first prove the main theorem.

Theorem 1 *Let $f \in \mathscr{A}_{(G,H)}^{(\alpha)}$. Then f is quadratic or one of the following properties holds.*

(i) *$\alpha = 0$ and f is constant.*

(ii) *$\alpha = -1$ and there exists $a \in G$ such that $f(a) \neq 0$ and*

$$f(a^n) = \begin{cases} 0 & \text{if } 3 \mid n, \\ f(a) & \text{otherwise} \end{cases}$$

for all integers n.

(iii) *$\alpha = -2$ and there exists $a \in G$ such that $f(a) \neq 0$ and*

$$f(a^n) = \begin{cases} 0 & \text{if } n \text{ is even}, \\ f(a) & \text{otherwise} \end{cases}$$

for all integers n.

Proof: We consider the possible cases of $f(e)$ as follows. If $f(e) \neq 0$, then Lemma 1 gives the property (i). Assume that $f(e) = 0$. If $f(x^2) = 4f(x)$ for all $x \in G$, then by Lemma 6, we obtain that f is quadratic. If there exists $a \in G$ such that $f(a^2) \neq 4f(a)$, then we get properties (ii) and (iii) by Lemma 5. □

In other words, the above theorem indicates that when $\alpha \in \{-1, -2\}$, there exists a function $f \in \mathscr{A}_{(G,H)}^{(\alpha)}$ but f is not necessarily quadratic. We will give the following two examples.

Example 1 Let $f : \mathbb{Z} \to H$ be such that

$$f(n) = \begin{cases} 0 & \text{if } 3 \mid n, \\ k & \text{otherwise} \end{cases}$$

for all $n \in \mathbb{Z}$ and for some $k \in H \backslash \{0\}$.

Observe that

$$f(1+1)+f(1-1)=k\neq 4k=2f(1)+2f(1).$$

Let x and y be integers. If $3\mid x$ and $3\mid y$, then

$$f(x+y)+f(x-y)=0=2f(x)+2f(y);$$

else if $3\mid x$ and $3\nmid y$, or $3\nmid x$ and $3\mid y$, then

$$f(x+y)+f(x-y)=2k=2f(x)+2f(y);$$

otherwise,

$$f(x+y)+f(x-y)=k=-f(x)+2f(y).$$

Therefore, $f\in\mathscr{A}^{(-1)}_{(\mathbb{Z},H)}$ but f is not quadratic.

Example 2 Let $f:\mathbb{Z}\to H$ be such that

$$f(n)=\begin{cases}0 & \text{if } n \text{ is even,}\\ k & \text{otherwise}\end{cases}$$

for all $n\in\mathbb{Z}$ and for some $k\in H\backslash\{0\}$.

Observe that

$$f(1+1)+f(1-1)=0\neq 4k=2f(1)+2f(1).$$

Let x and y be integers. If x and y have the same parity, then

$$f(x+y)+f(x-y)=0=-2f(x)+2f(y);$$

otherwise,

$$f(x+y)+f(x-y)=2k=2f(x)+2f(y).$$

Therefore, $f\in\mathscr{A}^{(-2)}_{(\mathbb{Z},H)}$ but f is not quadratic.

Example 1 and Example 2 indicate that if $G=\mathbb{Z}$ is a cyclic group, then there exist non-quadratic solutions of (3) in the case when $\alpha=-1$ and $\alpha=-2$, respectively. However, to the best of our knowledge, it is not known whether there exist non-quadratic solutions of (3) when G is a non-cyclic group.

The next theorem will establish the equivalence of the alternative quadratic functional equation (3) and the quadratic functional equation (2) for a function defined on a 6-divisible abelian group.

Theorem 2 *Let G be a 6-divisible abelian group and $f\in\mathscr{A}^{(\alpha)}_{(G,H)}$. If $\alpha\neq 0$, then f is quadratic.*

Proof: It should be noted that f is even by Lemma 2. Assume that $\alpha\neq 0$. Lemma 1 gives $f(e)=0$. We will prove that $f(x^2)=4f(x)$ for all $x\in G$ by contradiction. Suppose that there exists $a\in G$ such that $f(a^2)\neq 4f(a)$. By Lemma 5, we obtain that $f(a)\neq 0$ and

(i) $\alpha=-1$ and

$$f(a^n)=\begin{cases}0 & \text{if } 3\mid n,\\ f(a) & \text{otherwise}\end{cases}$$

for all integers n, or

(ii) $\alpha=-2$ and

$$f(a^n)=\begin{cases}0 & \text{if } n \text{ is even,}\\ f(a) & \text{otherwise}\end{cases}$$

for all integers n.

Case (i): Assume that property (i) holds. Since G is 6-divisible, there exists $b\in G$ such that $b^3=a$, i.e., $f(b^3)=f(a)$. By the alternatives in $\mathscr{P}f_b^{(-1)}(b)$, we get

$$f(b^2)\in\{4f(b),f(b)\}. \tag{31}$$

We first consider the case when $f(b^2)=4f(b)$. By Lemma 4, we obtain that $f(b^3)=3^2f(b)$ and $f(b^9)=9^2f(b)$. Since $f(b^9)=f(a^3)=0$, we get $f(b)=0$. Hence, $f(a)=f(b^3)=0$, a contradiction. Next, we consider the case when $f(b^2)=f(b)$. By the alternatives in $\mathscr{P}f_b^{(-1)}(b^2)$, we conclude that $f(b)=\frac{1}{3}f(a)$ and so is $f(b^2)$. The alternatives in $\mathscr{P}f_{b^2}^{(-1)}(b^2)$ gives

$$f(b^4)\in\Big\{\frac{4}{3}f(a),-\frac{2}{3}f(a)\Big\}.$$

Thus we get a contradiction by the alternatives in $\mathscr{P}f_b^{(-1)}(b^3)$.

Case (ii): Assume that property (ii) holds. Since G is 6-divisible, there exists $b\in G$ such that $b^2=a$, i.e., $f(b^2)=f(a)$. The alternatives in $\mathscr{P}f_b^{(-2)}(b)$ gives $f(b)=\frac{1}{4}f(a)$. The alternatives in $\mathscr{P}f_b^{(-2)}(b^2)$ gives

$$f(b^3)\in\Big\{\frac{9}{4}f(a),-\frac{11}{4}f(a)\Big\}.$$

By the alternatives in $\mathscr{P}f_b^{(-2)}(b^3)$, we get

$$f(b^4)\in\{4f(a),-6f(a)\}.$$

Since $b^4=a^2$, we have $f(b^4)=f(a^2)=0$. Hence, $f(a)=0$, a contradiction.

Therefore, we must have $f(x^2)=4f(x)$ for all $x\in G$ and by Lemma 6, we conclude that f is quadratic as desired. □

Next, we will give the general solution of the alternative quadratic functional equation (3) on an infinite cyclic group and a finite cyclic group as in two following theorems.

Theorem 3 *Let $(G,\cdot)$ be an infinite cyclic group with $G=\langle g\rangle$. A function $f\in\mathscr{A}^{(\alpha)}_{(G,H)}$ if and only if f is quadratic or one of the following properties holds.*

(i) *$\alpha=0$ and f is constant.*

(ii) $\alpha = -1$ *and*

$$f(g^n) = \begin{cases} 0 & \text{if } 3 \mid n, \\ k & \text{otherwise} \end{cases}$$

for all integers n and for some $k \in H \backslash \{0\}$.

(iii) $\alpha = -2$ *and*

$$f(g^n) = \begin{cases} 0 & \text{if } n \text{ is even}, \\ k & \text{otherwise} \end{cases}$$

for all integers n and for some $k \in H \backslash \{0\}$.

Proof: Assume that $f \in \mathscr{A}^{(\alpha)}_{(G,H)}$ and f is not quadratic. By Theorem 1, we conclude that one of the following properties holds.

(i) $\alpha = 0$ and f is constant.

(ii) $\alpha = -1$ and there exists $g^i \in G$ for some an integer $i \neq 0$ such that $f(g^i) \neq 0$ and

$$f(g^{in}) = \begin{cases} 0 & \text{if } 3 \mid n, \\ f(g^i) & \text{otherwise} \end{cases} \tag{32}$$

for all integers n.

(iii) $\alpha = -2$ and there exists $g^i \in G$ for some an integer $i \neq 0$ such that $f(g^i) \neq 0$ and

$$f(g^{in}) = \begin{cases} 0 & \text{if } n \text{ is even}, \\ f(g^i) & \text{otherwise} \end{cases} \tag{33}$$

for all integers n.

It should be noted that f is even by Lemma 2. We first assume that property (ii) holds. Suppose $f(g^2) \neq f(g)$. The alternatives in $\mathscr{P}f^{(-1)}_g(g)$ gives $f(g^2) = 4f(g)$. Lemma 4 gives $f(g^{3i}) = (3i)^2 f(g)$, while (32) gives $f(g^{3i}) = 0$. Hence, $f(g) = 0$ and then $f(g^2) = 0$, a contradiction to $f(g^2) \neq f(g)$. Thus, we must have $f(g^2) = f(g)$. The alternatives in $\mathscr{P}f^{(-1)}_g(g^2)$ gives

$$f(g^3) \in \{3f(g), 0\}.$$

Suppose $f(g^3) \neq 0$. Hence $f(g^3) = 3f(g)$. The alternatives in $\mathscr{P}f^{(-1)}_g(g^3)$ gives

$$f(g^4) \in \{7f(g), -2f(g)\},$$

while the alternatives in $\mathscr{P}f^{(-1)}_{g^2}(g^2)$ gives

$$f(g^4) \in \{4f(g), f(g)\}.$$

We obtain that $f(g) = 0$ and then $f(g^3) = 0$, a contradiction to $f(g^3) \neq 0$. We must have $f(g^3) = 0$. Observe that the alternatives in $\mathscr{P}f^{(-1)}_{g^3}(g^3)$ gives $f(g^6) = 0$, and the alternatives in $\mathscr{P}f^{(-1)}_g(g^3)$ and $\mathscr{P}f^{(-1)}_{g^2}(g^3)$ gives $f(g^4) = f(g^5) = f(g)$. Then, the alternatives in $\mathscr{P}f^{(-1)}_{g^6}(g^3)$ gives $f(g^9) = 0$, and the alternatives in $\mathscr{P}f^{(-1)}_{g^4}(g^3)$ and $\mathscr{P}f^{(-1)}_{g^5}(g^3)$ gives $f(g^7) = f(g^8) = f(g)$. For each a positive integer n, we use the alternatives in $\mathscr{P}f^{(-1)}_{a^{3n}}(g^3)$ to prove $f(g^{3n+3}) = 0$ and we use the alternatives in $\mathscr{P}f^{(-1)}_{a^{3n-2}}(g^3)$ and $\mathscr{P}f^{(-1)}_{a^{3n-1}}(g^3)$ to prove $f(a^{3n+1}) = f(g^{3n+2}) = f(g)$. By induction and the evenness of f, we get the property (ii) in the theorem.

Next, we assume that property (iii) holds. Suppose $f(g^2) \neq 0$. The alternatives in $\mathscr{P}f^{(-2)}_g(g)$ gives $f(g^2) = 4f(g)$. Lemma 4 gives $f(g^{2i}) = (2i)^2 f(g)$, while (33) gives $f(g^{2i}) = 0$. Hence, $f(g) = 0$ and then $f(g^2) = 0$, a contradiction to $f(g^2) \neq 0$. Thus, we must have $f(g^2) = 0$. Observe that the alternatives in $\mathscr{P}f^{(-2)}_{g^2}(g^2)$ gives $f(g^4) = 0$ and the alternatives in $\mathscr{P}f^{(-2)}_g(g^2)$ gives $f(g^3) = f(g)$. Then, the alternatives in $\mathscr{P}f^{(-2)}_{g^4}(g^2)$ gives $f(g^6) = 0$ and the alternatives in $\mathscr{P}f^{(-2)}_{g^3}(g^2)$ gives $f(g^5) = f(g)$. For each a positive integer n, we use the alternatives in $\mathscr{P}f^{(-2)}_{g^{2n}}(g^2)$ to prove $f(g^{2n+2}) = 0$ and we use alternatives in $\mathscr{P}f^{(-2)}_{g^{2n-1}}(g^2)$ to prove $f(g^{2n+1}) = f(g)$. By induction and the evenness of f, we get the property (iii) in the theorem.

Conversely, we can directly prove that if one of the properties in the theorem holds, then $f \in \mathscr{A}^{(\alpha)}_{(G,H)}$. □

Theorem 4 *Let* $(G,\cdot)$ *be a finite cyclic group of order* $m \geqslant 2$ *with* $G = \langle g \rangle$. *A function* $f \in \mathscr{A}^{(\alpha)}_{(G,H)}$ *if and only if f is quadratic or one of the following properties holds.*

(i) $\alpha = 0$ *and f is constant.*

(ii) $\alpha = -1$, $3 \mid m$ *and*

$$f(g^n) = \begin{cases} 0 & \text{if } 3 \mid n, \\ k & \text{otherwise} \end{cases}$$

for all integers n and for some $k \in H \backslash \{0\}$.

(iii) $\alpha = -2$, *m is even and*

$$f(g^n) = \begin{cases} 0 & \text{if } n \text{ is even}, \\ k & \text{otherwise} \end{cases}$$

for all integers n and for some $k \in H \backslash \{0\}$.

Proof: Assume that $f \in \mathscr{A}^{(\alpha)}_{(G,H)}$ and f is not quadratic. Thus we must have the possibilities in Theorem 3. However, some cases are admissible with some additional conditions. First, we assume that $\alpha = -1$, and

$$f(g^n) = \begin{cases} 0 & \text{if } 3 \mid n, \\ k & \text{otherwise} \end{cases}$$

for all integers n and for some $k \in H \backslash \{0\}$. Suppose $3 \nmid m$. Then, there exists an integer i such that $f(g^i) = 0$. Since $3 \nmid m$, $f(g^{i+m}) = k$. From m is the order of G, we get $g^i = g^{i+m}$, i.e., $f(g^i) = f(g^{i+m})$. Hence, $k = 0$, a contradiction. Therefore, we must have $3 \mid m$. Next, we assume that $\alpha = -2$ and

$$f(g^n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ k & \text{otherwise} \end{cases}$$

for all integers n and for some $k \in H \setminus \{0\}$. Suppose that m is odd. we get

$$0 = f(e) \text{ and } f(g^m) = k.$$

From m is the order of G, we get $g^m = e$, i.e., $f(g^m) = f(e)$. We get $k = 0$, a contradiction. Hence, m must be even.

Conversely, we can directly prove that if one of the properties in the theorem holds, then $f \in \mathscr{A}^{(\alpha)}_{(G,H)}$. □

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