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The properties of meromorphic functions shared values with their difference operators

Zhuo Wang[a,*], **Weichuan Lin**[b], **Shanhua Lin**[c]

[a] School of Mathematics, Renmin University of China, Beijing 100872 China
[b] Fujian Preschool Education College, Fuzhou 350007 China
[c] School of Mathematics and Computer Science, Quanzhou Normal University, Quanzhou, Fujian 362000 China

*Corresponding author, e-mail: zhuowangmaths@163.com

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ABSTRACT: In this paper, we focus on the property of the meromorphic functions when it has shared values with its difference operators, from which we obtain the transcendence of the meromorphic functions, and then we prove that the meromorphic functions has the value distribution property, and finally we derive the expressions of the meromorphic functions.

KEYWORDS: Meromorphic functions, Difference operators, Shared values, Transcendental functions

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INTRODUCTION

We assume that the reader is familiar with the fundamental concepts of Nevanlinna's value distribution theory (see [1–3]). Throughout this paper, a meromorphic function will always mean meromorphic in the whole complex plane.

Let f and g be meromorphic functions and a be a complex number. Let $E(a, f)$ be the set of all zeros of $f(z)-a$ with counting multiplicities (CM). If $E(a, f) \subseteq E(a, g)$, we say $f(z)$ and $g(z)$ partially share a CM, and if $E(a, f) = E(a, g)$, then $f(z)$ and $g(z)$ share a CM.

If the meromorphic function $\alpha(z)(\not\equiv \infty)$ is satisfied, it follows that $T(r, \alpha) = o(T(r, f)), r \to \infty, r \notin E$, where $E \subset [0, \infty)$ is a set of real numbers with finite measures, that is, $T(r, \alpha) = S(r, f)$, then α is called a small function of $f(z)$.

For a meromorphic function $f(z)$, we define its shift by $f(z+c)$ and its difference operators by

$$\Delta_c f(z) := f(z+c) - f(z),$$
$$\Delta_c^n f(z) := \Delta_c^{n-1}(\Delta_c f(z)), \quad n \in \mathbb{N},\ n \geqslant 2.$$

Let f be a non-constant meromorphic in $\mathbb{C}$. Then the order $\rho(f)$ and the lower order $\mu(f)$ of f are defined in turn as follow:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r},$$
$$\mu(f) = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

The study of functional expressions with specific properties has always been a hot issue for mathematicians, and the uniqueness of meromorphic functions and their derivatives with shared values is even more important for complex analysts. In 1986, Jank et al [4] characterized functional expressions from the perspective of shared values and proved that

Theorem 1 ([4]) *Let f be a non-constant meromorphic function, a is a finite non-zero constant. If f, f' and f'' CM shared a, then $f = Ae^z$, where A is a finite non-zero constant.*

In recent years, with the difference analogue of the lemma on the logarithmic derivative, many researchers naturally considered the value distribution issues for meromorphic functions and its difference operators [5–11] and references therein. Then, some complex researchers [12–16] obtained difference analogue of the result of Theorem 1, and characterized the properties of $f(z)$ from the perspective of shared values and proved the following results.

Theorem 2 ([12]) *Let $f(z)$ be a non-constant entire function of finite order, and let $a(z)(\not\equiv 0) \in S(r, f)$ be a periodic entire function with period c. If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share $a(z)$ CM, then $\Delta_c^2 f \equiv \Delta_c f$.*

Theorem 3 ([15]) *Let $f(z)$ be a non-periodic entire function of finite order, and let $a(z)(\not\equiv 0) \in S(r, f)$ be a periodic entire function with period c. If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share $a(z)$ CM, then $\Delta_c f(z) \equiv f(z)$.*

Theorem 4 ([16]) *Let $f(z)$ be a non-constant entire function of finite order, and let $a(z)(\not\equiv 0) \in S(r, f)$ be an entire function with $\rho(a) < 1$. If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share $a(z)$ CM, then $\Delta_c f(z) \equiv f(z)$.*

In Theorem 4 set $a(z) \equiv 1$, then $f(z) = 1 + e^{Az+B}$, where $A(\neq 0)$, B are two constants. Let c satisfy $e^{Ac} = 1$, then $\Delta_c^2 f(z) \equiv \Delta_c f(z) \equiv 0$. It is easy to find $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share 1 CM, but $\Delta_c f(z) \not\equiv f(z)$.

Therefore, combining with the above results, we naturally have the following question:

Can we characterize the expressions of $f(z)$ when the meromorphic function f, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ shared a value?

Firstly, we obtain that $f(z)$ be transcendental meromorphic functions when $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ shared a value.

Theorem 5 *If a function $f(z)$ and $\Delta_c f(z)$ share 1 CM, then f is not a polynomial function. Furthermore, if $f(z)$ and $\Delta_c f(z)$ share 1 and ∞ CM, then $f(z)$ is a transcendental meromorphic function.*

Remark 1 The condition of "CM" sharing is accurate. For example, let $f(z) = z^2 + 1$, then

$$\Delta_c f(z) = (z+c)^2 + 1 - z^2 - 1 = 2zc + c^2,$$

let $c^2 = 1$, that is, $c = \pm 1$, then $f(z)$ and $\Delta_c f(z)$ share 1 IM.

Secondly, by Theorem 5, $f(z)$ is a transcendental meromorphic function. Thus, we have obtained the following value distribution properties.

Theorem 6 *Let $f(z)$ be a non-constant entire function of finite order, and $\Delta_c^2 f(z) \not\equiv \Delta_c f(z)$. If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share 1 and ∞ CM, then the following holds:*

(i) $T(r,f) = N\left(r, \dfrac{1}{f(z)-1}\right) + S(r,f)$;

(ii) $T(r,f) = N\left(r, \dfrac{1}{\Delta_c f(z)-1}\right) + S(r,f)$;

(iii) $T(r,f) = N\left(r, \dfrac{1}{\Delta_c^2 f(z)-1}\right) + S(r,f)$.

Finally, by Theorem 6, we characterize the expressions of $f(z)$ from the perspective of shared values.

Theorem 7 *Let f be a meromorphic function of finite order, and c be a non-zero constant. If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share 1 and ∞ CM, then one of the following situations is true:*

(i) $f(z) = \mathrm{e}^{az} g(z)$, *where $g(z)$ is a meromorphic function with a period of c and $\mathrm{e}^{ac} = 2$, a is a finite non-zero constant;*

(ii) $f(z) = 1 + \mathrm{e}^{Az+B}$, *where $A (\neq 0)$, B are two constants and $\mathrm{e}^{Ac} = 1$.*

Corollary 1 *Let f be an entire function of finite order, c is a non-zero constant. If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share 1 CM, then $f(z) = 1 + \mathrm{e}^{Az+B}$, where $A (\neq 0)$, B are two constants and $\mathrm{e}^{Ac} = 1$.*

Remark 2 The result $f(z) = \mathrm{e}^{az} g(z)$ under the situation of Theorem 7(i) is hold. For example, let $f(z) = \mathrm{e}^{\frac{z}{c}\ln 2} \sin \frac{2\pi}{c} z$, then $\Delta_c f(z) = \mathrm{e}^{\frac{z}{c}\ln 2} \sin \frac{2\pi}{c} z$, $\Delta_c^2 f(z) = \mathrm{e}^{\frac{z}{c}\ln 2} \sin \frac{2\pi}{c} z$. Obviously, $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ CM share 1 and $f(z) = \mathrm{e}^{\frac{\ln 2}{c} z} g(z)$, where $g(z) = \sin \frac{2\pi}{c} z$ is an entire function with a period of c.

LEMMAS

In order to prove our main results, we shall recall some lemmas as follows.

Lemma 1 (Theorem 2.1 in [10]) *Let $f(z)$ be a meromorphic function with order $\rho = \rho(f)$, $\rho < \infty$, and let c be a fixed non-zero complex number, then for each $\varepsilon > 0$, we have*

$$T(r, f(z+c)) = T(r, f(z)) + O(r^{\rho - 1 + \varepsilon}) + O(\log r).$$

Lemma 2 (Lemma 2.3 in [11]) *Let $c \in \mathbb{C}$, $n \in \mathbf{N}$, and let f be a meromorphic function of finite order. Then for any small periodic function $a \in S(r,f)$, we have*

$$m\left(r, \frac{\Delta_c^n f}{f - a}\right) = S(r,f),$$

where the exceptional set associated with $S(r,f)$ is of at most finite logarithmic measure.

Lemma 3 (Lemma 1.10, p. 82 in [3]) *Let $f_1(z)$ and $f_2(z)$ be non-constant meromorphic functions in the complex plane and c_1, c_2, c_3 be non-zero constants. If $c_1 f_1 + c_2 f_2 \equiv c_3$, then*

$$T(r, f_1) < \overline{N}(r, \frac{1}{f_1}) + \overline{N}(r, \frac{1}{f_2}) + \overline{N}(r, f_1) + S(r, f_1).$$

Lemma 4 (Theorem 1.15, p. 31 in [3]) *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions in the complex plane with $\rho(f)$ and $\rho(g)$ as their orders, respectively. If $\rho(f) < \rho(g)$, then*

$$\rho(fg) = \rho(f+g) = \rho(g).$$

Lemma 5 (Theorem 1.14, p. 30 in [3]) *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. If the order of $f(z)$ and $g(z)$ is $\rho(f)$ and $\rho(g)$ respectively, then*

$$\rho(f \cdot g) \leqslant \max\{\rho(f), \rho(g)\},$$
$$\rho(f+g) \leqslant \max\{\rho(f), \rho(g)\}.$$

Lemma 6 (p. 65 in [3]) *Let $h(z)$ be a non-constant entire function and $f(z) = \mathrm{e}^{h(z)}$. Let ρ and μ be the order and the lower order of $f(z)$, respectively. We have the following holds:*

(i) *If $h(z)$ is a polynomial of degree p, then $\rho = \mu = p$;*

(ii) *If $h(z)$ is a transcendental entire function, then $\rho = \mu = \infty$.*

Lemma 7 *Let $\alpha(z)$ be a non-constant polynomial and c be a non-zero constant. Then the following holds:*

(i) $\deg \alpha(z+2c) = \deg \alpha(z+c) = \deg \alpha(z)$;

(ii) $\deg \Delta_c \alpha(z) = \deg \alpha(z) - 1$;

(iii) $\deg(\alpha(z+2c) - \alpha(z)) = \deg \alpha(z) - 1$.

Proof: Set $\alpha(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_0$, where $a_n(\neq 0)$, a_{n-1}, $\cdots$, a_0 are constants. For $\alpha(z+c)$, we have,

$$\begin{aligned}\alpha(z+c)&=a_n(z+c)^n+a_{n-1}(z+c)^{n-1}+\cdots+a_1(z+c)+a_0\\&=a_n(z^n+C_n^1z^{n-1}c+C_n^2z^{n-2}c^2+\cdots+c^n)+a_{n-1}(z^{n-1}\\&\quad+C_{n-1}^1z^{n-2}c+C_{n-1}^2z^{n-3}c^2+\cdots+c^{n-1})+\cdots+a_1(z+c)+a_0\\&=a_nz^n+(C_n^1a_nc+a_{n-1})z^{n-1}+(C_n^2a_nc^2+C_{n-1}^1a_{n-1}c\\&\quad+a_{n-2})z^{n-2}+\cdots+(a_nc^n+a_{n-1}c^{n-1}+\cdots+a_1c+a_0).\end{aligned}$$

Similarly, we get,

$$\begin{aligned}\alpha(z+2c)=a_nz^n&+(C_n^1a_n(2c)+a_{n-1})z^{n-1}+(C_n^2a_n(2c)^2\\&+C_{n-1}^1a_{n-1}(2c)+a_{n-2})z^{n-2}+\cdots+(a_n(2c)^n\\&+a_{n-1}(2c)^{n-1}+\cdots+a_1(2c)+a_0),\end{aligned}$$

hence, from the relationship between expansions of the highest order items of $\alpha(z+2c)$, $\alpha(z+c)$ and $\alpha(z)$, we obtain,

$$\deg\alpha(z+2c)=\deg\alpha(z+c)=\deg\alpha(z).$$

This is (i) of Lemma 7.

For $\Delta_c\alpha(z)$, we have

$$\begin{aligned}\Delta_c\alpha(z)&=\alpha(z+c)-\alpha(z)\\&=nca_nz^{n-1}+(C_n^2a_nc^2+C_{n-1}^1a_{n-1}c)z^{n-2}\\&\quad+\cdots+(a_nc^n+a_{n-1}c^{n-1}+\cdots+a_1c),\end{aligned}$$

then,

$$\deg\Delta_c\alpha(z)=\deg\alpha(z)-1.$$

This is (ii) of Lemma 7.

For $\alpha(z+2c)-\alpha(z)$, we have

$$\begin{aligned}\alpha(z+2c)-\alpha(z)&=2nca_nz^{n-1}+(C_n^2a_n(2c)^2+C_{n-1}^1a_{n-1}(2c))z^{n-2}\\&\quad+\cdots+(a_n(2c)^n+a_{n-1}(2c)^{n-1}+\cdots+a_1(2c)),\end{aligned}$$

therefore,

$$\deg(\alpha(z+2c)-\alpha(z))=\deg\alpha(z)-1.$$

This is (iii) o Lemma 7.

This completes the proof of Lemma 7. □

Lemma 8 *Let f be a non-constant meromorphic function of finite order. If $f(z)$, $\Delta_cf(z)$ and $\Delta_c^2f(z)$ share 1 CM and satisfy the following*

$$E(\infty,\Delta_c^2f)\subseteq E(\infty,f),\quad E(\infty,\Delta_cf)\subseteq E(\infty,f),$$

then, we have $\Delta_c^2f(z)=\Delta_cf(z)+\varphi(z)(f(z)-1)$, where $T(r,\varphi(z))=S(r,f)$.

Proof: Since $f(z)$ be a non-constant meromorphic function of finite order, and

$$\Delta_cf(z)=f(z+c)-f(z),$$

$$\Delta_c^2f(z)=\Delta_c(\Delta_cf(z))=f(z+2c)-2f(z+c)+f(z).$$

Therefore, by Lemma 1, we have

$$\begin{aligned}T(r,\Delta_cf)&=T(r,f(z+c)-f(z))\\&\leqslant 2T(r,f)+O(r^{\rho-1+\epsilon})+O(\log r),\\T(r,\Delta_c^2f)&=T(r,f(z+2c)-2f(z+c)+f(z))\\&\leqslant 3T(r,f)+O(r^{\rho-1+\epsilon})+O(\log r).\end{aligned}$$

Hence,

$$\begin{aligned}\rho\left(\Delta_cf\right)&=\limsup_{r\to\infty}\frac{\log^+T\left(r,\Delta_cf(z)\right)}{\log r}\\&\leqslant\limsup_{r\to\infty}\frac{\log^+T\left(r,f\right)}{\log r}=\rho(f),\end{aligned}$$

and

$$\begin{aligned}\rho\left(\Delta_c^2f\right)&=\limsup_{r\to\infty}\frac{\log^+T\left(r,\Delta_c^2f(z)\right)}{\log r}\\&\leqslant\limsup_{r\to\infty}\frac{\log^+T\left(r,f\right)}{\log r}=\rho(f),\end{aligned}$$

therefore, $\Delta_cf(z)$, $\Delta_c^2f(z)$ are also non-constant meromorphic functions of finite order.

Since $f(z)$, $\Delta_cf(z)$ and $\Delta_c^2f(z)$ share 1 CM and satisfy $E(\infty,\Delta_c^2f)\subseteq E(\infty,f)$, $E(\infty,\Delta_cf)\subseteq E(\infty,f)$; then, we have

$$\begin{aligned}\frac{\Delta_c^2f(z)-1}{f(z)-1}&=R_1(z)\,\mathrm{e}^{\alpha(z)},\\\frac{\Delta_cf(z)-1}{f(z)-1}&=R_2(z)\,\mathrm{e}^{\beta(z)},\end{aligned}\tag{1}$$

where $R_1(z)$, $R_2(z)$ are entire functions, $\alpha(z)$, $\beta(z)$ are non-constant polynomials.

Set

$$\varphi(z)=\frac{\Delta_c^2f(z)-\Delta_cf(z)}{f(z)-1},$$

that is, $\Delta_c^2f(z)=\Delta_cf(z)+\varphi(z)(f(z)-1)$.

By (1), we get that $\varphi(z)=R_1(z)\,\mathrm{e}^{\alpha(z)}-R_2(z)\,\mathrm{e}^{\beta(z)}$. Then, by Lemma 2 and $T(r,\varphi(z))=m(r,\varphi(z))+N(r,\varphi(z))$, we deduce that

$$\begin{aligned}m(r,\varphi(z))&\leqslant m\left(r,\frac{\Delta_c^2f(z)}{f(z)-1}\right)+m\left(r,\frac{\Delta_cf(z)}{f(z)-1}\right)+\log 2\\&=S(r,f).\end{aligned}$$

Moreover, by $\varphi(z)=R_1(z)e^{\alpha(z)}-R_2(z)e^{\beta(z)}$ and $E(\infty,\Delta_c^2f)\subseteq E(\infty,f)$, $E(\infty,\Delta_cf)\subseteq E(\infty,f)$, we have

$$N(r,\varphi(z))=N\left(r,\frac{\Delta_c^2f(z)-\Delta_cf(z)}{f(z)-1}\right)=S(r,f),$$

that is,

$$T(r,\varphi(z))=S(r,f).$$

This completes the proof of Lemma 8. □

PROOF OF THEOREMS

First we give the proof of Theorem 5.

Proof of Theorem 5

Suppose that $f(z)$ be a non-constant rational function; then

$$f(z)=\frac{P(z)}{Q(z)},$$

where $P(z)$, $Q(z)$ are non-zero relatively prime polynomials. Since $f(z)$ and $\Delta_c f(z)$ share 1 and ∞ CM; then

$$\frac{\Delta_c f(z)-1}{f(z)-1}=K, \tag{2}$$

where K be a non-zero constant. Let P_c, Q_c represents

$$P_c:=P(z+c), \quad Q_c:=Q(z+c),$$

hence, P_c, Q_c are also relatively prime polynomials. Therefore, we have

$$\begin{aligned}\Delta_c f(z)&=f(z+c)-f(z)\\&=\frac{P(z+c)}{Q(z+c)}-\frac{P(z)}{Q(z)}\\&=\frac{P_cQ-Q_cP}{Q_cQ}=\frac{P_c\frac{Q}{Q_c}-P}{Q}.\end{aligned}$$

Substituting the above formula into (2), we have

$$\frac{P_c\frac{Q}{Q_c}-P-Q}{Q}=K\frac{P-Q}{Q},$$

that is,

$$P_c\frac{Q}{Q_c}=(K+1)P+(1-K)Q, \tag{3}$$

noting that the right of (3) is a polynomial. Thus, we discuss the left side of this equation, the zeros of Q_c are all zeros of Q, since Q_c and Q have a same coefficient of the highest order; then, we get $Q_c=Q$, where $Q(z)$ is a non-zero constant.

By Lemma 7 we have $f(z)$ is a polynomial, then $\deg\Delta_c f=\deg f-1$. Since $f(z)$ and $\Delta_c f(z)$ share 1 CM, by (2), $f(z)-1$ and $\Delta_c f-1$ have the same zeros and the same number of degree, that is, $f(z)$ is a constant, so $f(z)$ is not a rational function.

This completes the proof of Theorem 5. □

Proof of Theorem 6

Since transcendental meromorphic functions $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share 1 and ∞ CM; then

$$\frac{\Delta_c^2 f(z)-1}{f(z)-1}=e^{\alpha(z)}, \qquad \frac{\Delta_c f(z)-1}{f(z)-1}=e^{\beta(z)}, \tag{4}$$

where $\alpha(z)$, $\beta(z)$ are non-zero polynomials.

By $\Delta_c^2 f(z)\not\equiv\Delta_c f(z)$ and (4), we arrive at $\varphi(z)=e^{\alpha(z)}-e^{\beta(z)}\neq 0$; then, $\frac{e^{\alpha(z)}}{\varphi(z)}-\frac{e^{\beta(z)}}{\varphi(z)}=1$. By Lemma 3, we set $f_1=\frac{e^{\alpha}}{\varphi}$, $f_2=\frac{e^{\beta}}{\varphi}$; then,

$$\begin{aligned}T(r,\frac{e^{\alpha}}{\varphi})&\leqslant\overline{N}(r,\frac{\varphi}{e^{\alpha}})+\overline{N}(r,\frac{\varphi}{e^{\beta}})+\overline{N}(r,\frac{e^{\alpha}}{\varphi})+S(r,\frac{e^{\alpha}}{\varphi})\\&=S(r,f)+S(r,\frac{e^{\alpha}}{\varphi}).\end{aligned}$$

By Lemma 8, we have

$$\begin{aligned}T(r,e^{\alpha})&\leqslant T(r,\frac{e^{\alpha}}{\varphi}\varphi)\\&\leqslant T(r,\frac{e^{\alpha}}{\varphi})+T(r,\varphi)=S(r,f).\end{aligned} \tag{5}$$

Hence, $T(r,e^{\alpha})=S(r,f)$. Similarly, we can obtain that $T(r,e^{\beta})=S(r,f)$.

From (4), we get that $\frac{1}{f(z)-1}=\frac{\Delta_c^2 f(z)}{f(z)-1}-e^{\alpha(z)}$; thus, by Lemma 2 and (5), we have

$$\begin{aligned}m\big(r,\frac{1}{f(z)-1}\big)&=m\big(r,\frac{\Delta_c^2 f(z)}{f(z)-1}-e^{\alpha(z)}\big)\\&\leqslant m\big(r,\frac{\Delta_c^2 f(z)}{f(z)-1}\big)+m(r,e^{\alpha(z)})+S(r,f)\\&=S(r,f).\end{aligned} \tag{6}$$

Combining the first fundamental Nevanlinna theorem with (6), we get that

$$\begin{aligned}T(r,f)&=T\big(r,\frac{1}{f(z)-1}\big)+O(1)\\&=N\big(r,\frac{1}{f(z)-1}\big)+m\big(r,\frac{1}{f(z)-1}\big)+O(1)\\&=N\big(r,\frac{1}{f(z)-1}\big)+S(r,f),\end{aligned}$$

this is (i) of Theorem 6.

Since $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share 1 and ∞ CM, thus (ii) and (iii) of Theorem 6 are also established. □

Proof of Theorem 7

According to $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share 1 and ∞ CM; thus, we have

Assertion 1: $\Delta_c^2 f(z)\equiv\Delta_c f(z)$.

The following is a description of expressions of the meromorphic function f. Suppose that $\Delta_c^2 f(z)\not\equiv\Delta_c f(z)$, since $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share 1 and ∞ CM; then, by Lemma 8, (4) can be rewritten as $\Delta_c f(z)=e^{\beta(z)}(f(z)-1)+1$, it follows that

$$f(z+c)=(1+e^{\beta(z)})f(z)-e^{\beta(z)}+1, \tag{7}$$

that is,

$$\begin{aligned}\Delta_c^2 f(z) &= \Delta_c(e^{\beta(z)}(f(z)-1)+1)\\ &= e^{\beta(z+c)}(f(z+c)-1)+1-e^{\beta(z)}(f(z)-1)-1\\ &= e^{\beta(z+c)}(f(z+c)-1)-e^{\beta(z)}(f(z)-1), \quad (8)\end{aligned}$$

substituting (7) into (8), we have

$$\begin{aligned}\Delta_c^2 f(z) &= e^{\beta(z+c)}(f(z)+e^{\beta(z)}f(z)-e^{\beta(z)})\\ &\quad -e^{\beta(z)}f(z)+e^{\beta(z)}\\ &= (e^{\beta(z+c)+\beta(z)}+e^{\beta(z+c)}-e^{\beta(z)})f(z)\\ &\quad -e^{\beta(z+c)+\beta(z)}+e^{\beta(z)}. \quad (9)\end{aligned}$$

Thus, (9) can be rewritten as

$$\Delta_c^2 f(z) = \gamma(z)f(z)+\delta(z), \quad (10)$$

where

$$\gamma(z) = e^{\beta(z+c)+\beta(z)}+e^{\beta(z+c)}-e^{\beta(z)}, \quad (11)$$

$$\delta(z) = -e^{\beta(z+c)+\beta(z)}+e^{\beta(z)} = -\gamma(z)+e^{\beta(z+c)}, \quad (12)$$

then, $T(r,e^{\beta}) = S(r,f)$; thus, we get that $T(r,\gamma(z)) = S(r,f)$ and $T(r,\delta(z)) = S(r,f)$. Then, we rewrite (10) as

$$\Delta_c^2 f(z)-1-\gamma(z)(f(z)-1) = \gamma(z)+\delta(z)-1. \quad (13)$$

Suppose that $\gamma(z)+\delta(z)-1 \not\equiv 0$. Let z_0 be a k order zero of $f(z)-1$; since $f(z)$, $\Delta_c^2 f(z)$ share 1 CM, so z_0 is also k order zero of $\Delta_c^2 f(z)-1$. Thus, z_0 is at least k order zero of $\Delta_c^2 f(z)-1-\gamma(z)(f(z)-1)$. Then, combining Theorem 6 with (13), we have that

$$\begin{aligned}N\Big(r,\frac{1}{\gamma(z)+\delta(z)-1}\Big) &= N\Big(r,\frac{1}{\Delta_c^2 f(z)-1-\gamma(z)(f(z)-1)}\Big)\\ &\geqslant N\Big(r,\frac{1}{f(z)-1}\Big) = T(r,f)+S(r,f). \quad (14)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}N\Big(r,\frac{1}{\gamma(z)+\delta(z)-1}\Big) &\leqslant T\Big(r,\frac{1}{\gamma(z)+\delta(z)-1}\Big)\\ &= S(r,f), \quad (15)\end{aligned}$$

then, incorporating (14) with (15), we have that $T(r,f) = S(r,f)$, which is a contradiction.

Hence, $\gamma(z)+\delta(z)-1 \equiv 0$. Combining with (12), we have

$$e^{\beta(z+c)} \equiv 1,$$

then, $\beta(z+c) = 2k\pi i$, $k \in \mathbb{Z}$. Therefore, $e^{\beta(z)} \equiv e^{\beta(z+c)} \equiv 1$.

From (13), we get that $\frac{\Delta_c^2 f-1}{f-1} = \gamma(z)$, combining this with (4), we have that $e^{\alpha} \equiv 1$, which is $e^{\alpha} \equiv e^{\beta}$; thus, $\Delta_c^2 f \equiv \Delta_c f$, which is contradiction with assumptions. This completes the proof of Assertion 1.

Set,

$$\psi(z) = \Delta_c f(z)-f(z), \quad (16)$$

then,

$$\begin{aligned}\psi(z) &= \Delta_c f(z)-f(z) = f(z+c)-2f(z),\\ \psi(z+c) &= \Delta_c f(z+c)-f(z+c)\\ &= f(z+2c)-2f(z+c),\end{aligned}$$

therefore,

$$\begin{aligned}\psi(z)+f(z) &= \Delta_c f(z)-f(z)\\ &= f(z+c)-f(z) = \Delta_c f(z),\\ \psi(z+c)+f(z) &= \Delta_c f(z+c)-f(z+c)\\ &= f(z+2c)-2f(z+c)+f(z)\\ &= \Delta_c^2 f(z),\end{aligned}$$

hence, $\Delta_c^2 f(z) \equiv \Delta_c f(z)$, that is, $\psi(z+c) = \psi(z)$, we get that $\psi(z)$ is a meromorphic periodic function of period c. From (16), we have

$$\Delta_c f(z)-1 = f(z)-1+\psi(z),$$

since $f(z)$ be non-constant meromorphic functions; then,

$$\frac{\Delta_c f(z)-1}{f(z)-1} = 1+\frac{\psi(z)}{f(z)-1}. \quad (17)$$

By Theorem 6, there be transcendental functions $f(z)$, and $f(z)$, $\Delta_c f(z)$ share 1 and ∞ CM; then, there exists polynomials $\beta(z)$ satisfying

$$\frac{\Delta_c f(z)-1}{f(z)-1} = e^{\beta(z)}.$$

Combining this with (17), we have

$$1+\frac{\psi(z)}{f(z)-1} = e^{\beta(z)}.$$

We have the following two cases:

Case 1: For $e^{\beta(z)} \equiv 1$; then, $\Delta_c f(z) \equiv f(z)$, that is, $f(z+c) = 2f(z)$. Firstly, we have the following hold:

Assertion 2: The necessary and sufficient condition for $f(z+c) = 2f(z)$ is $f(z) = e^{az}g(z)$, where $e^{ac} = 2$ and $g(z)$ is meromorphic functions with a period of c, and a is a finite non-zero complex number.

Adequacy:

$$f(z+c) = e^{az}e^{ac}g(z) = 2f(z),$$

where $e^{ac} = 2$.

Necessity: Set $a = \ln 2/c$; then,

$$g(z) = \frac{f(z)}{e^{az}},$$

where $g(z)$ is a meromorphic function.

Secondly, let us prove $g(z)$ is a meromorphic function with a period of c,

$$g(z+c)=\frac{f(z+c)}{e^{a(z+c)}}=\frac{2f(z)}{2e^{az}}=g(z).$$

Thus the form (i) of Theorem 7 is proved.

Case 2: For $e^{\beta(z)}\not\equiv 1$; then, we deduce that

$$f(z)=\frac{\psi(z)}{e^{\beta(z)}-1}+1. \tag{18}$$

Since $\psi(z)$ is a meromorphic function with a period c; then, we get that

$$\begin{aligned}\Delta_c f(z)&=f(z+c)-f(z)\\&=\frac{\psi(z+c)}{e^{\beta(z+c)}-1}+1-\frac{\psi(z)}{e^{\beta(z)}-1}-1\\&=\psi(z)\Delta_c\Big(\frac{1}{e^{\beta(z)}-1}\Big),\end{aligned} \tag{19}$$

and

$$\begin{aligned}\Delta_c^2 f(z)&=f(z+2c)-2f(z+c)+f(z)\\&=\frac{\psi(z+2c)}{e^{\beta(z+2c)}-1}+1-2\frac{\psi(z+c)}{e^{\beta(z+c)}-1}-2+\frac{\psi(z)}{e^{\beta(z)}-1}+1\\&=\frac{\psi(z+2c)}{e^{\beta(z+2c)}-1}-2\frac{\psi(z+c)}{e^{\beta(z+c)}-1}+\frac{\psi(z)}{e^{\beta(z)}-1}\\&=\psi(z)\Delta_c^2\Big(\frac{1}{e^{\beta(z)}-1}\Big).\end{aligned}$$

From Assertion 1, for $\Delta_c^2 f(z)\equiv\Delta_c f(z)$, we have

$$\Delta_c^2\Big(\frac{1}{e^{\beta(z)}-1}\Big)=\Delta_c\Big(\frac{1}{e^{\beta(z)}-1}\Big),$$

that is,

$$\frac{1}{e^{\beta(z+2c)}-1}-3\frac{1}{e^{\beta(z+c)}-1}+2\frac{1}{e^{\beta(z)}-1}=0, \tag{20}$$

hence,

$$\begin{aligned}&e^{\beta(z+c)+\beta(z)}-3e^{\beta(z+2c)+\beta(z)}+2e^{\beta(z+2c)+\beta(z+c)}\\&\qquad-3e^{\beta(z+c)}+2e^{\beta(z)}+e^{\beta(z+2c)}=0,\end{aligned}$$

then,

$$\begin{aligned}&e^{\beta(z+c)}-3e^{\beta(z+2c)}+2e^{\beta(z+2c)+\Delta_c\beta(z)}\\&\qquad-3e^{\Delta_c\beta(z)}+2+e^{\beta(z+2c)-\beta(z)}=0,\end{aligned}$$

therefore, we get that

$$\begin{aligned}&e^{\beta(z+c)}+\big(2e^{\Delta_c\beta(z)}-3\big)e^{\beta(z+2c)}\\&\qquad=3e^{\Delta_c\beta(z)}-2-e^{\beta(z+2c)-\beta(z)}.\end{aligned} \tag{21}$$

The following is the discussions of $\beta(z)$:

Subcase 2.1: For $e^{\beta(z)}\not\equiv e^{\beta(z+c)}$; then, for (21), we have

$$\begin{aligned}&\rho\Big(e^{\beta(z+c)}+\big(2e^{\Delta_c\beta(z)}-3\big)e^{\beta(z+2c)}\Big)\\&\qquad=\rho\Big(3e^{\Delta_c\beta(z)}-2-e^{\beta(z+2c)-\beta(z)}\Big),\end{aligned} \tag{22}$$

by Lemma 6, we get that $\rho(e^{\beta(z)})=\deg\beta(z)$. Moreover, combining Lemma 5 with Lemma 7, we get that

$$\begin{aligned}&\rho\Big(3e^{\Delta_c\beta(z)}-2-e^{\beta(z+2c)-\beta(z)}\Big)\\&\qquad\leqslant\max\{\rho(3e^{\Delta_c\beta(z)}-2),\rho(-e^{\beta(z+2c)-\beta(z)})\}\\&\qquad=\deg\beta(z)-1.\end{aligned} \tag{23}$$

Similarly,

$$\begin{aligned}&\rho\Big(e^{\beta(z+c)}+(2e^{\Delta_c\beta(z)}-3\Big)e^{\beta(z+2c)})\\&\qquad\leqslant\max\{\rho(e^{\beta(z+c)}),\rho((2e^{\Delta_c\beta(z)}-3)e^{\beta(z+2c)})\}\\&\qquad=\deg\beta(z+c)=\deg\beta(z).\end{aligned} \tag{24}$$

If $\rho\Big(e^{\beta(z+c)}+(2e^{\Delta_c\beta(z)}-3)e^{\beta(z+2c)}\Big)<\deg\beta(z)$; then, by Lemma 4,

$$\begin{aligned}\deg\beta(z)&=\rho\Big(\frac{e^{\beta(z+c)}+(2e^{\Delta_c\beta(z)}-3)e^{\beta(z+2c)}}{e^{\beta(z+c)}}\Big)\\&=\rho\Big(1+\big(2e^{\Delta_c\beta(z)}-3\big)e^{\beta(z+2c)-\beta(z+c)}\Big)\\&\leqslant\deg\beta(z)-1,\end{aligned} \tag{25}$$

this is impossible. Thus,

$$\rho\Big(e^{\beta(z+c)}+\big(2e^{\Delta_c\beta(z)}-3\big)e^{\beta(z+2c)}\Big)=\deg\beta(z).$$

Combining (22) with (23), we arrive at $\deg\beta(z)\leqslant\deg\beta(z)-1$, which is a contradiction.

Subcase 2.2: For $e^{\beta(z)}\equiv e^{\beta(z+c)}$; then, by (19), we get that $\Delta_c f(z)\equiv 0$. Next, we assert that $f(z)=1+e^{Az+B}$, where $A(\neq 0),B$ are two constants and $e^{Ac}=1$.

Since $\frac{\Delta_c f(z)-1}{f(z)-1}=e^{\beta(z)}$; then,

$$f(z)=1-e^{-\beta(z)},$$

where $\beta(z)$ is a non-zero polynomial.

The following is a discussion of the degree of polynomials $\beta(z)$. For $\deg\beta(z)\geqslant 2$; then, from Lemma 7, $\deg(\beta(z+c)-\beta(z))=\deg(\beta(z))-1\geqslant 1$. Also by $\Delta_c f(z)\equiv 0$, there is $e^{\beta(z+c)-\beta(z)}\equiv 1$; hence,

$$(\beta'(z+c)-\beta'(z))e^{\beta(z+c)-\beta(z)}\equiv 0,$$

thus, we get that $\beta'(z+c)-\beta'(z)\equiv 0$, which contradicts with the above $A(\neq 0)$.

So, $\deg(\beta(z))=1$; thus, $f(z)=1+e^{Az+B}$, where $A(\neq 0)$, B are two constants. Noticing that $\Delta_c f(z)\equiv 0$, we can deduce that $e^{Ac}=1$.

This completes the proof of form (ii) of Theorem 7. Thus, Theorem 7 is proved. □

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REFERENCES

1. Hayman WK (1964) *Meromorphic Functions*, Clarendon Press, Oxford.
2. Nevanlinna R (1929) *Le Théorème de Picard-Borel et la Théorie des Fonctions Méromorphes*, Paris.
3. Yang CC, Yi HX (2003) *Uniqueness Theory of Meromorphic Function*, Kluwer Academic Publishers, Dordrecht.
4. Jank G, Mues E, Volkmann L (1986) Meromorphe Funktionen, die mit ihrer und zweiten ableitung einen endlichen wert teilen. *Complex Var Theory Appl* **6**, 51–71.
5. Ahamed MB (2019) An investigation on the conjecture of Chen and Yi. *Results Math* **74**, 1–28.
6. Ahamed MB (2021) On the Fermat-type difference equation $f^3(z)+[c_1f(z+c)+c_0f(z)]^3=\mathrm{e}^{\alpha z+\beta}$. *J Contemp Math Anal* **56**, 255–269.
7. Ahamed MB (2023) The class of meromorphic functions sharing values with their difference polynomials. *Indian J Pure Appl Math* **54**, 1158–1169.
8. Banerjee A, Ahamed MB (2019) Uniqueness of meromorphic function with its shift operator under the purview of two or three shared sets. *Math Slovaca* **69**, 557–572.
9. Banerjee A, Ahamed MB (2020) Results on meromorphic function sharing two sets with its linear-difference operator. *J Contemp Math Anal* **55**, 143–155.
10. Chiang YM, Feng SJ (2006) On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane. *Ramanujan J* **16**, 105–129.
11. Halburd RG, Korhonen RJ (2006) Nevanlinna theory for the difference operator. *Ann Acad Sci Fenn Math* **31**, 463–487.
12. Chen BQ, Chen ZX, Li S (2012) Uniqueness theorems on entire functions and their difference operators or shifts. *Abstr Appl Anal* **2012**, 906893.
13. Deng BM, Liu D, Fang ML (2018) Unicity of meromorphic functions and their difference operators. *Math Ann Ser A* **39**, 341–348.
14. Deng BM, Liu D, Gu YY, Fang ML (2018) Meromorphic functions that share a polynomial with their difference operators. *Adv Differ Equ* **2018**, 194.
15. Farissi AE, Latreuch Z, Asiri A (2016) On the uniqueness theory of entire functions and their difference operators. *Complex Anal Oper Theory* **10**, 1317–1327.
16. Lü F, Wang YF, Xu JF (2017) Entire functions sharing a small function with their two difference operators. *Adv Differ Equ* **2017**, 216.