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Numerical simulation of a nonlinear model in finance by Broyden's method

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ABSTRACT: In this paper, we study the stationary Black-Scholes model arising in finance with transaction costs. This model becomes interesting when the time does not play a role such as, for instance, in perpetual options. The equation describing this model is a nonlinear second-order boundary value problem and there is no analytic solutions in closed form for such a nonlinear equation. After discretization via the centered finite difference formula we have to solve a nonlinear algebraic system which would be a serious problem when we use a small discretization mesh. We solve this nonlinear system by the residual-based Broyden's method, which is an efficient quasi-Newton method and is convenient to implement by a desk computer. We give a convergence analysis of the Broyden's method by assuming a lower and upper bound of the converged solution of the Black-Scholes model. Numerical results are given to show that the convergence rate of the method is robust with respect to the discretization mesh and the problem parameters.

KEYWORDS: Black-Scholes model, nonlinearity, centered finite difference, Broyden's method, convergence analysis

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INTRODUCTION

The Black-Scholes (BS) model, which was proposed in 1973 by Black and Scholes [1] and Merton [2], provides an approximate description of the behavior of the underlying assets. This model becames fundamental for the valuation of financial derivatives in a complete frictionless market [3]. The classical Black-Scholes model assumes that the hedging portfolio is continuously adjusted by transacting the underlying asset of the derivative, in order to replicate exactly the returns of a certain derivative. This can only happen if no transaction costs exist when buying or selling the assets. Otherwise, a continuous adjustment would imply that those costs, such as taxes or fees, would become infinitely large [4–6]. Hence, considering transaction costs in the model is an important issue which has motivated the work of several authors and has led to the study of new Black-Scholes model [7–10].

The BS model can be studied by both analytically and numerically. For example, Black and Scholes [1] first found the solution based on previous research on option pricing that gave an idea of what the solution would look like. Company, Gonzalez and Jodar [11] solved the BS model which was modified with discrete dividend. They utilized a delta-characterizing grouping of generalized Dirac-Delta function and connected the Mellin transformation established in [12, 13] to acquire an integral formula. However, in these studies the closed form for the analytic solutions is only available for BS equations with constant coefficients. For BS model with time-dependent coefficients [14–17] or nonlinear BS model there is no closed form for the analytic solution and we have to rely on numerical computations [18–21].

In this paper, we are interested in the study of a stationary BS model [9, 22], which is a nonlinear second-order differential equation that models the valuation of a call option in presence of transaction costs. These stationary solutions give the option value V as a function of the stock price, which can be interesting when dealing with a model where the time does not play a relevant role such as, for instance, in perpetual options. There is no closed formula for the analytic solution of such a nonlinear differential equation and we have to relay on numerical method for a quantitative study.

We discretize the stationary BS model by the centered finite difference method and after discretization we have to solve a nonlinear algebraic system which would be a serious problem when the discretization mesh size is small. We solve this nonlinear system by the residual-based Broyden's method [23–26]. By assuming the lower and upper bounds of the converged solution of the BS model, we present a convergence analysis of Broyden's method. Numerical results show that the proposed method is robust in terms of discrete grids and problem parameters.

NONLINEAR BS MODEL AND DISCRETIZATION

In the stationary case, the BS model taking into account the presence of transaction costs is a nonlinear second-order Dirichlet boundary problem [9, 22]:

$$\begin{cases} x^3V_{xx}^2+px^2V_{xx}+qxV_x-qV=0, \quad x\in[a,b],\\ V(a)=V_a,\ V(b)=V_b, \end{cases} \tag{1}$$

where $b>a>0$ and $p,q>0$ are constants. For the boundary conditions we consider $V_a \leqslant V_b$ which is a quite natural assumption in some financial settings, for instance, if we are dealing with call options. Problem (1) is related to financial option pricing model, namely the BS model introduced in 1973 [1]:

$$V_t+\tfrac{\sigma^2S^2}{2}V_{SS}+r(SV_S-V)=0, \tag{2}$$

where V represents the value of a call or put option, depending on the underlying asset S and time t, r is the interest short rate and σ is the volatility of the asset price. In the above model, S denotes a geometric Brownian motion, and no costs are considered when financial transactions hold.

The details for obtaining (1) from the classical BS model (2) are as follows. Suppose the transaction costs are included in the model under the assumption that they are a percentage of the transaction, given by a linear function h of the number of shares traded, that is $h(\xi)=a-b\xi$, with ξ being the number of shares traded and $a,b>0$. Under these assumptions, the following nonlinear BS equation is obtained

$$V_t+\tfrac{\sigma^2S^2}{2}V_{SS}-a\sigma S^2\sqrt{\tfrac{2}{\pi\Delta T}}|V_{SS}| + bS^3\sigma^2V_{SS}^2+r(SV_S-V)=0,$$

where ΔT is the interval between transactions. Then, if a is small enough and $V_{SS}>0$ from [5, 9] we obtain the following nonlinear version of the classical BS model (2), which is

$$V_t+\tfrac{\tilde\sigma^2S^2}{2}V_{SS}+bS^3\sigma^2V_{SS}^2+r(SV_S-V)=0, \tag{3}$$

where $\tilde\sigma=\sigma^2(1-2\frac{a}{\sigma}\sqrt{\frac{2}{\pi\Delta T}})>0$ is the so-called adjusted volatility. Now, if we consider the stationary version of (3), i.e., $V_t=0$, we obtain the above ordinary differential equation (1) with

$$p=\frac{\tilde\sigma^2}{2b\sigma^2}, \quad q=\frac{r}{b\sigma^2}.$$

The stationary solution gives the option value V as a function of the stock price x, which would be interesting in the situation that the time does not play a relevant role such as, for instance, in perpetual options.

For problem (1) with $\frac{V_b}{b}\leqslant\frac{V_a}{a}$, the following theorem states an existence and uniqueness result and also provides lower and upper bounds of the solution. The lower and upper bounds provide useful criterion for checking the numerical solutions. (For the case $\frac{V_b}{b}>\frac{V_a}{a}$, according to our best knowledge there is no lower and upper bounds in closed form for the solution $V(x)$.)

Theorem 1 ([9]) *For nonlinear Dirichlet boundary value problem (1) with $x\in[a,b]$, it holds*

- *The function $V(x)=\frac{V_a}{a}x$ is a solution of the problem (1) if and only if $\frac{V_a}{a}=\frac{V_b}{b}$.*
- *If $\frac{V_b}{b}<\frac{V_a}{a}$ the problem (1) has a convex solution V satisfying*

$$\frac{V_b}{b}x\leqslant V(x)\leqslant\frac{V_b-V_a}{b-a}x+\frac{bV_a-aV_b}{b-a}, \tag{4}$$

- *If $\frac{V_b}{b}<\frac{V_a}{a}$ and $k=\sqrt{\frac{q}{a^3}}$ is small, the problem (1) has a convex solution V satisfying*

$$\frac{V_b-V_a}{b^2-a^2}x^2+\frac{b^2V_a-a^2V_b}{b^2-a^2}\leqslant V(x) \leqslant\frac{V_b-V_a}{b-a}x+\frac{bV_a-aV_b}{b-a}. \tag{5}$$

Since the analytic solution of (1) is not available, we next introduce numerical method for solving (1). To this end, we first partition the computation domain $[a,b]$ by mesh size h and denote an arbitrary discrete point in $[a,b]$ by $x_j=a+jh$, where $j=0,1,\dots,N$ with $N=\frac{b-a}{h}$. On the discrete points x_j's we denote the numerical solution by $V_j\approx V(x_j)$. The numerical method studied here lies in approximating V_{xx} and V_x by the centered finite difference formula

$$\begin{aligned} V_{xx}(x_j)&=\frac{V(x_{j+1})-2V(x_j)+V(x_{j-1})}{h^2}+\tau_j,\\ V_x(x_j)&=\frac{V(x_{j+1})-V(x_{j-1})}{2h}+\tilde\tau_j, \end{aligned} \tag{6}$$

where $j=1,2,\dots,N-1$ and $V(x_0)=V_a$ and $V(x_N)=V_b$. The quantities τ_j and $\tilde\tau_j$ denote the truncation errors $\tau_j=O(h^2)$ and $\tilde\tau_j=O(h^2)$. Substituting (6) into (1) leads to the following finite difference method

$$x_j^3\left(\frac{V_{j+1}-2V_j+V_{j-1}}{h^2}\right)^2+px_j^2\frac{V_{j+1}-2V_j+V_{j-1}}{h^2} +qx_j\frac{V_{j+1}-V_{j-1}}{2h}-qV_j=0,$$

where $j=1,2,\dots,N-1$ and

$$V_j-V(x_j)=O(h^2). \tag{7}$$

We rewrite this difference equation as follows

$$x_j^3\left(V_{j+1}-2V_j+V_{j-1}\right)^2+ph^2x_j^2(V_{j+1}-2V_j+V_{j-1}) +qh^3x_j\tfrac{V_{j+1}-V_{j-1}}{2}-qh^4V_j=0. \tag{8}$$

Define

$$\begin{aligned}
\mathbf{V} &= (V_1, V_2, \dots, V_{N-1})^\top, \\
\mathbf{c} &= (V_a, 0, \dots, 0, V_b)^\top, \\
\tilde{\mathbf{c}} &= \frac{1}{2}(-V_a, 0, \dots, 0, V_b)^\top, \\
A &= \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}, \\
\tilde{A} &= \frac{1}{2}\begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}, \\
D_x &= \mathrm{diag}(x_1, \dots, x_{N-1}).
\end{aligned} \tag{9}$$

Then, we can represent (8) as

$$F(\mathbf{V}) = 0, \tag{10a}$$

where the nonlinear function F is

$$F(\mathbf{V}) := D_x^3(A\mathbf{V}+\mathbf{c})^2 + ph^2 D_x^2(A\mathbf{V}+\mathbf{c}) + qh^3 D_x(\tilde{A}\mathbf{V}+\tilde{\mathbf{c}}) - qh^4\mathbf{V}. \tag{10b}$$

The discrete equation (10a) is a nonlinear system with unknown solution $\mathbf{V}$. The computation of $\mathbf{V}$ would be a serious problem when h is small (using a small h is necessary if we want to get a more accurate solution, cf. (7).

In the next section we introduce the Broyden's method for such a nonlinear problem. We mention that many other methods can be used to solve (10a)–(10b), such as the fixed-point iteration [27], the classical Newton's method [28] and the splitting-iteration method [29], to name a few. The Broyden's method has the following advantages: compared to the fixed-point iteration and the waveform relaxation method it converges much faster (the convergence rate is very close to that of the classical Newton's method); compared to the classical Newton's method, we do not need to compute the Jacobian matrix which would be a serious computational burden if the mesh size h is small.

BROYDEN'S METHOD

In this section, we address how to solve $\mathbf{V}$ in (10a). In particular we use the residual-based Broyden's method [23] for (10a). The standard Newton's method for (10a) is

$$J_k \Delta\mathbf{V}^k = -F(\mathbf{V}_k), \quad \mathbf{V}_{k+1} = \mathbf{V}_k + \Delta\mathbf{V}_k, \tag{11}$$

where $k \geqslant 0$ is the iteration index, $\mathbf{V}_0$ is an initial guess and J_k is the Jacobian matrix of F

$$J_k := F'(\mathbf{V}_k) = 2D_x^3\mathrm{diag}(A\mathbf{V}_k+\mathbf{c})A + ph^2D_x^2A + qh^3D_x\tilde{A} - qh^4I. \tag{12}$$

In each iteration we have to solve a linear system which, in essence, is to compute the action of the inverse matrix J_k^{-1} on the residual $F(\mathbf{V}_k)$. This would be time consuming if N is large (i.e., h is small).

Broyden's method lies in replacing the Jacobian matrices $\{J_k\}$ by a series of matrices $\{B_k\}$, for which the inverse is defined recursively as

$$B_{k+1}^{-1} = B_k^{-1}\left(I - \frac{F_{k+1}(F_{k+1}-F_k)^\top}{\|F_{k+1}-F_k\|_2^2}\right), \tag{13}$$

with B_0 being an approximation of J_0^{-1} (i.e., $B_0 \approx J_0^{-1}$) and $F_k = F(\mathbf{V}_k)$. The resulted iterations for (10a) is

$$B_k\Delta\mathbf{V}^k = -F(\mathbf{V}_k), \ \mathbf{V}_{k+1} = \mathbf{V}_k + \Delta\mathbf{V}_k. \tag{14}$$

We now state the convergence property of the Broyden's method (14).

Theorem 2 *Let* $\mathbf{V}$ *be the unique solution of* F *in* (10a) *with* $F'(\mathbf{V})$ *being nonsingular. Assume that there exists some constant* $\omega \in (0, \infty)$ *such that the following Lipschitz condition holds*

$$\|F'(\tilde{\mathbf{V}}) - F'(\mathbf{V})\| \leqslant \omega\|\tilde{\mathbf{V}} - \mathbf{V}\|, \quad \forall\, \tilde{\mathbf{V}} \in D_0, \tag{15}$$

where $D_0 \subset \mathbb{R}^{N-1}$. *Then, there exists* $\epsilon > 0$ *and* $\delta > 0$ *such that if*

$$\|B_k - F'(\mathbf{V})\| \leqslant \delta, \quad \|\mathbf{V}_0 - \mathbf{V}\| \leqslant \epsilon,$$

Broyden's method starting from $\mathbf{V}_0$ *converges to* $\mathbf{V}$.

Proof: Let β be the bound of $\|F'(\mathbf{V})^{-1}\|$, i.e., $\|F'(\mathbf{V})^{-1}\| \leqslant \beta$. Choosing ϵ such that $\mathcal{N}(\mathbf{V}_0, \epsilon) \subset D_0$, where $\mathcal{N}(\mathbf{V}_0, \epsilon) = \{\tilde{\mathbf{V}} \in \mathbb{R}^{N-1} : \|\tilde{\mathbf{V}} - \mathbf{V}_0\| \leqslant \epsilon\}$. Moreover, we assume that δ and ϵ satisfies

$$\delta\beta < \frac{1}{2}, \quad \epsilon < \frac{1-2\delta\beta}{\beta\omega}. \tag{16}$$

With $\mathbf{V}_0$ and $\mathbf{B}_k$ selected above, from the Banach Lemma [30] we know that B_k^{-1} exists and can be bounded as

$$\|B_k^{-1}\| \leqslant \frac{\beta}{1-\beta\delta}. \tag{17}$$

Let

$$e_k = \|\mathbf{V}_k - \mathbf{V}\|.$$

We next derive a relationship between e_{k+1} and e_k. Since $F(\mathbf{V}) = 0$ and

$$\mathbf{V}_{k+1} = \mathbf{V}_k - B_k^{-1}F(\mathbf{V}_k),$$

we have

$$\begin{aligned} e_{k+1} &= e_k - B_k^{-1}[F(\mathbf{V}_k) - F(\mathbf{V})] \\ &= e_k - B_k^{-1}[F(\mathbf{V}_k) - F(\mathbf{V})] + B_k^{-1}F'(\mathbf{V})e_k - B_k^{-1}F'(\mathbf{V})e_k \\ &= -B_k^{-1}[F(\mathbf{V}_k) - F(\mathbf{V}) - F'(\mathbf{V})e_k] - B_k^{-1}[F'(\mathbf{V}) - B_k]e_k \\ &= -B_k^{-1}\left\{[F(\mathbf{V}_k) - F(\mathbf{V}) - F'(\mathbf{V})e_k] + [F'(\mathbf{V}) - B_k]e_k\right\}. \end{aligned} \tag{18}$$

By using Taylor's expansion we have

$$F(\mathbf{V}_k) - F(\mathbf{V}) = F'(\tilde{\mathbf{V}}_k)e_k,$$

where $\tilde{\mathbf{V}}_k$ is a suitable vector lying between $\mathbf{V}_k$ and $\mathbf{V}$. By using (17) it holds

$$\begin{aligned} \|e_{k+1}\| &\leqslant \|B_k^{-1}\|[\|F'(\tilde{\mathbf{V}}_k)e_k - F'(\mathbf{V})e_k\| + \|(F'(\mathbf{V}) - B_k)e_k\|] \\ &\leqslant \frac{\beta}{1-\delta\beta}[\|F'(\tilde{\mathbf{V}}_k)e_k - F'(\mathbf{V})e_k\| + \|F'(\mathbf{V}) - B_k\|\|e_k\|] \\ &\leqslant \frac{\omega\beta}{1-\delta\beta}\|e_k\|^2 + \frac{\delta\beta}{1-\delta\beta}\|e_k\|, \end{aligned} \tag{19}$$

where for the third inequality we have used $\|B_k - F'(\mathbf{V})\|_2 \leqslant \delta$.

We claim that $\|e_k\| \leqslant \epsilon$ for all $k \geqslant 0$. For $k = 0$, from (19) we have

$$\|e_1\| \leqslant \frac{\omega\beta}{1-\delta\beta}\|e_0\|^2 + \frac{\delta\beta}{1-\delta\beta}\|e_0\| \leqslant \frac{\omega\beta\epsilon^2}{1-\delta\beta} + \frac{\delta\beta\epsilon}{1-\delta\beta},$$

where we have used the assumption $\|e_0\| = \|\mathbf{V} - \mathbf{V}_0\| \leqslant \epsilon$. From (16) it holds

$$\begin{aligned} &\epsilon < \frac{1-2\delta\beta}{\beta\omega} \Rightarrow \beta\omega\epsilon + \delta\beta < 1 - \delta\beta \\ &\Rightarrow \beta\omega\epsilon^2 + \delta\beta\epsilon < (1-\delta\beta)\epsilon \Rightarrow \frac{\beta\omega\epsilon^2 + \delta\beta\epsilon}{1-\delta\beta} < \epsilon. \end{aligned} \tag{20}$$

Hence $\|e_1\| \leqslant \epsilon$. Clearly, for general $k > 0$ from (19) and (20) it holds $\|e_k\| \leqslant \epsilon$ as well. Substituting $\|e_k\| \leqslant \epsilon$ into the right hand-side of (19) gives

$$\begin{aligned} \|e_{k+1}\| &\leqslant \left(\frac{\omega\beta\epsilon}{1-\delta\beta} + \frac{\delta\beta}{1-\delta\beta}\right)\|e_k\| = \eta\|e_k\|, \\ \eta &= \frac{\omega\beta\epsilon + \delta\beta}{1-\delta\beta}, \end{aligned} \tag{21}$$

and thus $\|e_k\| \leqslant \eta^k\|e_0\|$. From (20) we have $\eta < 1$, which implies $\|e_k\| \to 0$ as $k \to \infty$. □

NUMERICAL RESULTS

In this section we solve the stationary nonlinear BS model with concrete data via the residual-based Broyden's method. All numerical results are implemented by Matlab R2016b installed in a desk computer with Mac OS and 2.7 GHz Intel Core i5. The initial guess of the Broyden's method is chosen randomly.

We first study whether the numerical solution satisfies Theorem 1 or not. To this end, we consider $(a, b) = (0.5, 2)$, $(p, q) = (0.25, 6.5)$ and two groups of (V_a, V_b):

$$(V_a, V_b) = (0.5, 2), \quad (V_a, V_b) = (1.5, 2). \tag{22}$$

For $(V_a, V_b) = (0.5, 2)$ it holds $\frac{V_b}{b} = \frac{V_a}{a}$ and thus according to Theorem 1 we know that the solution should be a straight line. For $(V_a, V_b) = (1.5, 2)$, it holds $\frac{V_b}{b} < \frac{V_a}{a}$ and the solution has explicit lower and upper bounds. In Fig. 1 we show the computed solution for these two groups of (V_a, V_b). We see that the numerical solution indeed confirms the theoretical prediction. Here we use a discretization mesh size $h = \frac{1}{256}$.

With the data given above we show in Fig. 2 the measured error for each iteration of the Broyden's method. Clearly, for both $\frac{V_b}{b} = \frac{V_a}{a}$ and $\frac{V_b}{b} < \frac{V_a}{a}$ the method converges with the same rate. We also tested the case $\frac{V_b}{b} = r\frac{V_a}{a}$ with several values of r and for each r the iteration number such that the global error arrives at 10^{-7} is shown in Table 1. The results in Table 1 indicate that the convergence rate of the Broyden's method is robust in terms of the problem parameters.

Table 1 For the case $\frac{V_b}{b} = r\frac{V_a}{a}$ with several values r, the iteration number of Broyden's method when the global error arrives at 10^{-7}.

$r =$	0.2	0.5	1	1.5	2
$(p,q) = (0.25, 6.5)$	20	21	21	21	21
$(p,q) = (6.5, 0.25)$	20	21	21	21	21

We next validate whether the bounds in (5) hold or not when $\sqrt{\frac{q}{a^3}}$ is small. Let

$$h = \tfrac{1}{128}, a = 0.5, b = 2, V_a = 1.5, V_b = 2, p = 0.25. \tag{23}$$

We then consider two values of q: $q = 1.25 \times 10^{-4}$ and $q = 0.125$. With this choice of q, it holds $\sqrt{\frac{q}{a^3}} = 0.01$ and $\sqrt{\frac{q}{a^3}} = 1$, respectively. The numerical solution is shown in Fig. 3 for these two values of q. Clearly, for q small (i.e., $\sqrt{\frac{q}{a^3}} = 0.01$) the solution indeed satisfies the bound given by (5), while for q relatively large (i.e., $\sqrt{\frac{q}{a^3}} = 1$) the solution can not be bounded by the lower bound. With the same data used for Fig. 3 the measured error at each iteration for Broyden's method is given by Fig. 4. Clearly, the convergence rate is robust in terms of the problem parameters as well.

At the end of this section, we validate the convergence properties of the Broyden's method given by Theorem 2. To this end, we consider the data in (23) and $q = 1.25 \times 10^{-4}$. We slect a special initial guess

$$\mathbf{V}^0 = \mathbf{V} + \texttt{random}(\texttt{'unif'}, -\epsilon, \epsilon, N-1, 1), \tag{24}$$

where $\epsilon = 0.02$. With the problem and discretization data given above, the quantities ω in (15) and $\beta =$

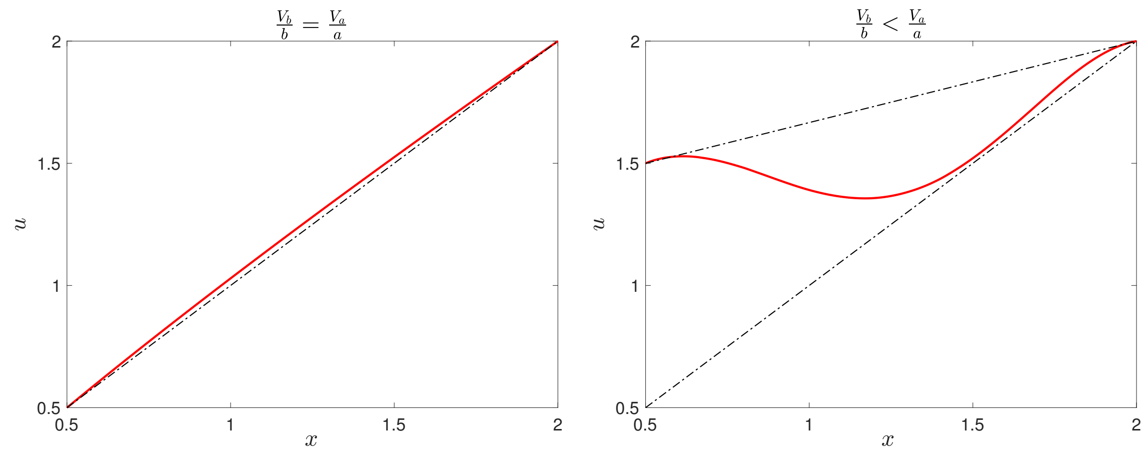

Fig. 1 With $h = \frac{1}{256}$, the computed solution u of nonlinear BS model for two situations. In the left subfigure $\frac{V_b}{b} = \frac{V_a}{a}$ and the solution indeed looks like a straight line. In the right subfigure $\frac{V_b}{b} < \frac{V_a}{a}$ and the solution indeed satisfies the upper and lower bounds given by Theorem 1 (cf. (4)).

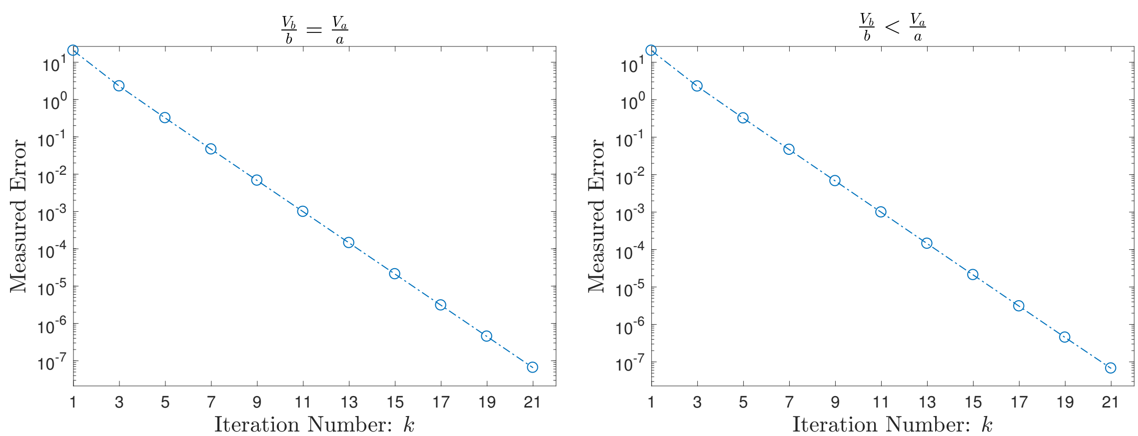

Fig. 2 With the same data used in Fig. 1, the measured global error for each iteration of the Broyden's method.

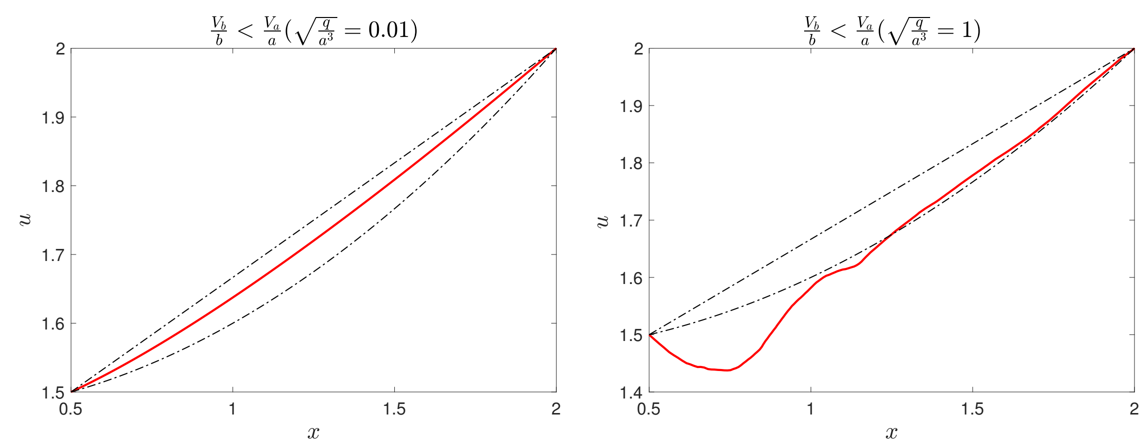

Fig. 3 Two representative cases for the BS model with $\frac{V_b}{b} < \frac{V_a}{a}$. Left: $\sqrt{\frac{q}{a^3}} = 0.01$ and the solution satisfies the upper bound and lower bound given by (5). Right: $\sqrt{\frac{q}{a^3}} = 1$ and the solution can not be bounded by the lower bound.

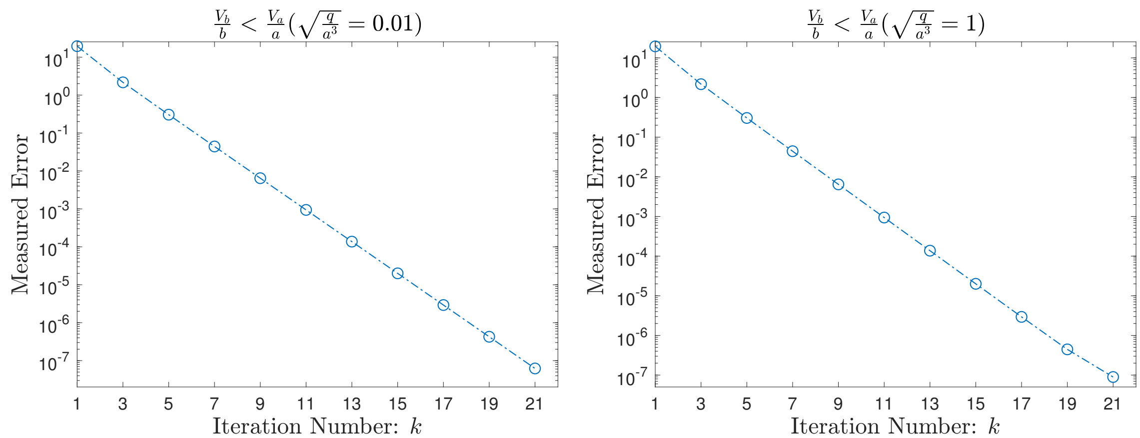

Fig. 4 With the same data used in Fig. 3, the measured global error for each iteration of the Broyden's method.

$\|F'(\mathbf{V})^{-1}\|_\infty$ measured numerically is

$$\omega = \max_{\tilde{\mathbf{V}}\in D_0} \frac{\|F'(\tilde{\mathbf{V}}) - F'(\mathbf{V})\|_\infty}{\|\tilde{\mathbf{V}} - \mathbf{V}\|_\infty} = 2.8813, \ \beta = 2.6613.$$

With the initial guess $\mathbf{V}^0$ given above, it holds $\|\mathbf{V}_0 - \mathbf{V}\|_\infty \leqslant \epsilon$. Let

$$\delta_k = \|B_k - F'(\mathbf{V})\|_\infty.$$

In Fig. 5 on the left we show δ_k for the first 21 iterations and it is clear that $\delta_k \leqslant 0.09$. Therefore, the quantity δ which bounds $\|B_k - F'(\mathbf{V})\|_\infty$ for all $k \geqslant 0$ is

$$\delta = \max_{k\geqslant 0} \delta_k = 0.09.$$

Now, we can calculate the contraction factor η in (21) as

$$\eta = \frac{\omega\beta\epsilon + \delta\beta}{1 - \delta\beta} = 0.4158. \tag{25}$$

With such a contraction factor we show in Fig. 5 the error measured in practice and the one predicted as

$$\|\mathbf{V}^0 - \mathbf{V}\|_\infty \times \eta^k, \quad k \geqslant 0. \tag{26}$$

The result shown in Fig. 5 on the left clearly indicates that the convergence rate of the Broyden's method analyzed in Theorem 2 is sharp and predicts the real convergence behavior very well.

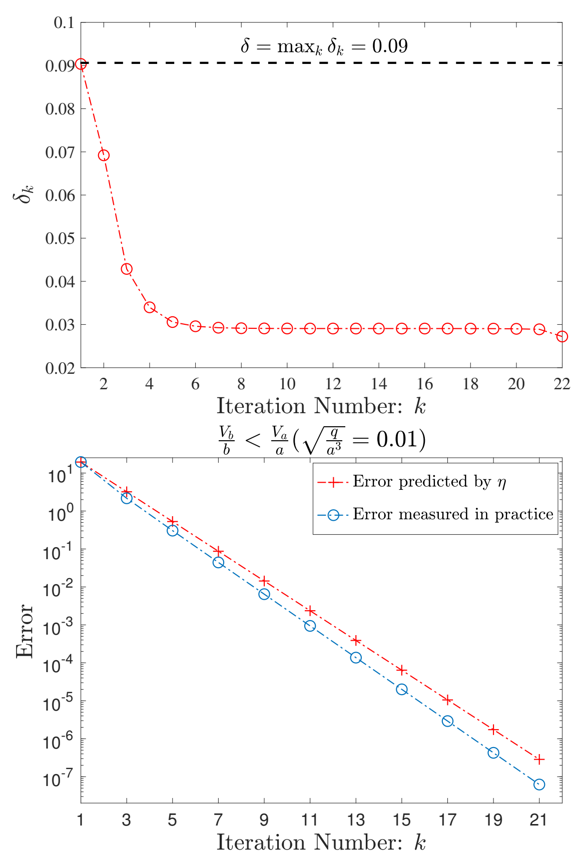

Fig. 5 Left: the quantity $\delta_k = \|B_k - F'(\mathbf{V})\|_\infty$ for each iteration. Right: the error predicted by the contraction factor η (cf. (26)) and the one measured in practice.

CONCLUSION

In this paper, we proposed a numerical method for solving a class of two-point nonlinear boundary value problems arising in finance, namely the stationary Black-Scholes model with transaction costs. We first discretize the continuous model in space via the centered finite difference method, which results in a large scale nonlinear algebraic system. We solve such a nonlinear system by the residual-based Broyden's method, where we do not need to invert any matrix during the iterations, thanks to the explicit recursive relationship between the matrices $\{B_k^{-1}\}$ cf. (13). Convergence of Broyden's method was proved and we also give an estimate of the convergence factor of the method cf.

(21). By using the proposed numerical method, we validated an interesting result concerning the bounds of the solution of the Black-Scholes model cf. Theorem 1. Numerical results indicate that the contraction factor is sharp and predicts the real convergence behavior very well.

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