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A refinement of a local limit theorem for a Poisson binomial random variable

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ABSTRACT: In this paper, a local limit theorem is proposed for a Poisson binomial random variable with a normal random variable as a limit distribution. Our result improves the convergence rate of previous research from $1/\sigma^2$ to $1/\sigma^3$, where σ^2 is a variance of the Poisson binomial random variable. The order of an error bound is equal to the best rate of convergence in the case of a symmetric binomial random variable. The methods for this work are the characteristic function and the correction term approaches.

KEYWORDS: Poisson binomial random variable, local limit theorem, normal density function, characteristic function, correction term

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INTRODUCTION

Let $X_1, X_2, \dots, X_n$ be integer-valued random variables and $S_n = \sum_{j=1}^{n} X_j$. One of interesting and powerful probabilities, called a point probability, is $P(S_n = k)$ for any integer k. A simple and easily calculated example of point probabilities is a binomial density function, $P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}$, where p is a chance of success and k is a number of success. Although we have an explicit formula for the density function, a cost for computing is a binomial coefficient $\binom{n}{k}$. In addition, if we sum non-identically Bernoulli random variables, called a Poisson binomial random variable, $P(S_n = k)$ is infeasible to compute. To solve this problem, one may approximates $P(S_n = k)$ instead of direct calculation. Any theorem that approximates a point probability by a density function of a certain random variable is called a local limit theorem. A classic example of the local limit theorems is the de Moivre–Laplace theorem (see history about the local limit theorem in [1]).

The local limit theorem is widely studied and applied in various directions. One direction is choosing a suitable limit distribution for approximations. If a chance of success for X_i is small, called rare event, Poisson and translated Poisson random variables are right and popular limit distributions (see [2, 3] for translated Poisson random variables with applications and see [4, 5] for Poisson random variables). Another famous and classic limit distribution is a normal distribution which is used when the chance of success is a constant (see [6] for the classic version of a normal distribution, [7] for an approximating a number of occupied urns by a normal approximation and [8] for a driftless Brownion motion).

In this work, we focus on a local limit theorem for a Poisson binomial random variable when a limit distribution is a normal density function. Let $X_1, X_2, \dots, X_n$ be independent Bernoulli random variables with parameters $p_1, p_2, \dots, p_n$ and $q_j = 1 - p_j$ for $j = 1, 2, \dots, n$. Denote $\mu = E(S_n) = \sum_{j=1}^{n} p_j$ and $\sigma^2 = Var(S_n) = \sum_{j=1}^{n} p_j q_j$. From our literature, we know that the rate of convergence of the bound for a symmetric binomial random variable is improved from order $1/\sigma^2$ to $1/\sigma^3$ by Petrov [6] (see [9, 10] for more details). However, the best rate of convergence of a bound for a Poisson binomial random variable obtained by Siripraparat and Neammanee [11] in 2021 is of order $1/\sigma^2$. Consequently, the bound for Poisson binomial random variable is improved in this work to be of order $1/\sigma^3$. The correction term is proposed to sharpen the bound as shown in the main theorem.

Theorem 1 *For $\sigma^2 > 1$, we have*

$$\max_{k \in \{0,1,2,\dots,n\}} \left| P(S_n = k) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}} - T \right| \leqslant \frac{0.0749}{\sigma^3} + \frac{0.2107}{\left(1 - \frac{3}{4\sigma}\right)^6 \sigma^3} + \frac{0.2123}{\left(1 - \frac{3}{4\sigma}\right)^4 \sigma^4} + 1.9979\, e^{-\frac{3}{2}\sigma} + \frac{e^{-\frac{\sigma^2}{2}}}{2\sigma^2}, \quad (1)$$

where $T = \frac{k-\mu}{6\sqrt{2\pi}\sigma^5}\left(3 - \frac{(k-\mu)^2}{\sigma^2}\right) \sum_{j=1}^{n} p_j q_j (p_j - q_j)\, e^{-\frac{(k-\mu)^2}{2\sigma^2}}$.

We make the implicit assumption that σ^2 surpasses a certain fixed threshold due to the favorable convergence of a normal density function as σ^2 tends towards infinity with increasing n, rendering it a suitable approximation for the point probability of a Poisson binomial random variable.

From Theorem 1, we immediately obtain a bound for a binomial random variable as shown in the following corollary.

Corollary 1 *Suppose that S_n is a binomial random variable with parameters n, p and $q = 1-p$. If $npq > 1$, then*

$$\max_{k\in\{0,1,2,\dots,n\}} \left| P(S_n = k) - \frac{1}{\sqrt{2npq\pi}} e^{-\frac{(k-np)^2}{2npq}} - T \right| \leqslant \frac{0.0749}{(npq)^{1.5}} + \frac{0.2107}{\left(1-\frac{3}{4\sqrt{npq}}\right)^6 (npq)^{1.5}} + \frac{0.2123}{\left(1-\frac{3}{4\sqrt{npq}}\right)^4 (npq)^2} + 1.9979\, e^{-\frac{3}{2}\sqrt{npq}} + \frac{e^{-\frac{npq}{2}}}{2npq},$$

where $T = \frac{(k-np)(p-q)}{6npq\sqrt{2npq\pi}} \left(3 - \frac{(k-np)^2}{npq}\right) e^{-\frac{(k-np)^2}{2npq}}$.

In the case of a symmetric binomial random variable, if we let $p = q = 1/2$ in Corollary 1, we have that

$$\max_{k\in\{0,1,2,\dots,n\}} \left| P(S_n = k) - \sqrt{\frac{2}{n\pi}}\, e^{-\frac{2(k-n/2)^2}{n}} \right| \leqslant \frac{0.5992}{n\sqrt{n}} + \frac{1.6856}{\left(1-\frac{3}{2\sqrt{n}}\right)^6 n\sqrt{n}} + \frac{3.3968}{\left(1-\frac{3}{2\sqrt{n}}\right)^4 n^2} + 1.9979\, e^{-\frac{3}{4}\sqrt{n}} + \frac{2\, e^{-\frac{n}{8}}}{n}.$$

Additionally, we can improve a bound for a symmetric binomial random variable as shown in the following theorem.

Theorem 2 *Suppose that S_n is a symmetric binomial random variable and $n > 4$, then*

$$\max_{k\in\{0,1,2,\dots,n\}} \left| P(S_n = k) - \sqrt{\frac{2}{n\pi}}\, e^{-\frac{2(k-n/2)^2}{n}} \right| \leqslant \frac{0.5992}{n\sqrt{n}} + 1.7326\, e^{-\frac{3}{4}\sqrt{n}} + \frac{2\, e^{-\frac{n}{8}}}{n}.$$

PROOF OF MAIN RESULTS

In this section, we prove our main theorems by using a characteristic function approach appeared in [12] and the technique from [11]. Let $\psi_1, \psi_2, \dots, \psi_n$ and ψ are characteristic functions of $X_1, X_2, \dots, X_n$ and S_n, respectively. Then, for $j = 1, 2, \dots, n$, we have

$$\psi_j(t) = q_j + p_j e^{it} = q_j + p_j \cos(t) + i p_j \sin(t) = \left|\psi_j(t)\right| e^{i\theta_j(t)},$$

$$\text{and} \quad \psi(t) = \prod_{j=1}^{n} (q_j + p_j e^{it}),$$

where $\tan\theta_j(t) = \frac{p_j \sin t}{q_j + p_j \cos t}$ and $t \in \mathbb{R}$. Denote

$$\theta(t) = \sum_{j=1}^{n} \theta_j(t) \pmod{2\pi},$$

$$\rho(t) = \prod_{j=1}^{n} \left|\psi_j(t)\right|,$$

$$\text{and} \quad \alpha(t) = \theta(t) - \mu t.$$

Before proving the main theorems, we notice the important fact provided by Siripraparat and Neammanee [11, p.113] that

$$P(S_n = k) = \frac{1}{\pi} \int_0^{\pi} \rho(t) \cos\left[(k-\mu)t - \alpha(t)\right] dt. \quad (2)$$

Next, we verify

$$\cos[(k-\mu)t - \alpha(t)]$$

as shown in the following lemma.

Lemma 1 *Let t be a positive real number such that $t \in \left[0, \min\left\{\sqrt{\frac{3}{\sigma}}, \frac{\pi}{2}\right\}\right]$. Then*

$$\cos\left[(k-\mu)t - \alpha(t)\right] = \cos[(k-\mu)t] + \frac{1}{6}\sum_{j=1}^{n} p_j q_j (p_j - q_j) t^3 \sin[(k-\mu)t] + A,$$

where $|A| \leqslant \frac{\sigma^2 t^5}{12\left(1-\frac{3}{4\sigma}\right)^4} + \frac{0.0352\sigma^4 t^6}{\left(1-\frac{3}{4\sigma}\right)^6}$.

Proof: By the trigonometric identity and the Taylor expansion of functions $\sin[\alpha(t)]$ and $\cos[\alpha(t)]$, we have that there are some $s_0, s_1 \in (0, t)$ such that

$$\begin{aligned} \cos\left[(k-\mu)t - \alpha(t)\right] &= \cos[(k-\mu)t]\cos[\alpha(t)] + \sin[(k-\mu)t]\sin[\alpha(t)] \\ &= \cos[(k-\mu)t]\left(1 - \tfrac{1}{2}\alpha^2(t)\cos(s_0)\right) \\ &\quad + \sin[(k-\mu)t]\left(\alpha(t) - \tfrac{1}{2}\alpha^2(t)\sin(s_1)\right) \\ &= \cos[(k-\mu)t] + A_1, \end{aligned}$$

where $A_1 = -\frac{1}{2}\cos[(k-\mu)t]\alpha^2(t)\cos(s_0) + \sin[(k-\mu)t]\left(\alpha(t) - \frac{1}{2}\alpha^2(t)\sin(s_1)\right)$.

By (9) and (10) in [11, Lemma 1, p.113], we have

$$\theta_j(t) = p_j t + \frac{1}{6} p_j q_j (p_j - q_j) t^3 + \frac{1}{24}\theta_j^{(4)}(t_j) t^4$$

for some t_j. This implies that

$$\alpha(t)=\frac{1}{6}\sum_{j=1}^{n}p_jq_j(p_j-q_j)t^3+\frac{1}{24}\sum_{j=1}^{n}\theta_j^{(4)}(t_j)t^4$$
$$=\frac{1}{6}\sum_{j=1}^{n}p_jq_j(p_j-q_j)t^3+A_2, \tag{3}$$

where $A_2=\frac{1}{24}\sum\limits_{j=1}^{n}\theta_j^{(4)}(t_j)t^4$. By the fact that

$$\left|\theta_j^{(4)}(t)\right|\leqslant\frac{2p_jq_jt}{\left(1-\frac{3}{4\sigma}\right)^4}$$

(see [13, p.722]), we have

$$|A_2|\leqslant\frac{\sigma^2t^5}{12\left(1-\frac{3}{4\sigma}\right)^4}. \tag{4}$$

By (3) and (4), we can rewrite A_1 to be

$$A_1=\alpha(t)\sin[(k-\mu)t]-\frac{1}{2}\alpha^2(t)\cos(s_0)\cos[(k-\mu)t]$$
$$-\frac{1}{2}\alpha^2(t)\sin(s_1)\sin[(k-\mu)t]$$
$$=\frac{1}{6}\sum_{j=1}^{n}p_iq_i(p_i-q_i)t^3\sin[(k-\mu)t]+A_2\sin[(k-\mu)t]$$
$$-\frac{1}{2}\alpha^2(t)\cos(s_0)\cos[(k-\mu)t]$$
$$-\frac{1}{2}\alpha^2(t)\sin(s_1)\sin[(k-\mu)t]$$
$$=\frac{1}{6}\sum_{j=1}^{n}p_iq_i(p_i-q_i)t^3\sin[(k-\mu)t]+A_3,$$

where $A_3=A_2\sin[(k-\mu)t]-\frac{1}{2}\alpha^2(t)\cos(s_0)\cos[(k-\mu)t]-\frac{1}{2}\alpha^2(t)\sin(s_1)\sin[(k-\mu)t]$. By (4) and the fact that $\left|\alpha^2(t)\right|\leqslant\frac{0.0352\sigma^4t^6}{(1-\frac{3}{4\sigma})^6}$, (see [11, p.113]), we obtain that

$$|A_3|\leqslant|A_2|+\left|\alpha^2(t)\right|\leqslant\frac{\sigma^2t^5}{12\left(1-\frac{3}{4\sigma}\right)^4}+\frac{0.0352\sigma^4t^6}{\left(1-\frac{3}{4\sigma}\right)^6}.$$

This completes the proof. □

Next, we prove Theorem 1.

Proof of Theorem 1: By (2), we first note that

$$P(S_n=k)$$
$$=\frac{1}{\pi}\int_0^{\sqrt{\frac{3}{\sigma}}}e^{-\frac{1}{2}\sigma^2t^2}\cos[(k-\mu)t-\alpha(t)]\,dt$$
$$+\frac{1}{\pi}\int_0^{\sqrt{\frac{3}{\sigma}}}\left(\rho(t)-e^{-\frac{1}{2}\sigma^2t^2}\right)\cos[(k-\mu)t-\alpha(t)]\,dt$$
$$+\frac{1}{\pi}\int_{\sqrt{\frac{3}{\sigma}}}^{\pi}\rho(t)\cos[(k-\mu)t-\alpha(t)]\,dt$$
$$=\frac{1}{\pi}\int_0^{\sqrt{\frac{3}{\sigma}}}e^{-\frac{1}{2}\sigma^2t^2}\cos[(k-\mu)t-\alpha(t)]\,dt+\Delta_1, \tag{5}$$

where

$$\Delta_1=\frac{1}{\pi}\int_0^{\sqrt{\frac{3}{\sigma}}}\left(\rho(t)-e^{-\frac{1}{2}\sigma^2t^2}\right)\cos[(k-\mu)t-\alpha(t)]\,dt$$
$$+\frac{1}{\pi}\int_{\sqrt{\frac{3}{\sigma}}}^{\pi}\rho(t)\cos[(k-\mu)t-\alpha(t)]\,dt. \tag{6}$$

Notice that, Siripraparat and Neammanee [11, pp.114–115] showed that

$$\left|\frac{1}{\pi}\int_0^{\sqrt{\frac{3}{\sigma}}}\left(\rho(t)-e^{-\frac{1}{2}\sigma^2t^2}\right)\cos[(k-\mu)t-\alpha(t)]\,dt\right|$$
$$\leqslant\frac{0.0749}{\sigma^3}. \tag{7}$$

To bound $\frac{1}{\pi}\int\limits_{\sqrt{\frac{3}{\sigma}}}^{\pi}\rho(t)\cos[(k-\mu)t-\alpha(t)]\,dt$, we utilize the technique from (20)–(21) in [11, p.114] to obtain the following general results: for $0<a\leqslant b\leqslant\frac{\pi}{2}$, we have

$$\frac{1}{\pi}\int\limits_a^b\rho(t)\,dt\leqslant\frac{\rho(a)}{\pi}(b-a)\leqslant\frac{b-a}{\pi}e^{-\frac{a^2\sigma^2}{2}+\frac{a^4\sigma^2}{24}}, \tag{8}$$

and for $0<a\leqslant b\leqslant\pi$, we have

$$\frac{1}{\pi}\int\limits_a^b\rho(t)\,dt\leqslant\frac{1}{\pi}\int\limits_a^{\pi}e^{-\frac{2t^2\sigma^2}{\pi^2}}\,dt$$
$$\leqslant\frac{\pi}{2a\sigma^2}\int\limits_{\frac{\sqrt{2}a\sigma}{\pi}}^{\infty}t\,e^{-t^2}\,dt=\frac{\pi}{4a\sigma^2}e^{-\frac{2a^2\sigma^2}{\pi^2}}. \tag{9}$$

If $\sqrt{\frac{3}{4\sigma}}\pi\wedge\frac{\pi}{2}=\sqrt{\frac{3}{4\sigma}}\pi$, then, from (8) with $a=\sqrt{\frac{3}{\sigma}}$ and $b=\sqrt{\frac{3}{4\sigma}}\pi$ and from (9) with $a=\sqrt{\frac{3}{4\sigma}}\pi$, we obtain

$$\left|\frac{1}{\pi}\int\limits_{\sqrt{\frac{3}{\sigma}}}^{\pi}\rho(t)\cos[(k-\mu)t-\alpha(t)]\,dt\right|$$
$$\leqslant\frac{1}{\pi}\int\limits_{\sqrt{\frac{3}{\sigma}}}^{\sqrt{\frac{3}{4\sigma}}\pi}\rho(t)\,dt+\frac{1}{\pi}\int\limits_{\sqrt{\frac{3}{4\sigma}}\pi}^{\pi}\rho(t)\,dt$$
$$\leqslant\frac{1}{\pi}e^{-\frac{3}{2}\sigma+\frac{9}{24}}\left(\sqrt{\frac{3}{4\sigma}}\pi-\sqrt{\frac{3}{\sigma}}\right)+\frac{e^{-\frac{3}{2}\sigma}}{2\sqrt{3}\sigma\sqrt{\sigma}}$$
$$\leqslant 1.2601\,e^{-\frac{3}{2}\sigma}+0.2887\,e^{-\frac{3}{2}\sigma}$$
$$=1.5488\,e^{-\frac{3}{2}\sigma}.$$

If $\sqrt{\frac{3}{4\sigma}}\pi \wedge \frac{\pi}{2} = \frac{\pi}{2}$ and $\sqrt{\frac{3}{\sigma}} \wedge \frac{\pi}{2} = \sqrt{\frac{3}{\sigma}}$, then by (8) with $a = \sqrt{\frac{3}{\sigma}}$ and $b = \frac{\pi}{2}$ and from (9) with $a = \frac{\pi}{2}$, we obtain that

$$\left|\frac{1}{\pi}\int_{\sqrt{\frac{3}{\sigma}}}^{\pi} \rho(t)\cos[(k-\mu)t-\alpha(t)]\,\mathrm{d}t\right| \leqslant \frac{1}{\pi}\int_{\sqrt{\frac{3}{\sigma}}}^{\frac{\pi}{2}} \rho(t)\,\mathrm{d}t + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} \rho(t)\,\mathrm{d}t \leqslant \frac{1}{\pi}\,\mathrm{e}^{-\frac{3}{2}\sigma+\frac{9}{24}}\left(\frac{\pi}{2}-\sqrt{\frac{3}{\sigma}}\right) + \frac{\mathrm{e}^{-\frac{\sigma^2}{2}}}{2\sigma^2} \leqslant 0.7275\,\mathrm{e}^{-\frac{3}{2}\sigma} + \frac{\mathrm{e}^{-\frac{\sigma^2}{2}}}{2\sigma^2}.$$

Suppose that $\sqrt{\frac{3}{4\sigma}}\pi \wedge \frac{\pi}{2} = \frac{\pi}{2}$ and $\sqrt{\frac{3}{\sigma}} \wedge \frac{\pi}{2} = \frac{\pi}{2}$. By the fact that $\frac{\pi}{2} \leqslant \sqrt{\frac{3}{\sigma}}$, $-\frac{1}{\pi^2} \leqslant -\frac{\sigma}{12}$ and (9), we obtain

$$\left|\frac{1}{\pi}\int_{\sqrt{\frac{3}{\sigma}}}^{\pi} \rho(t)\cos[(k-\mu)t-\alpha(t)]\,\mathrm{d}t\right| \leqslant \frac{1}{\pi}\int_{\sqrt{\frac{3}{\sigma}}}^{\pi} \rho(t)\,\mathrm{d}t \leqslant \frac{\pi}{4\sqrt{3}\sigma\sqrt{\sigma}} e^{-\frac{6}{\pi^2}\sigma} \leqslant \frac{\mathrm{e}^{-\frac{\sigma^2}{2}}}{2\sigma^2}.$$

Combining all cases, we obtain that

$$\left|\frac{1}{\pi}\int_{\sqrt{\frac{3}{\sigma}}}^{\pi} \rho(t)\cos[(k-\mu)t-\alpha(t)]\,\mathrm{d}t\right| \leqslant 1.5488\,\mathrm{e}^{-\frac{3}{2}\sigma} + \frac{\mathrm{e}^{-\frac{\sigma^2}{2}}}{2\sigma^2}. \quad (10)$$

Therefore, we conclude from (6), (7) and (10) that

$$|\Delta_1| \leqslant \frac{0.0749}{\sigma^3} + 1.5488\,\mathrm{e}^{-\frac{3}{2}\sigma} + \frac{\mathrm{e}^{-\frac{\sigma^2}{2}}}{2\sigma^2}. \quad (11)$$

Now, it remains to verify

$$\frac{1}{\pi}\int_0^{\sqrt{\frac{3}{\sigma}}} \mathrm{e}^{-\frac{1}{2}\sigma^2t^2}\cos[(k-\mu)t-\alpha(t)]\,\mathrm{d}t.$$

By Lemma 1, we have

$$\frac{1}{\pi}\int_0^{\sqrt{\frac{3}{\sigma}}} \mathrm{e}^{-\frac{1}{2}\sigma^2t^2}\cos[(k-\mu)t-\alpha(t)]\,\mathrm{d}t = \frac{1}{\pi}\int_0^{\sqrt{\frac{3}{\sigma}}} \mathrm{e}^{-\frac{1}{2}\sigma^2t^2}\cos[(k-\mu)t]\,\mathrm{d}t + \frac{1}{\pi}\int_0^{\sqrt{\frac{3}{\sigma}}} \mathrm{e}^{-\frac{1}{2}\sigma^2t^2}A\,\mathrm{d}t + \frac{1}{6\pi}\sum_{j=1}^{n} p_jq_j(p_j-q_j)\int_0^{\sqrt{\frac{3}{\sigma}}} \mathrm{e}^{-\frac{1}{2}\sigma^2t^2}t^3\sin[(k-\mu)t]\,\mathrm{d}t, \quad (12)$$

where $|A| \leqslant \frac{\sigma^2t^5}{12\left(1-\frac{3}{4\sigma}\right)^4} + \frac{0.0352\sigma^4t^6}{\left(1-\frac{3}{4\sigma}\right)^6}$. By the fact that

$$\int_0^{\infty} \mathrm{e}^{-at^2}\cos(bt)\,\mathrm{d}t = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,\mathrm{e}^{-\frac{b^2}{4a}}$$

for $a > 0$ (see [14, p. 488]), we have

$$\frac{1}{\pi}\int_0^{\sqrt{\frac{3}{\sigma}}} \mathrm{e}^{-\frac{1}{2}\sigma^2t^2}\cos[(k-\mu)t]\,\mathrm{d}t = \frac{1}{\sigma\sqrt{2\pi}}\,\mathrm{e}^{-\frac{(k-\mu)^2}{2\sigma^2}} + \Delta_2, \quad (13)$$

where $|\Delta_2| \leqslant \frac{0.1838}{\sigma\sqrt{\sigma}}\,\mathrm{e}^{-\frac{3}{2}\sigma}$ (see [11, p.115] for details).

Next, we consider

$$\frac{1}{6\pi}\sum_{j=1}^{n} p_jq_j(p_j-q_j)\int_0^{\sqrt{\frac{3}{\sigma}}} \mathrm{e}^{-\frac{1}{2}\sigma^2t^2}t^3\sin[(k-\mu)t]\,\mathrm{d}t.$$

It is well-known that

$$\int_0^{\infty} t\,\mathrm{e}^{-at^2}\sin(xt)\,\mathrm{d}t = \frac{x}{4a}\sqrt{\frac{\pi}{a}}\,\mathrm{e}^{-\frac{x^2}{4a}}$$

for $a > 0$ (see [14, p. 502]). Hence, we obtain

$$\begin{aligned}
\int_0^{\infty} t^3\mathrm{e}^{-at^2}\sin(xt)\,\mathrm{d}t &= -\int_0^{\infty} \frac{\partial}{\partial x} t^2\,\mathrm{e}^{-at^2}\cos(xt)\,\mathrm{d}t \\
&= -\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\infty} \frac{\partial}{\partial x} t\,\mathrm{e}^{-at^2}\sin(xt)\,\mathrm{d}t \\
&= -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_0^{\infty} t\,\mathrm{e}^{-at^2}\sin(xt)\,\mathrm{d}t \\
&= -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{x}{4a}\sqrt{\frac{\pi}{a}}\,\mathrm{e}^{-\frac{x^2}{4a}}\right) \\
&= \frac{x}{8a^2}\sqrt{\frac{\pi}{a}}\left(3-\frac{x^2}{2a}\right)\mathrm{e}^{-\frac{x^2}{4a}}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\int_0^{\sqrt{\frac{3}{\sigma}}} t^3\,\mathrm{e}^{-\frac{1}{2}\sigma^2t^2}\sin[(k-\mu)t]\,\mathrm{d}t
&= \int_0^{\infty} t^3\,\mathrm{e}^{-\frac{1}{2}\sigma^2t^2}\sin[(k-\mu)t]\,\mathrm{d}t - \int_{\sqrt{\frac{3}{\sigma}}}^{\infty} t^3\,\mathrm{e}^{-\frac{1}{2}\sigma^2t^2}\sin[(k-\mu)t]\,\mathrm{d}t \\
&= \frac{\sqrt{2\pi}(k-\mu)}{2\sigma^5}\left(3-\frac{(k-\mu)^2}{\sigma^2}\right)\mathrm{e}^{-\frac{(k-\mu)^2}{2\sigma^2}} - \int_{\sqrt{\frac{3}{\sigma}}}^{\infty} t^3\,\mathrm{e}^{-\frac{1}{2}\sigma^2t^2}\sin[(k-\mu)t]\,\mathrm{d}t.
\end{aligned}$$

This implies that

$$\frac{1}{6\pi}\sum_{j=1}^{n}p_jq_j(p_j-q_j)\int_0^{\sqrt{\frac{3}{\sigma}}}t^3\,\mathrm{e}^{-\frac{1}{2}\sigma^2t^2}\sin[(k-\mu)t]\,\mathrm{d}t$$
$$=\frac{\sqrt{2\pi}(k-\mu)}{12\pi\sigma^5}\Big(3-\frac{(k-\mu)^2}{\sigma^2}\Big)\sum_{j=1}^{n}p_jq_j(p_j-q_j)\,\mathrm{e}^{-\frac{(k-\mu)^2}{2\sigma^2}}+\Delta_3$$
$$=T_1+\Delta_3, \tag{14}$$

where

$$T_1=\frac{\sqrt{2\pi}(k-\mu)}{12\pi\sigma^5}\Big(3-\frac{(k-\mu)^2}{\sigma^2}\Big)\sum_{j=1}^{n}p_jq_j(p_j-q_j)\,\mathrm{e}^{-\frac{(k-\mu)^2}{2\sigma^2}},$$

and

$$\Delta_3=-\frac{1}{6\pi}\sum_{j=1}^{n}p_jq_j(p_j-q_j)\int_{\sqrt{\frac{3}{\sigma}}}^{\infty}t^3\,\mathrm{e}^{-\frac{1}{2}\sigma^2t^2}\sin((k-\mu)t)\,\mathrm{d}t.$$

By substitution, integration by parts techniques and the fact that $\sigma^2>1$, we have

$$\begin{aligned}|\Delta_3|&\leqslant\frac{1}{6\pi}\sum_{j=1}^{n}p_jq_j|p_j-q_j|\int_{\sqrt{\frac{3}{\sigma}}}^{\infty}t^3\,\mathrm{e}^{-\frac{1}{2}\sigma^2t^2}\,\mathrm{d}t\\&\leqslant\frac{\sigma^2}{6\pi}\int_{\sqrt{\frac{3}{\sigma}}}^{\infty}t^3\,\mathrm{e}^{-\frac{1}{2}\sigma^2t^2}\,\mathrm{d}t\\&=\frac{1}{\pi}\left(\frac{1}{2\sigma}+\frac{1}{3\sigma^2}\right)\mathrm{e}^{-\frac{3}{2}\sigma}\\&\leqslant 0.2653\,\mathrm{e}^{-\frac{3}{2}\sigma}.\end{aligned} \tag{15}$$

Finally, we want to bound $\left|\frac{1}{\pi}\int_0^{\sqrt{\frac{3}{\sigma}}}\mathrm{e}^{-\frac{1}{2}\sigma^2t^2}A\,\mathrm{d}t\right|$. By the facts that

$$\int_0^{\infty}t^{2m-1}\,\mathrm{e}^{-at^2}\,\mathrm{d}t=\frac{(m-1)!}{2a^m}$$
$$\text{and}\quad\int_0^{\infty}t^{2m}\,\mathrm{e}^{-at^2}\,\mathrm{d}t=\frac{(2m-1)!\sqrt{\pi}}{2^{2m}(m-1)!a^{m+\frac{1}{2}}}$$

for all $m\in\mathbb{N}$ and $a>0$, we have

$$\begin{aligned}&\left|\frac{1}{\pi}\int_0^{\sqrt{\frac{3}{\sigma}}}\mathrm{e}^{-\frac{1}{2}\sigma^2t^2}A\,\mathrm{d}t\right|\\&\leqslant\frac{1}{\pi}\int_0^{\infty}\mathrm{e}^{-\frac{1}{2}\sigma^2t^2}\left[\frac{\sigma^2t^5}{12\left(1-\frac{3}{4\sigma}\right)^4}+\frac{0.0352\sigma^4t^6}{\left(1-\frac{3}{4\sigma}\right)^6}\right]\mathrm{d}t\\&\leqslant\frac{0.2123}{\left(1-\frac{3}{4\sigma}\right)^4\sigma^4}+\frac{0.2107}{\left(1-\frac{3}{4\sigma}\right)^6\sigma^3}.\end{aligned} \tag{16}$$

From (5) and (11)–(16), we have the main theorem as required. □

Next, we modify the proof of Theorem 1 to prove Theorem 2.

Proof of Theorem 2: First, we will show that $\alpha(t)=0$ which is the key to success for improving the bound in the case of a symmetric binomial random variable. Assume that S_n is a symmetric binomial random variable, i.e., $p_1=p_2=\cdots=p_n=1/2$. Then,

$$\mu t=\sum_{j=1}^{n}p_jt=\frac{nt}{2},$$
$$\text{and}\quad\tan\theta_j(t)=\frac{\sin t}{1+\cos t}. \tag{17}$$

From (17), we obtain that

$$\cos\theta_j(t)=\frac{1+\cos t}{\sqrt{2(1+\cos t)}}=\sqrt{\frac{1+\cos t}{2}}=\left|\cos\left(\frac{t}{2}\right)\right|.$$

This implies that $\theta_j(t)=\frac{t}{2}+2m\pi$ for $m\in\mathbb{N}$. Consequently,

$$\theta(t)=\sum_{j=1}^{n}\theta_j(t)(\mathrm{mod}\ 2\pi)=\frac{nt}{2}.$$

This give an important fact for a symmetric binomial random variable, i.e.,

$$\alpha(t)=\theta(t)-\mu t=0.$$

Now, we use this fact to prove the theorem.

From (5) in the proof of Theorem 1, we have

$$P(S_n=k)=\frac{1}{\pi}\int_0^{\sqrt{\frac{3}{\sigma}}}\mathrm{e}^{-\frac{1}{2}\sigma^2t^2}\cos[(k-\mu)t]\,\mathrm{d}t+\Delta_1,$$

where $|\Delta_1|\leqslant\frac{0.0749}{\sigma^3}+1.5488\,\mathrm{e}^{-\frac{3}{2}\sigma}+\frac{\mathrm{e}^{-\frac{\sigma^2}{2}}}{2\sigma^2}$. By (13), we obtain

$$P(S_n=k)=\frac{1}{\sigma\sqrt{2\pi}}\,\mathrm{e}^{-\frac{(k-\mu)^2}{2\sigma^2}}+\Delta_1+\Delta_2,$$

where $|\Delta_2|\leqslant\frac{0.1838}{\sigma\sqrt{\sigma}}\,\mathrm{e}^{-\frac{3}{2}\sigma}$. Using the facts that $\mu=\frac{n}{2}$ and $\sigma^2=\frac{n}{4}$, we have

$$\begin{aligned}&\max_{k\in\{0,1,2,\dots,n\}}\left|P(S_n=k)-\sqrt{\frac{2}{n\pi}}\,\mathrm{e}^{-\frac{2(k-n/2)^2}{n}}\right|\\&\qquad\leqslant\frac{0.5992}{n\sqrt{n}}+1.7326\,\mathrm{e}^{-\frac{3}{4}\sqrt{n}}+\frac{2\,\mathrm{e}^{-\frac{n}{8}}}{n}\end{aligned}$$

as desired. □

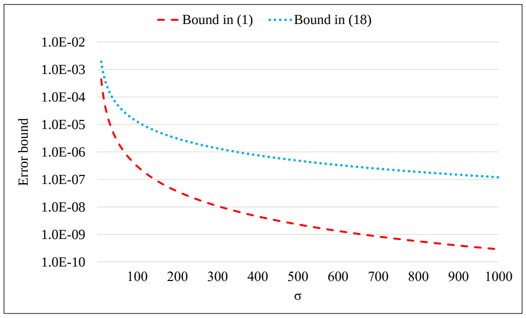

Fig. 1 Error bounds (1) and (18) in the case of a Poisson binomial random variable.

EXAMPLE

Before giving an example in this section, we first note from [11] that

$$\max_{k\in\{0,1,2,\dots,n\}}\left|P(S_n=k)-\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{(k-\mu)^2}{2\sigma^2}}\right| \leqslant \frac{0.1194}{\left(1-\frac{3}{4\sigma}\right)^3\sigma^2}+\left(0.0749+\frac{0.2107}{\left(1-\frac{3}{4\sigma}\right)^6}\right)\frac{1}{\sigma^3} + \left(\frac{0.4579}{\sqrt{\sigma}}+\frac{0.4725}{\sigma\sqrt{\sigma}}\right)e^{-\frac{3}{2}\sigma}, \quad (18)$$

where S_n is a Poisson binomial random variable.

Now, we compare the error bound in Theorem 1 with (18). Notice that, both bounds are valid when $\sigma^2>1$. Observe that, $1-\frac{3}{4\sigma}$ is approximately 1 when σ tend to infinity. Hence, the convergence rate of (18) is of order $\frac{1}{\sigma^2}$ whereas the convergence rate of (1) is of order $\frac{1}{\sigma^3}$. Consequently, (1) is significanly smaller than (18) when σ is large enough showing in Fig. 1.

CONCLUSION

In this paper, we have successfully established a local limit theorem for the Poisson binomial random variable, with a normal density function emerging as the limiting distribution. Our achievement in this endeavor can be attributed to the innovative methodologies employed, particularly the utilization of the characteristic function technique and the correction term approach. These novel approaches have significantly enhanced the precision of our approximation, exemplified by the remarkable convergence rate of $1/\sigma^3$. To further enhance the rate of convergence, the expansion of the methodology delineated in this paper to incorporate supplementary correction terms is a viable avenue, although it entails an associated increase in computational complexity.

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