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# Additive $\xi$-Lie $\sigma$-derivations on triangular algebras

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**ABSTRACT**: Let $\mathscr{A}$ and $\mathscr{B}$ be unital algebras over a field $\mathbb{F}$, $\mathscr{M}$ be a faithful $(\mathscr{A},\mathscr{B})$-bimodule, and let $\mathscr{U}=\mathrm{Tri}(\mathscr{A},\mathscr{M},\mathscr{B})$ be the triangular algebra. Assume that $x_0\in\mathscr{U}$ is some fixed element, $\xi\in\mathbb{F}$ and $\sigma$ is an additive automorphism of $\mathscr{U}$. It is shown that, under some mild conditions, if an additive map $L:\mathscr{U}\to\mathscr{U}$ satisfies $L(xy-\xi yx)=L(x)y-\xi\sigma(y)L(x)+\sigma(x)L(y)-\xi L(y)x$ for $x,y\in\mathscr{U}$ with $xy=x_0$, then $L$ is an additive $\sigma$-derivation if $\xi\neq 1$ and is the sum of an additive $\sigma$-derivation and a special central valued additive map if $\xi=1$; and based on this, all additive $\xi$-Lie $\sigma$-derivations for each possible $\xi$ on $\mathscr{U}$ are characterized completely. All these results generalize some known related ones from different directions.

**KEYWORDS**: Lie derivations, $\sigma$-derivations, Jordan derivations, triangular algebras

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## INTRODUCTION

Let $\mathscr{A}$ be an algebra over a commutative ring $\mathscr{R}$ and $\sigma$ an automorphism of $\mathscr{A}$. Recall that an $\mathscr{R}$-linear (an additive) map $L:\mathscr{A}\to\mathscr{A}$ is called a $\sigma$-derivation (or a skew derivation) if $L(xy)=L(x)y+\sigma(x)L(y)$ holds for all $x,y\in\mathscr{A}$; is called a Jordan $\sigma$-derivation if $L(xy+yx)=L(x)y+\sigma(x)L(y)+L(y)x+\sigma(y)L(x)$ holds for all $x,y\in\mathscr{A}$; is called a Lie $\sigma$-derivation if $L([x,y])=L(xy-yx)=L(x)y-\sigma(y)L(x)+\sigma(x)L(y)-L(y)x$ holds for all $x,y\in\mathscr{A}$. It is obvious that (Jordan or Lie) $\sigma$-derivations are usual (Jordan or Lie) derivations if $\sigma$ is an identity map (denoted by $id$). The structure of Lie (Jordan) $\sigma$-derivations has been studied (see [1–5] and the references therein).

Assume that $\mathscr{R}$ is a field $\mathbb{F}$ and $\xi\in\mathbb{F}$. For $x,y\in\mathscr{A}$, if $xy=\xi yx$, we say that $x$ commutes with $y$ up to a factor $\xi$. The notion of commutativity up to a factor for pairs of operators is an important concept and has been studied in the context of operator algebras and quantum groups (see [6,7]). Motivated by this, a binary operation $[x,y]_\xi=xy-\xi yx$, called $\xi$-Lie product of $x$ and $y$, and the concept of $\xi$-Lie derivations were introduced in [8]. The additive map $L$ is a $\xi$-Lie derivation if $L([x,y]_\xi)=[L(x),y]_\xi+[x,L(y)]_\xi$ holds for all $x,y\in\mathscr{A}$. It is clear that a $\xi$-Lie derivation is a derivation, a Lie derivation and a Jordan derivation if $\xi=0$, $\xi=1$ and $\xi=-1$, respectively. After that, some works about $\xi$-Lie derivations were done (for example, see [8–10] and the references therein). In [10], Yang and Zhu gave the concept of $\xi$-Lie $\sigma$-derivations which is a generalization of $\xi$-Lie derivations. Recall that the additive map $L$ is called a $\xi$-Lie $\sigma$-derivation if $L([x,y]_\xi)=L(x)y-\xi\sigma(y)L(x)+\sigma(x)L(y)-\xi L(y)x$ for all $x,y\in\mathscr{A}$. Obviously, $\xi$-Lie $\sigma$-derivations cover various kinds of derivations. For example, $\xi$-Lie $\sigma$-derivations are $\sigma$-derivations if $\xi=1$, are Jordan $\sigma$-derivations if $\xi=-1$ and are $\xi$-Lie derivations if $\sigma=id$.

The main purpose of this paper is to describe $\xi$-Lie $\sigma$-derivations by some local actions on the general triangular algebras. A lot of attention are paid to characterize the maps on triangular algebras. We mention here some results related to this paper. Let $\mathscr{A}$ and $\mathscr{B}$ be unital algebras over any commutative ring $\mathscr{R}$, and $\mathscr{M}$ be a faithful $(\mathscr{A},\mathscr{B})$-bimodule. The $\mathscr{R}$-algebra $\mathscr{U}=\mathrm{Tri}(\mathscr{A},\mathscr{M},\mathscr{B})=\{\begin{pmatrix} a & m\\ 0 & b\end{pmatrix}: a\in\mathscr{A}, m\in\mathscr{M}, b\in\mathscr{B}\}$ under the usual matrix operations is called a triangular algebra (see [11]). Han and Wei [12] proved that any $\mathscr{R}$-linear Jordan $\sigma$-derivation of $\mathscr{U}$ is a $\sigma$-derivation if $\mathscr{A}$ and $\mathscr{B}$ have only trivial idempotents. Yang and Zhu [10] proved that, under the assumption that $\mathscr{A}$ and $\mathscr{B}$ have only trivial idempotents, every additive $\xi$-Lie $\sigma$-derivation is the sum of an additive $\sigma$-derivation and a central valued additive map vanishing at all the commutators if $\xi=1$ and is an additive $\sigma$-derivation if $\xi\neq 1$. Note that this assumption that $\mathscr{A}$ and $\mathscr{B}$ have only trivial idempotents is very strong and is satisfied only by some special triangular algebras. The important nest algebras and (block) upper triangular matrix algebras do not satisfy the assumption. So the assumption is not natural. Benkovič in [1] tried to remove the condition "$\mathscr{A}$ and $\mathscr{B}$ have only trivial idempotents" and proved that every $\mathscr{R}$-linear Jordan $\sigma$-derivation of $\mathscr{U}$ is the sum of a $\sigma$-derivation and a special map. Recently, Benkovič in [13] gave sufficient conditions that every $\mathscr{R}$-linear Lie $\sigma$-derivation of $\mathscr{U}$ is the sum of a $\sigma$-derivation and a $\sigma$-central valued linear map vanishing at commutators.

In this paper, we will continue to discuss additive $\xi$-Lie $\sigma$-derivations for all possible $\xi$ on triangular algebras $\mathscr{U}$ without the assumption "$\mathscr{A}$ and $\mathscr{B}$ have only trivial idempotents". In fact, we first discuss such maps in a more general setting, that is, characterize all additive maps $L$ on $\mathscr{U}$ satisfying $L([x,y]_\xi) = L(x)y - \xi\sigma(y)L(x) + \sigma(x)L(y) - \xi L(y)x$ for $x, y \in \mathscr{U}$ with $xy = x_0$, where $x_0 \in \mathscr{U}$ is any fixed element. Based on these results, complete characterizations of additive $\xi$-Lie $\sigma$-derivations on $\mathscr{U}$ for all possible $\xi$ are obtained, which generalize many known related results.

## PRELIMINARIES AND MAIN RESULTS

In this section, we will state our main results in this paper.

Firstly, we fix some notations. In the rest of the paper, assume that $\mathscr{A}$, $\mathscr{B}$ are unital algebras over a field $\mathbb{F}$ with characteristic 0 and $\mathscr{M}$ is a faithful $(\mathscr{A},\mathscr{B})$-bimodule, that is, for any $a \in \mathscr{A}$ and $b \in \mathscr{B}$, $a\mathscr{M} = \{0\} \Rightarrow a = 0$ and $\mathscr{M}b = \{0\} \Rightarrow b = 0$. Let $\mathscr{U} = \mathrm{Tri}(\mathscr{A},\mathscr{M},\mathscr{B})$ be the triangular algebra. Denote by $e = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$ and $f = 1 - e = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}$, where 1, $1_A$ and $1_B$ are units of $\mathscr{U}$, $\mathscr{A}$ and $\mathscr{B}$, respectively. Then $e$ and $f$ are two nontrivial idempotents of $\mathscr{U}$ and $\mathscr{U}$ can be decomposed as $\mathscr{U} = e\mathscr{U}e + e\mathscr{U}f + f\mathscr{U}f$. So for every $x \in \mathscr{U}$, $x$ can be written as $x = exe + exf + fxf$. It is obvious that the faithfulness of $\mathscr{M}$ can be restated as:

(i) for $x \in \mathscr{U}$, $exe \cdot e\mathscr{U}f = \{0\}$ implies $exe = 0$;
(ii) for $x \in \mathscr{U}$, $e\mathscr{U}f \cdot fxf = \{0\}$ implies $fxf = 0$.

Let $\sigma : \mathscr{U} \to \mathscr{U}$ be a ring automorphism. It is easily seen that $\sigma(1) = 1$ and $\sigma(e), \sigma(f) \in \mathscr{U}$ are two idempotents. Note that

$$\begin{aligned} \sigma(f)\mathscr{U}\sigma(e) &= \sigma(f)\sigma(\mathscr{U})\sigma(e) \\ &= \sigma(f\mathscr{U}e) = \sigma(\{0\}) = \{0\}. \end{aligned}$$

So we can represent $\mathscr{U}$ as a triangular algebra of the form $\mathscr{U} = \sigma(e)\mathscr{U}\sigma(e) + \sigma(e)\mathscr{U}\sigma(f) + \sigma(f)\mathscr{U}\sigma(f)$. Since $\sigma(e)\mathscr{U}\sigma(e)$ and $e\mathscr{U}e$ are isomorphic, $\sigma(e)\mathscr{U}\sigma(f)$ and $e\mathscr{U}f$ are isomorphic, and $\sigma(f)\mathscr{U}\sigma(f)$ and $f\mathscr{U}f$ are isomorphic, the above (i)–(ii), respectively, result in

(i′) for $x \in \mathscr{U}$, $\sigma(e)x\sigma(e) \cdot \sigma(e)\mathscr{U}\sigma(f) = \{0\} \Rightarrow \sigma(e)x\sigma(e) = 0$;

(ii′) for $x \in \mathscr{U}$, $\sigma(e)\mathscr{U}\sigma(f) \cdot \sigma(f)x\sigma(f) = \{0\} \Rightarrow \sigma(f)x\sigma(f) = 0$.

Furthermore, as $\sigma(e)xe = e\sigma(e)xe + f(\sigma(e)x)e = e\sigma(e)xe \in e\mathscr{U}e$ and $\sigma(f)xf = \sigma(f)xf\sigma(e) + \sigma(f)xf\sigma(f) = \sigma(f)(xf)\sigma(f) \in \sigma(f)\mathscr{U}\sigma(f)$, (i) and (ii′) yield

$$\begin{aligned} \sigma(e)xe \cdot e\mathscr{U}f = \{0\} &\Rightarrow \sigma(e)xe = 0 \\ \sigma(e)\mathscr{U}\sigma(f) \cdot \sigma(f)xf = \{0\} &\Rightarrow \sigma(f)xf = 0. \end{aligned} \tag{1}$$

Also note that each $x \in \mathscr{U}$ can be uniquely written in the form

$$\begin{aligned} x = 1x1 &= (\sigma(e) + \sigma(f))x(e + f) \\ &= \sigma(e)xe + \sigma(e)xf + \sigma(f)xe + \sigma(f)xf. \end{aligned}$$

This decomposition will be often used in this paper.

Denote by $\mathscr{Z}_\sigma(\mathscr{U})$ the set $\{x \in \mathscr{U} : xy = \sigma(y)x$ for all $y \in \mathscr{U}\}$. Obviously, $\mathscr{Z}_{id}(\mathscr{U})$ is the usual center of $\mathscr{U}$.

The following proposition gives a characterization about the set $\mathscr{Z}_\sigma(\mathscr{U})$.

**Proposition 1 (Proposition 2.5 in [13])** *Let $\mathscr{U} = \mathrm{Tri}(\mathscr{A},\mathscr{M},\mathscr{B})$ be the triangular algebra and $\sigma : \mathscr{U} \to \mathscr{U}$ a ring automorphism. Then $\mathscr{Z}_\sigma(\mathscr{U}) = \{x \in \mathscr{U} : \sigma(e)xf = \sigma(f)xe = 0$ and $\sigma(e)xeyf = \sigma(eyf)xf$ for all $y \in \mathscr{U}\}$.*

The following proposition is interesting itself.

**Proposition 2** *Let $\mathscr{U} = \mathrm{Tri}(\mathscr{A},\mathscr{M},\mathscr{B})$ be the triangular algebra and $\sigma : \mathscr{U} \to \mathscr{U}$ a ring automorphism. Assume that $x \in \mathscr{U}$. If $x$ satisfies $\sigma(e)xeyf = \sigma(eyf)xf$ for all $y \in \mathscr{U}$, then $\sigma(e)xe \in \mathscr{Z}_\sigma(e\mathscr{U}e)$ and $\sigma(f)xf \in \mathscr{Z}_\sigma(f\mathscr{U}f)$.*

*Proof*: For any $y, z \in \mathscr{U}$, by the assumption, we have

$$\begin{aligned} \sigma(e)xezeyf &= \sigma(ezeyf)xf = \sigma(eze)\sigma(eyf)xf \\ &= \sigma(eze)\sigma(e)xeyf = \sigma(e)\sigma(eze)\sigma(e)xeyf, \end{aligned}$$

that is, $\sigma(e)(xez - \sigma(eze)\sigma(e)x)e \cdot eyf = 0$ for all $y \in \mathscr{U}$. It follows from Eq. (1) that $\sigma(e)xeze = \sigma(eze)\sigma(e)xe$ holds for all $z \in \mathscr{U}$. So $\sigma(e)xe \in \mathscr{Z}_\sigma(e\mathscr{U}e)$.

Similarly, one can prove $\sigma(f)xf \in \mathscr{Z}_\sigma(f\mathscr{U}f)$. □

Now we begin to state our main results in this paper.

For the case $\xi \neq 1$, we have

**Theorem 1** *Let $\mathscr{U} = \mathrm{Tri}(\mathscr{A},\mathscr{M},\mathscr{B})$ be the triangular algebra and $\sigma : \mathscr{U} \to \mathscr{U}$ a ring automorphism. Let $x_0 \in \mathscr{U}$ is any fixed element and $\xi \in \mathbb{F}$ with $\xi \neq 1$. Suppose that, for every $a \in \mathscr{A}$ and $b \in \mathscr{B}$, there exist integers $n, m$ such that $n1_A - a$ and $m1_B - b$ invertible in $\mathscr{A}$ and $\mathscr{B}$, respectively. Assume that $L : \mathscr{U} \to \mathscr{U}$ is an additive map satisfying $L([x,y]_\xi) = L(x)y - \xi\sigma(y)L(x) + \sigma(x)L(y) - \xi L(y)x$ whenever $xy = x_0$ for $x, y \in \mathscr{U}$.*

(i) *If $\xi \neq 0$, then $L(1) \in \mathscr{Z}_\sigma(\mathscr{U})$ and there exists an additive Jordan $\sigma$-derivation $\Delta$ on $\mathscr{U}$ such that $L(x) = \Delta(x) + L(1)x$ for all $x \in \mathscr{U}$.*
(ii) *If $\xi = 0$ and $\sigma(f)L(1)e = 0$, then $L(1) \in \mathscr{Z}_\sigma(\mathscr{U})$ and there exists an additive Jordan $\sigma$-derivation $\Delta$ on $\mathscr{U}$ such that $L(x) = \Delta(x) + L(1)x$ holds for all $x \in \mathscr{U}$.*

If $ex_0f = 0$ in Theorem 1, then the condition $\sigma(f)L(1)e = 0$ for the case $\xi = 0$ can be omitted.

**Theorem 2** *Let $\mathscr{U} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$ be the triangular algebra and $\sigma : \mathscr{U} \to \mathscr{U}$ a ring automorphism. Assume that $L : \mathscr{U} \to \mathscr{U}$ is an additive map and $x_0 = ex_0e + fx_0f \in \mathscr{U}$ is some fixed element. Suppose that, for every $a \in \mathscr{A}$ and $b \in \mathscr{B}$, there exist integers $n, m$ such that $n1_A - a$ and $m1_B - b$ invertible in $\mathscr{A}$ and $\mathscr{B}$, respectively. If $L$ satisfies $L(xy) = L(x)y + \sigma(x)L(y)$ whenever $xy = x_0$ for $x, y \in \mathscr{U}$, then $L(1) \in \mathscr{Z}_\sigma(\mathscr{U})$ and there exists an additive Jordan $\sigma$-derivation $\Delta$ on $\mathscr{U}$ such that $L(x) = \Delta(x) + L(1)x$ for all $x \in \mathscr{U}$.*

In [1], the author gives a characterization of additive Jordan $\sigma$-derivations on $\mathscr{U}$ and discusses sufficient conditions that a Jordan $\sigma$-derivation on a triangular algebra becomes a $\sigma$-derivation. Thus, by Theorems 3.1 and 4.1 in [1] and Theorems 1–2, the following corollary is immediate, which is also a generalization of Theorems 3.1 and 4.1 in Ref.1.

**Corollary 1** *Let $\mathscr{U} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$ be a triangular algebra, $\sigma : \mathscr{U} \to \mathscr{U}$ be a ring automorphism, $x_0 \in \mathscr{U}$ is any fixed element and $\xi \in \mathbb{F}$. Suppose that for every $a \in \mathscr{A}$ and $b \in \mathscr{B}$, there exist integers $n, m$ such that $n1_A - a$ and $m1_B - b$ invertible in $\mathscr{A}$ and $\mathscr{B}$, respectively. Assume that $L : \mathscr{U} \to \mathscr{U}$ is an additive map satisfying $L([x, y]_\xi) = L(x)y - \xi\sigma(y)L(x) + \sigma(x)L(y) - \xi L(y)x$ whenever $xy = x_0$ for $x, y \in \mathscr{U}$.*

- (i) *If $\xi \neq 0$, then $L(1) \in \mathscr{Z}_\sigma(\mathscr{U})$, and there exists an additive $\sigma$-derivation $\Delta : \mathscr{U} \to \mathscr{U}$ and an additive map $g : \mathscr{U} \to \sigma(f)e\mathscr{U}e$ vanishing on $e\mathscr{U}e + f\mathscr{U}f$ such that $L(x) = \Delta(x) + L(1)x + g(x)$ for all $x \in \mathscr{U}$;*
- (ii) *if $\xi = 0$ and either $ex_0f = 0$ or $\sigma(f)L(1)e = 0$, then $L(1) \in \mathscr{Z}_\sigma(\mathscr{U})$, and there exists an additive $\sigma$-derivation $\Delta : \mathscr{U} \to \mathscr{U}$ and an additive map $g : \mathscr{U} \to \sigma(f)e\mathscr{U}e$ vanishing on $e\mathscr{U}e + f\mathscr{U}f$ such that $L(x) = \Delta(x) + L(1)x + g(x)$ for all $x \in \mathscr{U}$.*

*Furthermore, if $\mathscr{U}$ also satisfies any one of the following conditions (a)–(e), then $g \equiv 0$.*

- (a) *$e\mathscr{U}e$ is not of a triangular form;*
- (b) *$f\mathscr{U}f$ is not of a triangular form;*
- (c) *$e\mathscr{U}e = \mathrm{Id}([e\mathscr{U}e, e\mathscr{U}e])$, the ideal of $e\mathscr{U}e$ generated by all commutators of $e\mathscr{U}e$;*
- (d) *$f\mathscr{U}f = \mathrm{Id}([f\mathscr{U}f, f\mathscr{U}f])$;*
- (e) *$e\mathscr{U}f$ is a loyal $(\mathscr{A}, \mathscr{B})$-bimodule, that is, $a\mathscr{M}b = \{0\}$ implies $a = 0$ or $b = 0$.*

Here, an algebra $\mathscr{A}$ is not of a triangular algebra if for each idempotent $e \in \mathscr{A}$, the condition $(1-e)\mathscr{A}e = \{0\}$ implies that $e\mathscr{A}(1-e) = \{0\}$. Such algebras include commutative algebras, algebras having only trivial idempotents and semiprime algebras.

For the case $\xi = 1$, we obtain the following theorem.

**Theorem 3** *Let $\mathscr{U} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$ be the triangular algebra and $\sigma : \mathscr{U} \to \mathscr{U}$ a ring automorphism. Suppose that, for every $a \in \mathscr{A}$ and $b \in \mathscr{B}$, there exist integers $n, m$ such that $n1_A - a$ and $m1_B - b$ invertible in $\mathscr{A}$ and $\mathscr{B}$, respectively; $e\mathscr{Z}_\sigma(\mathscr{U})e = \mathscr{Z}_\sigma(e\mathscr{U}e)$ and $f\mathscr{Z}_\sigma(\mathscr{U})f = \mathscr{Z}_\sigma(f\mathscr{U}f)$. Assume that $L : \mathscr{U} \to \mathscr{U}$ is an additive map and $x_0 = ex_0e \in \mathscr{U}$ (resp. $x_0 = fx_0f \in \mathscr{U}$) is some fixed element. Then $L$ satisfies*

$$L([x, y]) = L(x)y - \sigma(y)L(x) + \sigma(x)L(y) - L(y)x$$

*whenever $xy = x_0$ for $x, y \in \mathscr{U}$ if and only if $L(x) = \Delta(x) + h(x)$ for all $x \in \mathscr{U}$, where $\Delta$ is an additive $\sigma$-derivation on $\mathscr{U}$ and $h : \mathscr{U} \to \mathscr{Z}_\sigma(\mathscr{U})$ is an additive map satisfying $h([x, y]) = 0$ for $x, y \in \mathscr{U}$ with $xy = x_0$.*

Note that, if $L$ is an additive $\xi$-Lie $\sigma$-derivation, then $L$ must be satisfy the condition about $L$ in the above theorems. Also observe that, by the proofs in the next section, the condition "for every $a \in \mathscr{A}$ and $b \in \mathscr{B}$, there exist integers $n, m$ such that $n1_A - a$ and $m1_B - b$ are invertible in $\mathscr{A}$ and $\mathscr{B}$, respectively" can be deleted if $L$ is a $\xi$-Lie $\sigma$-derivation. Hence, by Theorems 1–3, Corollary 1, and their proofs, we can give a complete characterization of additive $\xi$-Lie $\sigma$-derivations on $\mathscr{U}$.

**Theorem 4** *Let $\mathscr{U} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$ be a triangular algebra, $\sigma : \mathscr{U} \to \mathscr{U}$ be a ring automorphism and $\xi \in \mathbb{F}$. Assume that $L : \mathscr{U} \to \mathscr{U}$ is an additive $\xi$-Lie $\sigma$-derivation.*

- (i) *If $\xi \neq 1$ and $\mathscr{U}$ satisfies any one of (a)–(e) in Corollary 1, then $L$ is an additive $\sigma$-derivation;*
- (ii) *if $\xi = 1$, $e\mathscr{Z}_\sigma(\mathscr{U})e = \mathscr{Z}_\sigma(e\mathscr{U}e)$ and $f\mathscr{Z}_\sigma(\mathscr{U})f = \mathscr{Z}_\sigma(f\mathscr{U}f)$, then $L(x) = \Delta(x) + h(x)$ for all $x \in \mathscr{U}$, where $\Delta : \mathscr{U} \to \mathscr{U}$ is an additive $\sigma$-derivation and $h : \mathscr{U} \to \mathscr{Z}_\sigma(\mathscr{U})$ is an additive map vanishing at all commutators.*

**Remark 1** Note that $\sigma(f)\mathscr{U}e = f\mathscr{U}e = \{0\}$ if $\sigma = id$. Thus the map $g$ in Corollary 1 is a zero map. So Corollary 1 and Theorems 3–4 are generalizations of corresponding results in [1, 8, 13, 14], respectively.

**Remark 2** Yang and Zhu [10] introduced more general concepts: $\xi$-Lie $(\alpha, \beta)$-derivations. Let $\mathscr{A}$ be a unital algebra over a field $\mathbb{F}$ and $\alpha, \beta$ be two automorphisms of $\mathscr{A}$. An additive map $L$ on $\mathscr{A}$ is called a $\xi$-Lie $(\alpha, \beta)$-derivation if $L([x, y]_\xi) = L(x)\alpha(y) - \xi\beta(y)L(x) + \beta(x)L(y) - \xi L(y)\alpha(x)$ for all $x, y \in \mathscr{A}$ (see also [12]). We remark that $\xi$-Lie $(\alpha, \beta)$-derivations can be reduced to $\xi$-Lie $\sigma$-derivations. Indeed, define $\sigma = \alpha^{-1} \circ \beta$. If $\alpha(\xi 1) = \xi 1$, then $L : \mathscr{A} \to \mathscr{A}$ is an additive $\xi$-Lie $(\alpha, \beta)$-derivation if and only if $L' = \alpha^{-1} \circ L$ is an additive $\xi$-Lie $\sigma$-derivation. So our results can be used to characterize additive $\xi$-Lie $(\alpha, \beta)$-derivations on triangular algebras in [10]. Since the condition "$\mathscr{A}$ and $\mathscr{B}$ have only trivial idempotents" is assumed in [10], our theorems also generalize all the corresponding results in [10].

At the end of this section, we give some applications of our results to some special algebras. For the simpleness, we only list the application of Theorem 4.

Recall that a nest $\mathscr{N}$ on a Banach space $X$ is a collection of closed (under norm topology) subspaces of $X$ containing $\{0\}$ and $X$, which is a chain under the inclusion relation, and is closed under the formation of arbitrary closed linear span (denoted by $\bigvee$) and intersection (denoted by $\bigwedge$). The nest algebra associated to the nest $\mathscr{N}$, denoted by $\mathrm{Alg}\mathscr{N}$, is the weakly closed operator algebra consisting of all operators that leave $\mathscr{N}$ invariant, i.e., $\mathrm{Alg}\mathscr{N} = \{T \in \mathscr{B}(X) : TN \subseteq N \text{ for all } N \in \mathscr{N}\}$. When $\mathscr{N} \neq \{\{0\}, X\}$, we say that $\mathscr{N}$ is non-trivial. Note that $\mathrm{Alg}\mathscr{N} = \mathscr{B}(X)$ if the nest $\mathscr{N}$ is trivial.

If $\mathscr{N}$ contains a non-trivial element $N_1$ complemented in $X$, then there exists an idempotent operator $E$ with $\mathrm{ran}(E) = N_1 \in \mathscr{N}$. It is clear that $E \in \mathrm{Alg}\mathscr{N}$. Decompose $X$ into the direct sum $X = \mathrm{ran}(E) \oplus \ker E$. With respect to this decomposition, we have $E = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. Let $\mathscr{N}_E = \{N \cap N_1 : \forall N \in \mathscr{N}\}$ and $\mathscr{N}_{I-E} = \{N \cap \ker E : \forall N \in \mathscr{N}\}$. Then $\mathscr{N}_E$ and $\mathscr{N}_{I-E}$ are nests on Banach spaces $N_1$ and $\ker E$, respectively. Thus, $E(\mathrm{Alg}\mathscr{N})E|_{N_1} = \mathrm{Alg}(\mathscr{N}_E)$, $(I-E)(\mathrm{Alg}\mathscr{N})(I-E)|_{\ker E} = \mathrm{Alg}(\mathscr{N}_{I-E})$ and

$$\mathrm{Alg}\mathscr{N} = \left\{ \begin{pmatrix} C & W \\ 0 & D \end{pmatrix} : \ C \in \mathrm{Alg}(\mathscr{N}_E), \ W \in \mathscr{B}(\ker E, \mathrm{ran}(E)), \ D \in \mathrm{Alg}(\mathscr{N}_{I-E}) \right\}.$$

It is easy to prove that $\mathscr{B}(\ker E, \mathrm{ran}(E))$ is a loyal $(\mathrm{Alg}(\mathscr{N}_E), \mathrm{Alg}(\mathscr{N}_{I-E}))$-bimodule (for example, see [8, 15]).

So, applying Theorem 4 to nest algebras, we have the following corollary.

**Corollary 2** *Let $X$ be a Banach space over the real or complex field $\mathbb{F}$ with dimension greater than 2. Let $\mathscr{N}$ be a nest on $X$ which contains a nontrivial element $N_1$ complemented in $X$. Let $\sigma : \mathrm{Alg}\mathscr{N} \to \mathrm{Alg}\mathscr{N}$ be an automorphism and $\xi \in \mathbb{F}$. Assume that $L : \mathrm{Alg}\mathscr{N} \to \mathrm{Alg}\mathscr{N}$ is an additive $\xi$-Lie $\sigma$-derivation.*

(i) *If $\xi \neq 1$, then $L$ is an additive $\sigma$-derivation.*

(ii) *If $\xi = 1$ and $\mathscr{Z}_\sigma(E(\mathrm{Alg}\mathscr{N})E) = E\mathscr{Z}_\sigma(\mathrm{Alg}\mathscr{N})E$, $\mathscr{Z}_\sigma((I-E)\mathrm{Alg}\mathscr{N}(I-E)) = (I-E)\mathscr{Z}_\sigma(\mathrm{Alg}\mathscr{N})(I-E)$, then $L(A) = \Delta(A) + h(A)$ for all $A \in \mathrm{Alg}\mathscr{N}$, where $\Delta$ is an additive $\sigma$-derivation on $\mathrm{Alg}\mathscr{N}$ and $h : \mathrm{Alg}\mathscr{N} \to \mathbb{F}I$ is an additive map vanishing at all commutators.*

Particularly, if $\sigma = id$, then the condition "$\mathscr{Z}_\sigma(E(\mathrm{Alg}\mathscr{N})E) = E\mathscr{Z}_\sigma(\mathrm{Alg}\mathscr{N})E$, $\mathscr{Z}_\sigma((I-E)\ \mathrm{Alg}\ \mathscr{N}\ (I-E)) = (I-E)\mathscr{Z}_\sigma(\mathrm{Alg}\mathscr{N})(I-E)$" is superfluous as the center of any nest algebras is $\mathbb{F}I$.

Let $H$ be a Hilbert space over a complex field $\mathbb{C}$ and $\dim H > 1$. Assume that $\mathscr{B}(H)$ is the algebra of bounded linear operators on $H$. It is well known that $\mathscr{B}(H) = \mathrm{Id}([\mathscr{B}(H), \mathscr{B}(H)])$. Thus, applying Theorem 4, the following result is obvious.

**Corollary 3** *Let $\mathscr{M}$ be any faithful $(\mathscr{B}(H), \mathscr{B}(H))$-bimodule and $\mathscr{U} = \mathrm{Tri}(\mathscr{B}(H), \mathscr{M}, \mathscr{B}(H))$ the triangular algebra. Assume that $\sigma : \mathscr{U} \to \mathscr{U}$ is a ring automorphism, $\xi \in \mathbb{C}$, and $L : \mathscr{U} \to \mathscr{U}$ is an additive $\xi$-Lie $\sigma$-derivation.*

(i) *If $\xi \neq 1$, then $L$ is an additive $\sigma$-derivation;*

(ii) *if $\xi = 1$, $e\mathscr{Z}_\sigma(\mathscr{U})e = \mathscr{Z}_\sigma(e\mathscr{U}e)$ and $f\mathscr{Z}_\sigma(\mathscr{U})f = \mathscr{Z}_\sigma(f\mathscr{U}f)$, then $L(x) = \Delta(x) + h(x)$ for all $x \in \mathscr{U}$, where $\Delta : \mathscr{U} \to \mathscr{U}$ is an additive $\sigma$-derivation and $h : \mathscr{U} \to \mathscr{Z}_\sigma(\mathscr{U})$ is an additive map vanishing at all commutators.*

Finally, we give an example to illustrate that assumptions $e\mathscr{Z}_\sigma(\mathscr{U})e = \mathscr{Z}_\sigma(e\mathscr{U}e)$ and $f\mathscr{Z}_\sigma(\mathscr{U})f = \mathscr{Z}_\sigma(f\mathscr{U}f)$ are necessary.

**Example 1** Let $\mathscr{T}_2(\mathbb{R})$ be an upper triangular matrix algebra over a real field $\mathbb{R}$ and let

$$\mathscr{A} = \left\{ \begin{pmatrix} t & s \\ 0 & t \end{pmatrix} : \ t, s \in \mathbb{R} \right\}.$$

Clearly, $\mathscr{A}$ is a subalgebra of $\mathscr{T}_2(\mathbb{R})$, and for every $A \in \mathscr{A}$, there is some integer $n$ such that $n1_A - A$ is invertible in $\mathscr{A}$. Define $\mathscr{U} = \begin{pmatrix} \mathscr{A} & \mathscr{T}_2(\mathbb{R}) \\ 0 & \mathscr{A} \end{pmatrix}$. Then $\mathscr{U}$ is a triangular algebra. Now, define two maps $\sigma, L : \mathscr{U} \to \mathscr{U}$ respectively by

$$\sigma \begin{pmatrix} t_1 & s_1 & x & z \\ 0 & t_1 & 0 & y \\ 0 & 0 & t_2 & s_2 \\ 0 & 0 & 0 & t_2 \end{pmatrix} = \begin{pmatrix} t_1 & -s_1 & -x & z \\ 0 & t_1 & 0 & -y \\ 0 & 0 & t_2 & -s_2 \\ 0 & 0 & 0 & t_2 \end{pmatrix}$$

and

$$L \begin{pmatrix} t_1 & s_1 & x & z \\ 0 & t_1 & 0 & y \\ 0 & 0 & t_2 & s_2 \\ 0 & 0 & 0 & t_2 \end{pmatrix} = \begin{pmatrix} 0 & s_2 & y & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & s_1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\begin{pmatrix} t_1 & s_1 \\ 0 & t_1 \end{pmatrix}, \begin{pmatrix} t_2 & s_2 \\ 0 & t_2 \end{pmatrix} \in \mathscr{A}$ and $\begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \in \mathscr{T}_2(\mathbb{R})$ are arbitrary. By direct calculations, $\sigma$ is an additive automorphism and $L$ is an additive Lie $\sigma$-derivation. In addition, it is easily checked that $e\mathscr{Z}_\sigma(\mathscr{U})e \neq \mathscr{Z}_\sigma(e\mathscr{U}e)$ since

$$\mathscr{Z}_\sigma(\mathscr{U}) = \{0\} \ \text{ and } \ e\mathscr{Z}_\sigma(\mathscr{U})e = \begin{pmatrix} 0 & \mathbb{R} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

However, by [13], $L$ is not of the form in Theorem 4.

**PROOFS OF MAIN RESULTS**

In this section, we will give the proofs of Theorems 1–4.

In the rest of this paper, we always assume that $\mathscr{U} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$ is the triangular algebra, which satisfies, for every $a \in \mathscr{A}$ and $b \in \mathscr{B}$, there exist integers $n, m$ such that $n1_A - a$ and $m1_B - b$ invertible in $\mathscr{A}$ and $\mathscr{B}$, respectively. Suppose that $\sigma : \mathscr{U} \to \mathscr{U}$ is a ring automorphism, $x_0 \in \mathscr{U}$ is any fixed element, $\xi \in \mathbb{F}$, and $L$ is an additive map satisfying

$$L([x, y]_\xi) = L(x)y - \xi\sigma(y)L(x) + \sigma(x)L(y) - \xi L(y)x$$

whenever $xy = x_0$ for $x, y \in \mathscr{U}$.

**Lemma 1** *The following statements hold:*
(i) $\sigma(e)L(1)f = 0$*;*
(ii) *if* $\xi \neq 0$*, then* $\sigma(f)L(1)e = 0$*;*
(iii) *if* $\xi \neq 1$*, then* $\sigma(f)L(exe)f = \sigma(e)L(fxf)e = 0$ *for all* $x \in \mathscr{U}$*;*
(iv) *if* $\xi = 1$*, then* $\sigma(f)L(e\mathscr{U}e)f \subseteq \mathscr{Z}_\sigma(f\mathscr{U}f)$ *and* $\sigma(e)L(f\mathscr{U}f)e \subseteq \mathscr{Z}_\sigma(e\mathscr{U}e)$*.*

*Proof*: For any nonzero $t = n1 \in \mathbb{F}$ with $n$ any integer, any invertible element $exe \in e\mathscr{U}e$ and any $fy_1f, fy_2f \in f\mathscr{U}f$ with $fy_1fy_2f = fx_0f$, since $(exe + tfy_1f)((exe)^{-1}x_0 + t^{-1}fy_2f) = x_0$, we have

$$\begin{aligned}
&L(x_0) - L(\xi(exe)^{-1}x_0exe) - L(\xi t(exe)^{-1}x_0fy_1f)\\
&-L(\xi fy_2fy_1f) = L(exe + tfy_1f)((exe)^{-1}x_0 + t^{-1}fy_2f)\\
&\qquad - \xi\sigma((exe)^{-1}x_0 + t^{-1}fy_2f)L(exe + tfy_1f)\\
&\qquad + \sigma(exe + tfy_1f)L((exe)^{-1}x_0 + t^{-1}fy_2f)\\
&\qquad - \xi L((exe)^{-1}x_0 + t^{-1}fy_2f)(exe + tfy_1f),
\end{aligned}$$

that is,

$$\begin{aligned}
&L(x_0) - L(\xi(exe)^{-1}x_0exe) - tL(\xi(exe)^{-1}x_0fy_1f)\\
&- L(\xi fy_2fy_1f) = L(exe)(exe)^{-1}x_0 + t^{-1}L(exe)fy_2f\\
&+ tL(fy_1f)(exe)^{-1}x_0 + L(fy_1f)fy_2f\\
&- \xi\sigma((exe)^{-1}x_0)L(exe) - \xi t^{-1}\sigma(fy_2f)L(exe)\\
&- \xi t\sigma((exe)^{-1}x_0)L(fy_1f) - \xi\sigma(fy_2f)L(fy_1f)\\
&+ \sigma(exe)L((exe)^{-1}x_0) + t^{-1}\sigma(exe)L(fy_2f)\\
&+ t\sigma(fy_1f)L((exe)^{-1}x_0) + \sigma(fy_1f)L(fy_2f)\\
&- \xi L((exe)^{-1}x_0)exe - \xi t^{-1}L(fy_2f)exe\\
&- \xi tL((exe)^{-1}x_0)fy_1f - \xi L(fy_2f)fy_1f. \qquad (2)
\end{aligned}$$

It follows that $L(exe)fy_2f - \xi\sigma(fy_2f)L(exe) + \sigma(exe)L(fy_2f) - \xi L(fy_2f)exe = 0$ holds for all invertible $exe \in e\mathscr{U}e$ and all invertible $fy_2f \in f\mathscr{U}f$ (in this case, $fy_1f = fx_0f(fy_2f)^{-1}$). Note that, for every $exe \in e\mathscr{U}e$ and $fyf \in f\mathscr{U}f$, there exist integers $n, m$ such that $ne - exe$ and $mf - fyf$ are invertible in $e\mathscr{U}e$ and $f\mathscr{U}f$, respectively. Then, the additivity of $L$ implies

$$\begin{aligned}
&L(exe)fyf - \xi\sigma(fyf)L(exe)\\
&\qquad + \sigma(exe)L(fyf) - \xi L(fyf)exe = 0 \qquad (3)
\end{aligned}$$

for all $x, y \in \mathscr{U}$.

Multiplying by $\sigma(f)$ and $f$ from the left and the right in Eq. (3), respectively, one gets $\sigma(f)L(exe)fyf - \xi\sigma(fyf)L(exe)f = 0$ for all $x, y \in \mathscr{U}$. If $\xi = 1$, then $\sigma(f)L(exe)f \cdot fyf = \sigma(fyf) \cdot \sigma(f)L(exe)f$ for all $x, y \in \mathscr{U}$, which implies $\sigma(f)L(exe)f \in \mathscr{Z}_\sigma(f\mathscr{U}f)$; if $\xi \neq 1$, by taking $y = f$, we obtain $(1-\xi)\sigma(f)L(exe)f = 0$, and so $\sigma(f)L(exe)f = 0$.

Similarly, multiplying by $\sigma(e)$ and $e$ from the left and the right in Eq. (3), respectively, one can show $\sigma(e)L(fyf)e = 0$ if $\xi \neq 1$, and $\sigma(e)L(f\mathscr{U}f)e \subseteq \mathscr{Z}_\sigma(e\mathscr{U}e)$ if $\xi = 1$.

Multiplying by $\sigma(e)$ and $f$ from the left and the right in Eq. (3), respectively, one has $\sigma(e)L(exe)fyf + \sigma(exe)L(fyf)f = 0$, and so

$$\begin{aligned}
\sigma(e)L(1)f &= 0,\\
\sigma(e)L(exe)f &= -\sigma(exe)\sigma(e)L(f)f, \qquad (4)\\
\sigma(e)L(fyf)f &= -\sigma(e)L(e)fyf;
\end{aligned}$$

multiplying by $\sigma(f)$ and $e$ from the left and the right in Eq. (3), respectively, one can obtain $\xi(\sigma(fyf)L(exe)e + \sigma(f)L(fyf)exe) = 0$, and so

$$\begin{aligned}
\sigma(f)L(1)e &= 0,\\
\sigma(f)L(exe)e &= -\sigma(f)L(f)exe, \qquad (5)\\
\sigma(f)L(fyf)e &= -\sigma(fyf)\sigma(f)L(e)e
\end{aligned}$$

if $\xi \neq 0$. The proof of the lemma is finished. □

Next, define a map $\delta : \mathscr{U} \to \mathscr{U}$ by

$$\delta(x) = L(x) - (\sigma(x)y_0 - y_0x) \ \text{ for all } \ x \in \mathscr{U},$$

where $y_0 = \sigma(e)L(e)f - \sigma(f)L(e)e$. Clearly, the map $x \mapsto \sigma(x)y_0 - y_0x$ for some fixed $y_0$ is a $\sigma$-derivation. So it is easy to check that $\delta$ is an additive map satisfying

$$\delta(xy - \xi yx) = \delta(x)y - \xi\sigma(y)\delta(x) + \sigma(x)\delta(y) - \xi\delta(y)x$$

whenever $xy = x_0$ for $x, y \in \mathscr{U}$. In addition, if $\sigma(f)L(1)e = 0$ when $\xi = 0$, then, by Lemma 1, $\delta$ also satisfies

$$\begin{cases}
\begin{cases}\delta(e) \in \sigma(e)\mathscr{U}e,\ \delta(f) \in \sigma(f)\mathscr{U}f,\\ \delta(1) \in \sigma(e)\mathscr{U}e + \sigma(f)\mathscr{U}f\end{cases} & \text{if } \xi \neq 1;\\
\delta(e), \delta(f), \delta(1) \in \sigma(e)\mathscr{U}e + \sigma(f)\mathscr{U}f & \text{if } \xi = 1.
\end{cases} \qquad (6)$$

Moreover, it is easily seen that Lemma 1 and Eqs. (4)–(5) still hold for the map $\delta$. Thus, we can achieve

$$\begin{cases}
\delta(e\mathscr{U}e) \subseteq \sigma(e)\mathscr{U}e,\ \delta(f\mathscr{U}f) \subseteq \sigma(f)\mathscr{U}f & \text{if } \xi \neq 1;\\
\begin{cases}\delta(e\mathscr{U}e),\ \delta(f\mathscr{U}f) \subseteq \sigma(e)\mathscr{U}e + \sigma(f)\mathscr{U}f\\ \sigma(f)\delta(e\mathscr{U}e)f \subseteq \mathscr{Z}_\sigma(f\mathscr{U}f),\\ \sigma(e)\delta(f\mathscr{U}f)e \subseteq \mathscr{Z}_\sigma(e\mathscr{U}e)\end{cases} & \text{if } \xi = 1.
\end{cases} \qquad (7)$$

Next, we will discuss the map $\delta$.

Take any integers $p, q$ and let $s = p1, t = q1 \in \mathbb{F}$. For any $x, y, z_1z_2 \in \mathscr{U}$ with $exe \in e\mathscr{U}e$ invertible in

$e\mathcal{U}e$ and $fz_1fz_2f = fx_0f$, let $v = exe + t(sexeyf + fz_1f)$ and $w = (exe)^{-1}x_0 - seyfz_2f + t^{-1}fz_2f$. Since $vw = x_0$, we have

$$\delta([v,w]_\xi) = \delta(v)w - \xi\sigma(w)\delta(v) + \sigma(v)\delta(w) - \xi\delta(w)v,$$

that is,

$$\begin{aligned}
&\delta(x_0) - \delta(\xi(exe)^{-1}x_0exe) - ts\delta(\xi(exe)^{-1}x_0exeyf)\\
&-t\delta(\xi(exe)^{-1}x_0fz_1f) + st\delta(\xi eyfz_2fz_1f) - \delta(\xi fz_2fz_1f)\\
&= \delta(exe)(exe)^{-1}x_0 - s\delta(exe)eyfz_2f + t^{-1}\delta(exe)fz_2f\\
&+ ts\delta(exeyf)(exe)^{-1}x_0 - ts^2\delta(exeyf)eyfz_2f\\
&+ s\delta(exeyf)fz_2f + t\delta(fz_1f)(exe)^{-1}x_0\\
&- ts\delta(fz_1f)eyfz_2f + \delta(fz_1f)fz_2f\\
&- \xi\sigma((exe)^{-1}x_0)\delta(exe) + \xi s\sigma(eyfz_2f)\delta(exe)\\
&- \xi t^{-1}\sigma(fz_2f)\delta(exe) - \xi ts\sigma((exe)^{-1}x_0)\delta(exeyf)\\
&+ \xi ts^2\sigma(eyfz_2f)\delta(exeyf) - \xi s\sigma(fz_2f)\delta(exeyf)\\
&- \xi t\sigma((exe)^{-1}x_0)\delta(fz_1f) + \xi ts\sigma(eyfz_2f)\delta(fz_1f)\\
&- \xi\sigma(fz_2f)\delta(fz_1f) + \sigma(exe)\delta((exe)^{-1}x_0)\\
&- s\sigma(exe)\delta(eyfz_2f) + t^{-1}\sigma(exe)\delta(fz_2f)\\
&+ ts\sigma(exeyf)\delta((exe)^{-1}x_0) - ts^2\sigma(exeyf)\delta(eyfz_2f)\\
&+ s\sigma(exeyf)\delta(fz_2f) + t\sigma(fz_1f)\delta((exe)^{-1}x_0)\\
&- ts\sigma(fz_1f)\delta(eyfz_2f) + \sigma(fz_1f)\delta(fz_2f)\\
&- \xi\delta((exe)^{-1}x_0)exe + \xi s\delta(eyfz_2f)exe\\
&- \xi t^{-1}\delta(fz_2f)exe - \xi ts\delta((exe)^{-1}x_0)exeyf\\
&- \xi s\delta(fz_2f)exeyf + \xi ts^2\delta(eyfz_2f)exeyf\\
&- \xi t\delta((exe)^{-1}x_0)fz_1f + \xi ts\delta(eyfz_2f)fz_1f\\
&- \xi\delta(fz_2f)fz_1f. \qquad (8)
\end{aligned}$$

Comparing the terms of coefficients $s$ and $ts^2$ in Eq. (8) gives

$$\begin{aligned}
0 &= \delta(exe)eyfz_2f - \xi\sigma(eyfz_2f)\delta(exe)\\
&+ \sigma(exe)\delta(eyfz_2f) - \xi\delta(eyfz_2f)exe - \delta(exeyf)fz_2f\\
&+ \xi\sigma(fz_2f)\delta(exeyf) - \sigma(exeyf)\delta(fz_2f)\\
&+ \xi\delta(fz_2f)exeyf \qquad (9)
\end{aligned}$$

and

$$\begin{aligned}
0 &= \delta(exeyf)eyfz_2f - \xi\sigma(eyfz_2f)\delta(exeyf)\\
&+ \sigma(exeyf)\delta(eyfz_2f) - \xi\delta(eyfz_2f)exeyf \qquad (10)
\end{aligned}$$

for all $x, y, z_1z_2 \in \mathcal{U}$ with $exe \in e\mathcal{U}e$ invertible and $fz_1fz_2f = fx_0f$.

Now, since for every $x \in \mathcal{U}$, there exist integers $n, m$ such that $ne - exe$ and $mf - fxf$ are invertible in $e\mathcal{U}e$ and $f\mathcal{U}f$, respectively. It is easy to check that Eqs. (9)–(10) are true for all $x, y, z_2 \in \mathcal{U}$.

**Lemma 2** *For any $\xi \in \mathbb{F}$, we have $\delta(1) \in \mathscr{Z}_\sigma(\mathcal{U})$.*

*Proof*: By taking $x = e$ and $z_2 = f$ in Eq. (9), we obtain

$$\begin{aligned}
0 &= \delta(e)eyf - \xi\sigma(eyf)\delta(e) + \sigma(e)\delta(eyf) - \xi\delta(eyf)e\\
&- \delta(eyf)f + \xi\sigma(f)\delta(eyf) - \sigma(eyf)\delta(f) + \xi\delta(f)eyf. \qquad (11)
\end{aligned}$$

Multiplying by $\sigma(e)$ and $f$ from the left and the right in Eq. (11), respectively, one gets

$$\begin{aligned}
&\sigma(e)\delta(e)eyf + \xi\sigma(e)\delta(f)eyf\\
&= \xi\sigma(eyf)\delta(e)f + \sigma(eyf)\delta(f)f \text{ for all } y \in \mathcal{U}.
\end{aligned}$$

If $\xi = 1$, then the above equation becomes $\sigma(e)\delta(1)eyf = \sigma(eyf)\delta(1)f$.

If $\xi \neq 1$, by Eq. (7), $\sigma(f)\delta(e)f = \sigma(e)\delta(f)e = 0$; and thus the above equation reduces to $\sigma(e)\delta(e)eyf = \sigma(eyf)\delta(f)f$, which also implies

$$\begin{aligned}
\sigma(e)\delta(1)eyf &= (\sigma(e)\delta(e)e + \sigma(e)\delta(f)e)eyf\\
&= \sigma(eyf)(\sigma(f)\delta(e)f + \sigma(f)\delta(f)f)\\
&= \sigma(eyf)\delta(1)f.
\end{aligned}$$

Now, it follows from Proposition 1 and Eq. (6) that $\delta(1) = \sigma(e)\delta(1)e + \sigma(f)\delta(1)f \in \mathscr{Z}_\sigma(\mathcal{U})$, completing the proof the lemma. □

**Lemma 3** *Assume that $\xi \neq 1$. Then for any $y \in \mathcal{U}$, we have $\delta(eyf)eyf + \sigma(eyf)\delta(eyf) = 0$.*

*Proof*: Letting $x = e$ and $z_2 = f$ in Eq. (10), we have

$$\begin{aligned}
0 &= \delta(eyf)eyf - \xi\sigma(eyf)\delta(eyf)\\
&+ \sigma(eyf)\delta(eyf) - \xi\delta(eyf)eyf,
\end{aligned}$$

which implies $\delta(eyf)eyf + \sigma(eyf)\delta(eyf) = 0$ as $\xi \neq 1$. □

**Lemma 4** *Assume that $\xi \neq 1$ and $\sigma(f)\delta(1)e = 0$ if $\xi = 0$. Then the following statements hold:*
(i) *$\delta(eyf) = \sigma(e)\delta(eyf)f + \sigma(f)\delta(eyf)e$ for all $y \in \mathcal{U}$;*
(ii) *$\sigma(f)\delta(exeyf)e = \sigma(f)\delta(eyf)exe$ and $\sigma(e)\delta(exeyf)f = \sigma(e)\delta(exe)eyf + \sigma(exe)\delta(eyf)f - \sigma(exeyf)\delta(f)f$ for all $x, y \in \mathcal{U}$;*
(iii) *$\sigma(f)\delta(eyfzf)e = \sigma(fzf)\delta(eyf)e$ and $\sigma(e)\delta(eyfzf)f = \sigma(eyf)\delta(fzf)f + \sigma(e)\delta(eyf)fzf - \sigma(e)\delta(e)eyf$ for all $y, z \in \mathcal{U}$;*
(iv) *$\delta(ex_1ex_2e) = \delta(ex_1e)ex_2e + \sigma(ex_1e)\delta(ex_2e) - \sigma(ex_1ex_2e)\delta(e)$ for all $x_1, x_2 \in \mathcal{U}$;*
(v) *$\delta(fz_1fz_2f) = \delta(fz_1f)fz_2f + \sigma(fz_1f)\delta(fz_2f) - \delta(f)fz_1fz_2f$ for all $z_1, z_2 \in \mathcal{U}$.*

*Proof*: Assume that $\xi \neq 1$ and $\sigma(f)\delta(1)e = 0$ if $\xi = 0$. By Eqs.(6)–(7), we have proved that

$$\begin{aligned}
&\delta(e\mathcal{U}e) \subseteq \sigma(e)\mathcal{U}e,\ \delta(f\mathcal{U}f) \subseteq \sigma(f)\mathcal{U}f\\
&\text{and } \delta(1) \in \sigma(e)\mathcal{U}e + \sigma(f)\mathcal{U}f. \qquad (12)
\end{aligned}$$

Firstly, note that Eq. (11) holds. Multiplying by $\sigma(e)$ and $e$ from the left and the right in Eq. (11), respectively, and by Eq. (12), one gets $\sigma(e)\delta(eyf)e = 0$; multiplying by $\sigma(f)$ and $f$ from the left and the right in Eq.(11), respectively, and by Eq. (12) again, one has $\sigma(f)\delta(eyf)f = 0$. So (i) is true.

Next, taking $z_2 = f$ in Eq.(9) gives

$$\begin{aligned}0 &= \delta(exe)eyf - \xi\sigma(eyf)\delta(exe) + \sigma(exe)\delta(eyf)\\ &\quad - \xi\delta(eyf)exe - \delta(exeyf)f + \xi\sigma(f)\delta(exeyf)\\ &\quad - \sigma(exeyf)\delta(f) + \xi\delta(f)exeyf\\ &= \sigma(e)\delta(exe)eyf + \sigma(exe)\delta(eyf)f - \xi\sigma(f)\delta(eyf)exe\\ &\quad - \sigma(e)\delta(exeyf)f + \xi\sigma(f)\delta(exeyf)e - \sigma(exeyf)\delta(f)f,\end{aligned}$$

which implies

$$\sigma(f)\delta(exeyf)e = \sigma(f)\delta(eyf)exe$$

and

$$\sigma(e)\delta(exeyf)f = \sigma(e)\delta(exe)eyf + \sigma(exe)\delta(eyf)f - \sigma(exeyf)\delta(f)f$$

for all $x, y \in \mathscr{U}$. So, for any $x_1, x_2 \in \mathscr{U}$, we have

$$\sigma(e)\delta(ex_1ex_2eyf)f = \sigma(e)\delta(ex_1ex_2e)eyf + \sigma(ex_1ex_2e)\delta(eyf)f - \sigma(ex_1ex_2eyf)\delta(f)f$$

and

$$\begin{aligned}&\sigma(e)\delta(ex_1ex_2eyf)f\\ &= \sigma(e)\delta(ex_1e)ex_2eyf + \sigma(ex_1e)\delta(ex_2eyf)f\\ &\quad - \sigma(ex_1ex_2eyf)\delta(f)f\\ &= \sigma(e)\delta(ex_1e)ex_2eyf + \sigma(ex_1e)\delta(ex_2e)eyf\\ &\quad + \sigma(ex_1e)\sigma(ex_2e)\delta(eyf)f - \sigma(ex_1e)\sigma(ex_2eyf)\delta(f)f\\ &\quad - \sigma(ex_1ex_2eyf)\delta(f)f.\end{aligned}$$

Comparing the two equations above yields

$$(\delta(ex_1ex_2e) - \delta(ex_1e)ex_2e - \sigma(ex_1e)\delta(ex_2e) + \sigma(ex_1ex_2e)\delta(e))eyf = 0$$

for all $y \in \mathscr{U}$ by using Eq. (12) and Lemma 2. It follows from Eq. (1) that $\delta(ex_1ex_2e) - \delta(ex_1e)ex_2e - \sigma(ex_1e)\delta(ex_2e) + \sigma(ex_1ex_2e)\delta(e) = 0$. Hence (ii) and (iv) hold.

Finally, by taking $x = e$ in Eq. (9), analogously to what was done above, one can obtain

$$\sigma(fzf)\delta(eyf)e = \sigma(f)\delta(eyfzf)e,$$

$$\sigma(e)\delta(eyfzf)f = \sigma(eyf)\delta(fzf)f + \sigma(e)\delta(eyf)fzf - \sigma(e)\delta(e)eyfzf$$

and

$$\sigma(eyf)(\delta(fz_1fz_2f) - \delta(fz_1f)fz_2f - \sigma(fz_1f)\delta(fz_2f) + \delta(f)fz_1fz_2f) = 0$$

for all $y, z, z_1, z_2 \in \mathscr{U}$. Note that

$$\delta(fz_1fz_2f) - \delta(fz_1f)fz_2f - \sigma(fz_1f)\delta(fz_2f) + \delta(f)fz_1fz_2f \in \sigma(f)\mathscr{U}f.$$

It follows from Eq. (1) again that $\delta(fz_1fz_2f) - \delta(fz_1f)fz_2f - \sigma(fz_1f)\delta(fz_2f) + \delta(f)fz_1fz_2f = 0$, and so (iii), (v) are true. The proof of the lemma is completed. □

**Lemma 5** *Assume that $\xi \neq 1$ and $\sigma(f)\delta(1)e = 0$ if $\xi = 0$. Then there exists an additive Jordan $\sigma$-derivation $\tau$ on $\mathscr{U}$ such that $\delta(x) = \tau(x) + \delta(1)x$ holds for all $x \in \mathscr{U}$.*

*Proof*: For any $x = exe + exf + fxf \in \mathscr{U}$, by Lemmas 2–4, a direct calculation leads to

$$\delta(x^2) = \delta(x)x + \sigma(x)\delta(x) - \delta(1)x^2.$$

Define $\tau$ by $\tau(x) = \delta(x) - \delta(1)x$ for all $x \in \mathscr{U}$. Obviously, $\tau$ is an additive map on $\mathscr{U}$. Note that $\delta(1) \in \mathscr{Z}_\sigma(\mathscr{U})$. It is easy to check that $\tau(x^2) = \tau(x)x + \sigma(x)\tau(x)$ holds for all $x \in \mathscr{U}$, that is, $\tau$ is a Jordan $\sigma$-derivation. □

**Lemma 6** *Assume that $\xi = 1$ and $x_0 = ex_0e$ (or $x_0 = fx_0f$). If $\mathscr{Z}_\sigma(e\mathscr{U}e) = \sigma(e)\mathscr{Z}_\sigma(\mathscr{U})e$ and $\mathscr{Z}_\sigma(f\mathscr{U}f) = \sigma(f)\mathscr{Z}_\sigma(\mathscr{U})f$, then there exists an additive $\sigma$-derivation $\tau : \mathscr{U} \to \mathscr{U}$ and an additive map $h : \mathscr{U} \to \mathscr{Z}_\sigma(\mathscr{U})$ satisfying $h([x, y]) = 0$ for $x, y \in \mathscr{U}$ with $xy = x_0$ such that $\delta(x) = \tau(x) + h(x)$ for all $x \in \mathscr{U}$.*

*Proof*: Here we only give the proof for the case $x_0 = ex_0e$. The proof of other case $x_0 = fx_0f$ is similar.

In fact, by Eqs. (6)–(7) and Lemma 2, we have proved that

$$\begin{aligned}&\delta(e\mathscr{U}e), \delta(f\mathscr{U}f) \subseteq \sigma(e)\mathscr{U}e + \sigma(f)\mathscr{U}f,\\ &\delta(1) = \sigma(e)\delta(1)e + \sigma(f)\delta(1)f \in \mathscr{Z}_\sigma(\mathscr{U})\end{aligned} \tag{13}$$

and

$$\sigma(f)\delta(e\mathscr{U}e)f \subseteq \mathscr{Z}_\sigma(f\mathscr{U}f),\ \sigma(e)\delta(f\mathscr{U}f)e \subseteq \mathscr{Z}_\sigma(e\mathscr{U}e).$$

Then the assumption about $\mathscr{Z}_\sigma(\mathscr{U})$ gives

$$\begin{aligned}&\sigma(f)\delta(exe)f \in \sigma(f)\mathscr{Z}_\sigma(\mathscr{U})f\\ \text{and}\quad &\sigma(e)\delta(fyf)e \in \sigma(e)\mathscr{Z}_\sigma(\mathscr{U})e.\end{aligned}$$

So there exist two elements $h_A(exe), h_B(fyf) \in \mathscr{Z}_\sigma(\mathscr{U})$ such that

$$\begin{aligned}&\sigma(f)\delta(exe)f = \sigma(f)h_A(exe)f\\ \text{and}\quad &\sigma(e)\delta(fyf)e = \sigma(e)h_B(fyf)e.\end{aligned} \tag{14}$$

For any $x \in \mathscr{U}$, since $x_0e = (x_0 + exf)e = x_0$, we have

$$0 = \delta([x_0, e]) = \delta(x_0)e - \sigma(e)\delta(x_0) + \sigma(x_0)\delta(e) - \delta(e)x_0$$

and

$$\delta([x_0+exf,e])=\delta(x_0+exf)e-\sigma(e)\delta(x_0+exf) + \sigma(x_0+exf)\delta(e)-\delta(e)(x_0+exf).$$

Comparing the two equations above yields

$$-\delta(exf)=\delta(exf)e-\sigma(e)\delta(exf)+\sigma(exf)\delta(e)-\delta(e)exf.$$

Multiplying by $\sigma(e)$ and $e$ from the left and the right in the above equation, respectively, and by Eq. (13), one gets $\sigma(e)\delta(exf)e=0$; multiplying by $\sigma(f)$ from the left in the above equation, and by Eq. (13), one has $-\sigma(f)\delta(exf)=\sigma(f)\delta(exf)e$, and so $\sigma(f)\delta(exf)e=\sigma(f)\delta(exf)f=0$. Hence,

$$\delta(exf)=\sigma(e)\delta(exf)f\in\sigma(e)\mathscr{U}f. \tag{15}$$

Now define two maps $h:\mathscr{U}\to\mathscr{Z}_\sigma(\mathscr{U})$ and $\tau:\mathscr{U}\to\mathscr{U}$ respectively by

$$h(x)=h_A(exe)+h_B(fxf) \quad \text{and}$$
$$\tau(x)=\delta(x)-h(x) \quad \text{for all} \quad x\in\mathscr{U}.$$

In the sequel, we will prove that $\tau$ is an additive $\sigma$-derivation by five steps.

**Step 1.** $\tau(e\mathscr{U}e)\subseteq\sigma(e)\mathscr{U}e$, $\tau(e\mathscr{U}f)\subseteq\sigma(e)\mathscr{U}f$ and $\tau(f\mathscr{U}f)\subseteq\sigma(f)\mathscr{U}f$.

By the definition of $\tau$ and Eq. (15), it is clear that $\tau(e\mathscr{U}f)\subseteq\sigma(e)\mathscr{U}f$.

For any $x\in\mathscr{U}$, by Proposition 1, Eqs. (13)–(14), we have

$$\begin{aligned}\tau(exe)&=\delta(exe)-h(exe)\\&=\sigma(e)\delta(exe)e+\sigma(f)\delta(exe)f-h_A(exe)\\&=\sigma(e)\delta(exe)e+\sigma(f)h_A(exe)f-h_A(exe)\\&=\sigma(e)\delta(exe)e+\sigma(e)h_A(exe)e,\end{aligned}$$

which implies $\tau(exe)\in\sigma(e)\mathscr{U}e$.

Similarly, one can prove $\tau(f\mathscr{U}f)\subseteq\sigma(f)\mathscr{U}f$.

**Step 2.** $\tau$ and $h$ are additive.

It is obvious that $\tau(e\mathscr{U}f)=\delta(e\mathscr{U}f)$ and $h(e\mathscr{U}f)=\{0\}$. So we only need to check that $\tau$ is additive on $e\mathscr{U}e$ and $f\mathscr{U}f$.

To do this, take any $x,y\in\mathscr{U}$. Then,

$$\begin{aligned}\tau(exe+eye)&=\delta(exe+eye)-h(exe+eye)\\&=\delta(exe)+\delta(eye)-h(exe+eye)\end{aligned}$$

and

$$\tau(exe)+\tau(eye)=\delta(exe)+\delta(eye)-h(exe)-h(eye).$$

So $\tau(exe+eye)-\tau(exe)-\tau(eye)=h(exe)+h(eye)-h(exe+eye)\in\mathscr{Z}_\sigma(\mathscr{U})$. Note that $\tau(exe+eye)-\tau(exe)-\tau(eye)\in\sigma(e)\mathscr{U}e$ (Step 1). It follows from Proposition 1 and Eq. (1) that $\tau(exe+eye)-\tau(exe)-\tau(eye)=0$. That is, $\tau$ is additive on $e\mathscr{U}e$. A similar argument leads to the proof that $\tau$ is additive on $f\mathscr{U}f$. Hence $\tau$ is additive on $\mathscr{U}$, and so $h$ is additive, too.

**Step 3.** For any $x,y\in\mathscr{U}$, we have $\tau(exeyf)=\tau(exe)eyf+\sigma(exe)\tau(eyf)$ and $\tau(exeye)=\tau(exe)eye+\sigma(exe)\tau(eye)$.

For any $x\in\mathscr{U}$ and any invertible $eye\in e\mathscr{U}e$, since $x_0(eye)^{-1}eye=x_0=(x_0(eye)^{-1}+exf)eye$, we have

$$\begin{aligned}&\delta([x_0(eye)^{-1},eye])\\&\quad=\delta(x_0(eye)^{-1})eye-\sigma(eye)\delta(x_0(eye)^{-1})\\&\qquad+\sigma(x_0(eye)^{-1})\delta(eye)-\delta(eye)x_0(eye)^{-1}\end{aligned}$$

and

$$\begin{aligned}\delta([x_0(eye)^{-1}+exf,eye])&=\delta(x_0(eye)^{-1}+exf)eye\\&\quad-\sigma(eye)\delta(x_0(eye)^{-1}+exf)\\&\quad+\sigma(x_0(eye)^{-1}+exf)\delta(eye)-\delta(eye)(x_0(eye)^{-1}+exf).\end{aligned}$$

These imply

$$\begin{aligned}\delta(-eyexf)&=\delta(exf)eye-\sigma(eye)\delta(exf)\\&\quad+\sigma(exf)\delta(eye)-\delta(eye)(exf)\end{aligned}$$

for all $x\in\mathscr{U}$ and all invertible $eye\in e\mathscr{U}e$. By the assumption about $e\mathscr{U}e$, it is true that

$$\begin{aligned}\delta(-eyexf)&=\delta(exf)eye-\sigma(eye)\delta(exf)\\&\quad+\sigma(exf)\delta(eye)-\delta(eye)(exf)\end{aligned}$$

holds for all $x,y\in\mathscr{U}$. Replacing $\delta$ by $\tau+h$ in the above equation, and noting that $\sigma(exf)h(eye)=h(eye)(exf)$, we obtain

$$\begin{aligned}\tau(-eyexf)&=\tau(exf)eye-\sigma(eye)\tau(exf)\\&\quad+\sigma(exf)\tau(eye)-\tau(eye)(exf).\end{aligned}$$

It follows from Step 1 that

$$\tau(eyexf)=\tau(eye)(exf)+\sigma(eye)\tau(exf)$$

holds for all $x,y\in\mathscr{U}$.

Now, for any $y_1,y_2\in\mathscr{U}$, one has

$$\tau(ey_1ey_2exf)=\tau(ey_1ey_2e)(exf)+\sigma(ey_1ey_2e)\tau(exf)$$

and

$$\begin{aligned}\tau(ey_1ey_2exf)&=\tau(ey_1e)ey_2exf+\sigma(ey_1e)\tau(ey_2exf)\\&=\tau(ey_1e)ey_2exf+\sigma(ey_1e)\tau(ey_2e)(exf)\\&\quad+\sigma(ey_1e)\sigma(ey_2e)\delta(exf),\end{aligned}$$

which imply $(\tau(ey_1ey_2e)-\tau(ey_1e)ey_2e-\sigma(ey_1e)\tau(ey_2e))exf=0$ for all $x\in\mathscr{U}$. By Eq. (1), we get $\tau(ey_1ey_2e)-\tau(ey_1e)ey_2e-\sigma(ey_1e)\tau(ey_2e)=0$, completing the proof the step.

**Step 4.** For any $x, y \in \mathscr{U}$, we have $\tau(exfyf) = \tau(exf)fyf + \sigma(exf)\tau(fyf)$ and $\tau(fxfyf) = \tau(fxf)fyf + \sigma(fxf)\tau(fyf)$.

The proof is similar to that of Step 3 and we omit it here.

**Step 5.** For any $x, y \in \mathscr{U}$, we have $\tau(xy) = \tau(x)y + \sigma(x)\tau(y)$, that is, $\tau$ is an additive $\sigma$-derivation.

By using Steps 2–4, this is a direct calculation.

Finally, for any $x, y \in \mathscr{U}$ with $xy = x_0$, by the property of $\delta$, Step 5 and Proposition 1, we have

$$\begin{aligned} h([x,y]) &= \delta([x,y]) - \tau([x,y]) \\ &= \delta(x)y - \sigma(y)\delta(x) + \sigma(x)\delta(y) - \delta(y)x \\ &\quad - \tau(x)y - \sigma(x)\tau(y) + \tau(y)x + \sigma(y)\tau(x) \\ &= h(x)y - \sigma(y)h(x) + \sigma(x)h(y) - h(y)x = 0. \end{aligned}$$

The proof of the lemma is finished. □

Now, we are at a position to the proofs of Theorems 1–4

**Proof of Theorem 1**

Note that $L(x) = \delta(x) + (\sigma(x)y_0 - y_0x)$ for all $x \in \mathscr{U}$. It is clear that $L(1) = \delta(1) \in \mathscr{Z}_\sigma(\mathscr{U})$ by Lemma 2. Let $\Delta(x) = \tau(x) + (\sigma(x)y_0 - y_0x)$ for each $x$. By Lemma 5, $\Delta$ is also an additive Jordan $\sigma$-derivation on $\mathscr{U}$ and $L(x) = \Delta(x) + L(1)x$. Therefore, Theorem 1 holds. □

**Proof of Theorem 2**

We first claim $\sigma(f)L(1)e = 0$ in the case $ex_0f = 0$. In fact, for $t \in \{1,2,3\}$, since $(tf + ex_0e)(t^{-1}fx_0f + e) = ex_0e + fx_0f = x_0$, we have $L(x_0) = L(tf + ex_0e)(t^{-1}fx_0f + e) + \sigma(tf + ex_0e)L(t^{-1}fx_0f + e)$, that is,

$$\begin{aligned} L(x_0) &= L(f)fx_0f + tL(f)e + t^{-1}L(ex_0e)fx_0f \\ &\quad + L(ex_0e)e + \sigma(f)L(fx_0f) + t\sigma(f)L(e) \\ &\quad + t^{-1}\sigma(ex_0e)L(fx_0f) + \sigma(ex_0e)L(e). \end{aligned}$$

By taking three different values of $t$, one can obtain $L(f)e + \sigma(f)L(e) = 0$. Multiplying by $\sigma(f)$ and $e$ from the left and the right in the equation, respectively, one gets $\sigma(f)L(f)e + \sigma(f)L(e)e = 0$, that is, $\sigma(f)L(1)e = 0$.

Now by the same argument as that of Theorem 1, the theorem is true. □

**Proof of Theorem 3**

The "if" part is a direct calculation. For the "only if" part, by Lemma 6, $L(x) = \delta(x) + (\sigma(x)y_0 - y_0x) = \tau(x) + h(x) + (\sigma(x)y_0 - y_0x)$ for all $x \in \mathscr{U}$. Let $\Delta(x) = \tau(x) + (\sigma(x)y_0 - y_0x)$ for each $x$. It is obvious that $\Delta$ is an additive $\sigma$-derivation and $L(x) = \Delta(x) + h(x)$, completing the proof. □

**Proof of Theorem 4**

By Corollary 1 and Theorem 3, $L$ is of the form $L(x) = \Delta(x) + L(1)x$ if $\xi \neq 1$ and $L(x) = \Delta(x) + h(x)$ if $\xi = 1$, where $\Delta$ is an additive $\sigma$-derivation and $h$ is an additive map into $\mathscr{Z}_\sigma(\mathscr{U})$. To complete the proof, one only needs to check $L(1) = 0$ if $\xi \neq 1$ and $h([x,y]) = 0$ for all $x, y$ if $\xi = 1$.

In fact, if $\xi \neq 1$, we have

$$\begin{aligned} L([x,y]_\xi) &= L(x)y - \xi\sigma(y)L(x) + \sigma(x)L(y) - \xi L(y)x \\ &= \Delta(x)y + L(1)xy - \xi\sigma(y)\Delta(x) - \xi\sigma(y)L(1)x \\ &\quad + \sigma(x)\Delta(y) + \sigma(x)L(1)y - \xi\Delta(y)x - 2\xi L(1)yx \\ &= \Delta(xy) + 2L(1)xy - \xi\Delta(yx) - \xi L(1)yx \end{aligned}$$

and

$$L([x,y]_\xi) = \Delta([x,y]_\xi) + L(1)[x,y]_\xi,$$

which imply $L(1)[x,y]_\xi = 0$ for all $x, y$. Particularly, $(1-\xi)L(1) = 0$, and so $L(1) = 0$.

If $\xi = 1$, then

$$\begin{aligned} h([x,y]) &= L([x,y]) - \Delta([x,y]) \\ &= L(x)y - \sigma(y)L(x) + \sigma(x)L(y) - L(y)x \\ &\quad - \Delta(x)y - \sigma(x)\Delta(y) + \Delta(y)x + \sigma(y)\Delta(x) \\ &= h(x)y - \sigma(y)h(x) + \sigma(x)h(y) - h(y)x = 0. \end{aligned}$$

The proof is completed. □

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**REFERENCES**


1. Benkovič D (2016) Jordan $\sigma$-derivations of triangular algebras. *Linear Multilinear Algebra* **64** 143–155.
2. Bergen J (2000) Skew derivations whose invariants satisfy a polynomial identity. *J Algebra* **228**, 710–737.
3. Brešar M, Villena AR (2002) The noncommutative Singer-Wermer conjecture and $\phi$-derivations. *J London Math Soc* **66**, 710–720.
4. Chuang CL, Lee TK (2005) Identities with a single skew derivation. *J Algebra* **288**, 59–77.
5. Wang Y (2018) A note on Jordan $\sigma$-derivations of triangular algebras. *Linear Multilinear Algebra* **66**, 639–644.
6. Brooke JA, Busch P, Pearson B (2002) Commutativity up to a factor of bounded operators in complex Hilbert spaces. *Proc R Soc Lond A* **458**, 109–118.
7. Kassel C (1995) *Quantum Groups*, Springer-Verlag, New York.
8. Qi X-F, Hou J-C (2009) Additive Lie ($\xi$-Lie) derivations and generalized Lie ($\xi$-Lie) derivations on nest algebras. *Linear Algebra Appl* **431**, 843–854.
9. Qi X-F, Cui J-L, Hou J-C (2011) Characterizing additive $\xi$-Lie derivations of prime algebras by $\xi$-Lie zero products. *Linear Algebra Appl* **434**, 669–682.
10. Yang W-L, Zhu J (2013) Characterizations of additive (generalized) $\xi$-Lie $(\alpha,\beta)$ derivations on triangular algebras. *Linear Multilinear Algebra* **61**, 811–830.

11. Cheung WS (2000) Mappings on triangular algebras. PhD dissertation, University of Victoria, Victoria.
12. Han D, Wei F (2011) Jordan $(\alpha,\beta)$-derivations on triangular algebras and related mappings. *Linear Algebra Appl* **434**, 259–284.
13. Benkovič D (2022) Lie $\sigma$-derivations of triangular algebras. *Linear Multilinear Algebra* **70**, 2966–2983.
14. Ji P-S, Qi W-Q (2011) Characterizations of Lie derivations of triangular algebras. *Linear Algebra Appl* **435**, 1137–1146.
15. Benkovič D, Eremita D (2004) Commuting traces and commutativity preserving maps on triangular algebras. *J Algebra* **280**, 797–824.