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The study of solutions of several systems of the product type nonlinear partial differential difference equations

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ABSTRACT: The purpose of this article is to investigate the solutions of several systems of the nonlinear partial differential difference equations (PDDEs) (including second order partial differential, mixed partial differential and complex difference)

$$\begin{cases} f(z+c)(g_{z_1}+g_{z_1z_1})=1, \\ g(z+c)(f_{z_1}+f_{z_1z_1})=1, \end{cases}$$

$$\begin{cases} f(z+c)(g_{z_1}+g_{z_1z_2})=1, \\ g(z+c)(f_{z_1}+f_{z_1z_2})=1, \end{cases}$$

and

$$\begin{cases} f(z+c)(f_{z_1}+g_{z_1z_2})=1, \\ g(z+c)(g_{z_1}+f_{z_1z_2})=1, \end{cases}$$

where $c=(c_1,c_2)\in\mathbb{C}^2$. We establish some theorems concerning the forms of the pair of solutions for these systems of PDDEs which are some improvements and generalization of the previous results given by Gao, Liu and Xu. Moreover, some examples show that the forms of solutions of our theorems are precise to some extent.

KEYWORDS: partial differential difference equation, second order, Nevanlinna theory

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INTRODUCTION

The classical result about the solution of the eikonal (eiconal) equation in $\mathbb{C}^2$

$$\left(u_{z_1}\right)^2+\left(u_{z_2}\right)^2=1 \tag{1}$$

is that any entire solution of (1) must be linear of the form $u=c_1z_1+c_2z_2+c_0$, where $c_1^2+c_2^2=1$, which was given by Khavinson in [1]. This result can also be found in [2]. Equation (1) can be seen as a typical partial differential equation (can be written as PDE in short). Later, Saleeby [3] and Li [4] proved the same conclusion by using two different methods (see [3, 4]). In the past two decades, many mathematics scholars including B.Q. Li, D.C. Chang, E.G. Saleeby, Q. Han and F. Lü discussed the solutions of the eikonal equation and its variants, and obtained a number of interest and important results (see [3–15]).

Theorem A ([16]) *Let $P(z_1;z_2)$ and $Q(z_1;z_2)$ be arbitrary polynomials in $\mathbb{C}^2$. Then u is an entire solution of the equation*

$$(Pu_{z_1})^2+(Qu_{z_2})^2=1 \tag{2}$$

if and only if $u=c_1z_1+c_2z_2+c_3$ is a linear function, where c_j are constants, and exactly one of the following holds:
(i) $c_1=0$ *and Q is a constant satisfying that* $(c_2Q)^2=1$;
(ii) $c_2=0$ *and P is a constant satisfying that* $(c_1P)^2=1$;
(iii) $c_1c_2\neq 0$ *and P, Q are both constant satisfying that* $(c_1P)^2+(c_2Q)^2=1$.

As Khavinson and Li mentioned in [1, 4], by taking the linear transformation $z_1=x+iy$ and $z_2=x-iy$, equation (1) can be reduced to $u_{z_1}u_{z_2}=1$. Differentiating this new equation with respect to z_1,z_2, respectively, we have that $u_{z_1z_1}u_{z_2}=-u_{z_1}u_{z_2z_1}$ and $u_{z_1z_2}u_{z_2}=-u_{z_1}u_{z_2z_2}$, this leads to $u_{z_1z_1}u_{z_2z_2}-u_{z_1z_2}^2=0$. This equation can be seen as a degenerated Monge-Ampère equation, which has the linear function solutions. For the non-degenerated Monge-Ampère equation

$$A(u_{z_1z_1}u_{z_2z_2}-u_{z_1z_2}^2)+Bu_{z_1z_1}+Cu_{z_1z_2}+Du_{z_2z_2}+E=0,$$

where A,B,C,D,E are functions depending only on $z_1,z_2,u,u_{z_1},u_{z_2}$, it is usually difficult to find solutions of a non-degenerate Monge-Ampère equation. There is a great number references focusing on the study of this class equation.

Inspired by the remark of Khavinson [1] and Li [4], Lü [17] paid the attention on entire solutions of a variation of the eikonal equation with product form PDEs.

Theorem B ([17]) *Let g be a polynomial in $\mathbb{C}^2$, and let m be a non-negative integer. Then u is an entire solution of the partial differential equation $u_x u_y = x^m \mathrm{e}^g$ in $\mathbb{C}^2$ if and only if the following assertions hold:*
(i) *$u = \phi_1(x) + \phi_2(y)$, where $\phi_1'(x) = x^m \mathrm{e}^{\alpha(x)}$ and $\phi_2'(y) = \mathrm{e}^{\beta(y)}$ satisfying $\alpha(x) + \beta(y) = g(x, y)$;*
(ii) *$u = F(y + Ax^{m+1})$, where A is a non-zero constant and $(m+1)AF'^2(y + Ax^{m+1}) = \mathrm{e}^g$;*
(iii) *$u = (x^{k+1}/(k+1))\mathrm{e}^{ay+b} + C$, where $(a/(k+1))\mathrm{e}^{2(ay+b)} = \mathrm{e}^g$, $m = 2k+1$ and $a(\neq 0)$, b, C are constant.*

In 2022, Chen and Han [18] further investigated the entire solutions for a series of product type nonlinear partial differential equations, and obtained:

Theorem C ([18]) *Let $p(z, w) \neq 0$ be a polynomial in $\mathbb{C}^2$, and let $l \geqslant 0$ and $m, n \geqslant 1$ be integers. $u(z, w)$ in $\mathbb{C}^2$ is an entire solution to the nonlinear first-order partial differential equation*

$$(u^l u_z)^m (u^l u_w)^n = p(z, w) \tag{3}$$

if and only if one of the following situations occurs.
(i) *$l = 0$, $p(z, w) = q^m(z) r^n(w)$ for some nonzero polynomials $q(z)$, $r(w)$ in $\mathbb{C}$, and $u(z, w) = c_1 \int q(z)\,\mathrm{d}z + c_2 \int r(w)\,\mathrm{d}w + c_0$ for some constants c_0, c_1, c_2 satisfying $c_1^m c_2^n = 1$; in particular, when $p(z, w) = K$ for a constant $K(\neq 0)$, then $u(z, w)$ is affine.*
(ii) *$l \geqslant 0$ and $u(z, w) = \Big\{(l+1)\ \big(c_1 \int q(z, w)\,\mathrm{d}z + c_2 \int r(z, w)\,\mathrm{d}w - c_1 \int\int q_w(z, w)\,\mathrm{d}z\,\mathrm{d}w\big)\Big\}^{1/l+1}$ for some constants c_1, c_2 with $c_1^m c_2^n = 1$, where $q(z, w)$, $r(z, w)$ are nonzero polynomials in $\mathbb{C}^2$ such that $c_1 q_w(z, w) = c_2 r_z(z, w) \not\equiv 0$ and $p(z, w) = q^m(z, w) r^n(z, w)$.*

In 2023, Xu, Xu and Liu [19] investigated the entire solutions of some systems of the product form partial differential equations and obtained:

Theorem D ([19]) *et $D := ad - bc \neq 0$ and (f, g) be a pair of transcendental entire solutions with finite order for system*

$$\begin{cases} \left(af_{z_1} + bf_{z_2}\right)\left(cg_{z_1} + dg_{z_2}\right) = 1, \\ \left(ag_{z_1} + bg_{z_2}\right)\left(cf_{z_1} + df_{z_2}\right) = 1. \end{cases} \tag{4}$$

Then (f, g) is one of the following forms
(i) *$(f(z), g(z)) = \left(\frac{1}{a}F_1(z_1), \frac{1}{c}G_1(z_1)\right)$;*
(ii) *$(f(z), g(z)) = \left(\frac{1}{b}F_2(z_2), \frac{1}{d}G_2(z_2)\right)$;*
(iii) *$(f(z), g(z)) = \left(\frac{aA^{-1}-c}{D}F_3\left(z_2 - \frac{b-dA}{a-cA}z_1\right), \frac{cA-a}{D} G_3\left(z_2 - \frac{b-dA}{a-cA}z_1\right)\right)$, where $A \in \mathbb{C} - \{0\}$, $\varphi_j(t)$, $j = 1, 2, 3$ are nonconstant polynomial in $\mathbb{C}$ and $F_j'(t) = \mathrm{e}^{\varphi_j(t)}$, $G_j'(t) = \mathrm{e}^{-\varphi_j(t)}$, $j = 1, 2, 3$.*

With the rapid development of the difference Nevanlinna theory of several complex variables [20–22], Xu and Cao [23, 24] in 2020 discussed the transcendental solutions of several partial differential difference equations. In general, an equation is called as a partial differential difference equation, if this equation includes the partial derivatives, shifts and differences of f, which can be denoted PDDE for short.

Theorem E ([23]) *Let $c = (c_1, c_2) \in \mathbb{C}^2$. Then any transcendental entire solution with finite order of the partial differential difference equation*

$$\left(f_{z_1}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1 \tag{5}$$

has the form of $f(z_1, z_2) = \sin(Az_1 + B)$, where A is a constant on $\mathbb{C}$ satisfying $A\mathrm{e}^{\mathrm{i}Ac_1} = 1$, and B is a constant on $\mathbb{C}$; in the special case whenever $c_1 = 0$, we have $f(z_1, z_2) = \sin(z_1 + B)$.

In the same year, Xu, Liu and Li [25] studied the finite order transcendental entire solutions when equation (1) turn to the system of Fermat type PDDEs, and obtained:

Theorem F ([25]) *Let $c = (c_1, c_2) \in \mathbb{C}^2$. Then any pair of transcendental entire solutions with finite order for the system of Fermat type partial differential-difference equations*

$$\begin{cases} \left(f_{z_1}\right)^2 + g(z_1 + c_1, z_2 + c_2)^2 = 1, \\ \left(g_{z_1}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1 \end{cases} \tag{6}$$

have the following forms

$$(f, g) = \left(\frac{\mathrm{e}^{L(z)+B_1} + \mathrm{e}^{-(L(z)+B_1)}}{2}, \frac{A_{21}\mathrm{e}^{L(z)+B_1} + A_{22}\mathrm{e}^{-(L(z)+B_1)}}{2}\right),$$

where $L(z) = a_1 z_1 + a_2 z_2$, B_1 is a constant in $\mathbb{C}$, and a_1, c, A_{21}, A_{22} satisfy one of the following cases
(i) *$A_{21} = -\mathrm{i}$, $A_{22} = \mathrm{i}$, and $a_1 = \mathrm{i}$, $L(c) = (2k + \frac{1}{2})\pi\mathrm{i}$, or $a_1 = -\mathrm{i}$, $L(c) = (2k - \frac{1}{2})\pi\mathrm{i}$;*
(ii) *$A_{21} = \mathrm{i}$, $A_{22} = -\mathrm{i}$, and $a_1 = \mathrm{i}$, $L(c) = (2k - \frac{1}{2})\pi\mathrm{i}$, or $a_1 = -\mathrm{i}$, $L(c) = (2k + \frac{1}{2})\pi i$;*
(iii) *$A_{21} = 1$, $A_{22} = 1$, and $a_1 = \mathrm{i}$, $L(c) = 2k\pi\mathrm{i}$, or $a_1 = -\mathrm{i}$, $L(c) = (2k+1)\pi\mathrm{i}$;*
(iv) *$A_{21} = -1$, $A_{22} = -1$, and $a_1 = \mathrm{i}$, $L(c) = (2k+1)\pi\mathrm{i}$, or $a_1 = -\mathrm{i}$, $L(c) = 2k\pi\mathrm{i}$.*

In 2022, Tang, Zhang and Xu [26] discussed the solutions of several second partial differential equations and obtained:

Theorem G ([26]) *Let $c=(c_1,c_2)\in\mathbb{C}^2$ and $c_1c_2\neq 0$. If the second order Fermat type partial differential difference equation*

$$\left(f_{z_1z_2}\right)^2+f(z_1+c_1,z_2+c_2)^2=1 \tag{7}$$

admits a transcendental entire solution with finite order $f(z_1,z_2)$, then $f(z_1,z_2)$ has the following form

$$f(z_1,z_2)=\eta\frac{\mathrm{e}^{\mathrm{i}(a_1z_1+a_2z_2+B)}+\mathrm{e}^{-\mathrm{i}(a_1z_1+a_2z_2+B)}}{2},$$

where η, c_1, c_2, a_1, a_2, B are constants in $\mathbb{C}$, and satisfy one of the following cases
(i) $L(c)=2k\pi+\frac{1}{2}\pi$, $a_1a_2=1$, *and* $\eta=-1$;
(ii) $L(c)=2k\pi-\frac{1}{2}\pi$, $a_1a_2=-1$, *and* $\eta=1$.

The above theorems suggest the following question:

Question 1 *What will happen about the solutions if the equation (system) is of the product type and includes the difference and the second order partial differential or second order mixed partial differential?*

RESULTS AND EXAMPLES

Motivated by Question 1, we mainly describe the entire solutions of several systems of nonlinear PDEs and PDDEs in $\mathbb{C}^2$. As far as we know, there is few reference concerning this subject in the fields of complex analysis. In this paper, let us assume that the readers are familiar with the Nevanlinna theory and difference Nevanlinna theory with several complex variables, including some basic theorems and the difference version of logarithmic derivative lemma for meromorphic functions on $\mathbb{C}^m$ (can refer to Korhonen [21] and improved by Korhonen, Cao [20,27]). Here and below, we denote $z+w=(z_1+w_1,z_2+w_2)$ and $az=(az_1,az_2)$ for any $z=(z_1,z_2)$, $w=(w_1,w_2)$ and $a\in\mathbb{C}$.

Theorem 1 *Let $c=(c_1,c_2)\in\mathbb{C}^2$, c_1, $c_2\in\mathbb{C}$ and $c_2\neq 0$, and assume that f, g is a pair of finite order transcendental entire solutions of system*

$$\begin{cases} f(z+c)(g_{z_1}+g_{z_1z_1})=1,\\ g(z+c)(f_{z_1}+f_{z_1z_1})=1. \end{cases} \tag{8}$$

Then (f,g) must be the form of

$$(f,g)=\left(\frac{1}{A_1(A_1+1)}\,\mathrm{e}^{L(z)-B_2},\frac{1}{A_1(A_1-1)}\,\mathrm{e}^{-L(z)-B_1}\right),$$

where $L(z)=A_1z_1+A_2z_2$, $L(c)=A_1c_1+A_2c_2$, A_1, A_2, B_1, $B_2\in\mathbb{C}$ satisfy $A_1\neq 0,\pm 1$ and

$$\mathrm{e}^{2L(c)}=\frac{A_1+1}{A_1-1},\qquad \mathrm{e}^{2(B_1+B_2)}=\frac{1}{A_1^2(A_1^2-1)}. \tag{9}$$

The following examples show the existence of transcendental entire solutions of equation (8) for every case in Theorem 1.

Example 1 Let

$$(f,g)=\left(\frac{9}{2}\,\mathrm{e}^{\frac{5}{3}z_1+\frac{2}{3}z_2},\frac{1}{10}\,\mathrm{e}^{-\frac{5}{3}z_1-\frac{2}{3}z_2}\right).$$

Thus, (f,g) is a pair of transcendental entire solutions of equation (8) for the case $c_1=\log 2$, $c_2=-\log 2$ and $\rho(f,g)=1$.

Example 2 Let

$$(f,g)=\left(\frac{16}{45}\,\mathrm{e}^{\frac{5}{4}z_1+z_2+\log 15+\pi\mathrm{i}},\frac{16}{5}\,\mathrm{e}^{-\frac{5}{4}z_1-z_2-\log 16}\right).$$

Thus, (f,g) is a pair of transcendental entire solutions of equation (8) for the case $c_1=\frac{4}{5}\pi\mathrm{i}$, $c_2=\log 3$ and $\rho(f,g)=1$.

The following example shows that the condition $c_2\neq 0$ in Theorem 1 can not be removed.

Example 3 Let

$$(f,g)=\left(\frac{4}{\sqrt{5}}e^{\frac{3}{2}z_1+z_2^3},\frac{4}{3}e^{-\frac{3}{2}z_1-z_2^3}\right).$$

Thus, (f,g) is a pair of transcendental entire solutions of equation (8) for the case $c_1=\frac{\log 5}{3}$, $c_2=0$. Noting that $\rho(f,g)=3$, the forms of this solution can not be included in the forms stated as in Theorem 1.

By observing (9) in Theorem 1, we can get the following corollary for $c_1=c_2=0$.

Corollary 1 *The system*

$$\begin{cases} f(z)(g_{z_1}+g_{z_1z_1})=1,\\ g(z)(f_{z_1}+f_{z_1z_1})=1 \end{cases}$$

has not any pair of nonconstant finite order transcendental entire solutions.

Theorem 2 *The system*

$$\begin{cases} f(z+c)(g_{z_1}+f_{z_1z_1})=1,\\ g(z+c)(f_{z_1}+g_{z_1z_1})=1 \end{cases} \tag{10}$$

has no any pair of finite order entire solutions.

Theorem 3 *Let $c=(c_1,c_2)\in\mathbb{C}^2$, and assume that (f,g) is a pair of finite order transcendental entire solutions of system*

$$\begin{cases} f(z+c)(g_{z_1}+g_{z_1z_2})=1,\\ g(z+c)(f_{z_1}+f_{z_1z_2})=1. \end{cases} \tag{11}$$

Then (f,g) must be the form of

$$(f,g)=\left(\frac{1}{A_1(A_2+1)}\,\mathrm{e}^{L(z)-B_2},\frac{1}{A_1(A_2-1)}\,\mathrm{e}^{-L(z)-B_1}\right),$$

where $L(z)=A_1z_1+A_2z_2$, $L(c)=A_1c_1+A_2c_2$, A_1, A_2, B_1, $B_2\in\mathbb{C}$ satisfy

$$\mathrm{e}^{2L(c)}=\frac{A_2+1}{A_2-1},\qquad \mathrm{e}^{2(B_1+B_2)}=\frac{1}{A_1^2(A_2^2-1)}. \tag{12}$$

The following examples show the existence of transcendental entire solutions of equation (11) for every case in Theorem 3.

Example 4 Let

$$(f, g) = \left(\frac{5}{2}\,\mathrm{e}^{z_1+\frac{3}{5}z_2}, \frac{\mathrm{i}}{2}\,\mathrm{e}^{-z_1-\frac{3}{5}z_2}\right).$$

Thus, (f, g) is a pair of transcendental entire solutions of equation (11) for the case $c_1 = \log 2$, $c_2 = \frac{5}{6}\pi\mathrm{i}$ and $\rho(f, g) = 1$.

Example 5 Let

$$(f, g) = \left(\frac{1}{\sqrt{3}}\,\mathrm{e}^{z_1+2z_2}, -\mathrm{e}^{-z_1-2z_2}\right).$$

Thus, (f, g) is a pair of transcendental entire solutions of equation (11) for the case $c_1 = \log 3$, $c_2 = -\frac{3}{4}\log 3$ and $\rho(f, g) = 1$.

By observing (12) in Theorem 3, we can get the following corollary for $c_1 = c_2 = 0$.

Corollary 2 *The system*

$$\begin{cases} f(z)(g_{z_1} + g_{z_1 z_2}) = 1, \\ g(z)(f_{z_1} + f_{z_1 z_2}) = 1 \end{cases}$$

has not any pair of nonconstant finite order transcendental entire solutions.

Theorem 4 *The system*

$$\begin{cases} f(z+c)(f_{z_1} + g_{z_1 z_2}) = 1, \\ g(z+c)(g_{z_1} + f_{z_1 z_2}) = 1 \end{cases} \tag{13}$$

has no any pair of finite order entire solutions.

Theorem 5 *The system*

$$\begin{cases} f(z+c)(f_{z_1} + g_{z_1 z_1}) = 1, \\ g(z+c)(g_{z_1} + f_{z_1 z_1}) = 1 \end{cases} \tag{14}$$

has no any pair of finite order entire solutions.

LEMMAS

The following lemmas play the key role in proving our results.

Lemma 1 ([28, 29]) *For an entire function F on $\mathbb{C}^n$, $F(0) \neq 0$ and put $\rho(n_F) = \rho < \infty$. Then there exist a canonical function f_F and a function $g_F \in \mathbb{C}^n$ such that $F(z) = f_F(z)\,\mathrm{e}^{g_F(z)}$. For the special case $n = 1$, f_F is the canonical product of Weierstrass.*

Remark 1 Here, denote $\rho(n_F)$ to be the order of the counting function of zeros of F.

Lemma 2 ([30]) *If g and h are entire functions on the complex plane $\mathbb{C}$ and $g(h)$ is an entire function of finite order, then there are only two possible cases: either (a) the internal function h is a polynomial and the external function g is of finite order; or else (b) the internal function h is not a polynomial but a function of finite order, and the external function g is of zero order.*

Lemma 3 *Let $g(u) = g(x, y)$ be a polynomial in $\mathbb{C}^2$, and $u_0 = (x_0, y_0)$, x_0, $y_0 \in \mathbb{C}$. If $g(u+u_0) - g(u) = g(x+x_0, y+y_0) - g(x, y)$ is a constant, then $g(u)$ can be represented as the form of*

$$g(x, y) = L(u) + H(s),$$

where $L(u) = \alpha x + \beta y$, α, β are constants, and $H(s)$ is a polynomial in s in $\mathbb{C}$, $s := y_0 x - x_0 y$.

Proof: From the assumption of this lemma, we can write $g(x, y)$ as the form

$$\begin{aligned} g(u) = g(x, y) &= \sum_{j=0}^{n} Q_j(y) x^j \\ &= Q_n(y)x^n + Q_{n-1}(y)x^{n-1} + \cdots + Q_1(y)x + Q_0(y), \end{aligned} \tag{15}$$

where $Q_j(y)$, $j = 0, 1, \ldots, n$ are polynomials in y. Since $g(u+u_0) - g(u) = g(x+x_0, y+y_0) - g(x, y)$ is a constant, let

$$\eta = g(u+u_0) - g(u) = g(x+x_0, y+y_0) - g(x, y). \tag{16}$$

Next, three cases will be considered.

Case 1. $x_0 \neq 0$, $y_0 = 0$. Thus, we have from (16) that

$$\begin{aligned} \eta &= g(x+x_0, y) - g(x, y) \\ &= \sum_{j=0}^{n} Q_j(y)\left[(x+x_0)^j - x^j\right] \\ &= \sum_{j=1}^{n} Q_j(y)[C_j^1 x_0 x^{j-1} + \cdots + C_j^j (x_0)^j], \end{aligned} \tag{17}$$

where $C_j^i = \frac{j(j-1)\cdots(j-i+1)}{i!}$. If $n = 0$, then $g(u) = Q_0(y)$. Obviously, $g(u) = H(s)$.

If $n \geqslant 1$, we have from (17) that

$$Q_n(y) \equiv Q_{n-1}(y) \equiv \cdots \equiv Q_2(y) \equiv 0, \tag{18}$$

and

$$Q_1(y) = \frac{\eta}{x_0}(\text{Const.}). \tag{19}$$

Thus, we conclude from (18) and (19) that

$$\begin{aligned} g(x, y) &= Q_1(y)x + Q_0(y) \\ &= \frac{\eta}{x_0}x + \gamma_m y^m + \gamma_{m-1}y^{m-1} + \cdots + \gamma_1 y + \gamma_0 \\ &= \alpha x + \beta y + H(s), \end{aligned} \tag{20}$$

where $\alpha=\frac{\eta}{x_0}$, $\beta=0$, $d_j=\frac{\gamma_j}{(-x_0)^j}$, $j=0,1,\dots,m$ and

$$H(s)=H(-x_0y)=d_ms^m+d_{m-1}s^{m-1}+\dots+d_1s+d_0.$$

Case 2. $y_0\neq 0$, $x_0=0$. Here we can rewrite $g(u)$ as the following form

$$g(u)=g(x,y)=\sum_{j=0}^{m}Q_m(x)y^m.$$

Using the same argument as in Case 1, we can prove that $g(x,y)$ is of the form $\alpha x+\beta y+H(s)$.

Case 3. $x_0\neq 0$, $y_0\neq 0$. We have

$$\begin{aligned}\eta&=g(u+u_0)-g(u)\\&=\sum_{j=0}^{n}\big[Q_j(y+y_0)(x+x_0)^j-Q_j(y)x^j\big]\\&=Q_n(y+y_0)(x+x_0)^n-Q_n(y)x^n\\&\quad+Q_{n-1}(y+y_0)(x+x_0)^{n-1}-Q_{n-1}(y)x^{n-1}+\cdots\\&\quad+Q_1(y+y_0)(x+x_0)-Q_1(y)x+Q_0(y+y_0)-Q_0(y).\end{aligned}\tag{21}$$

If $n\leqslant 1$, by analyzing the coefficients of x,y in both sides of (21), we have

$$Q_1(y+y_0)-Q_1(y)\equiv 0,\tag{22}$$
$$x_0Q_1(y+y_0)+Q_0(y+y_0)-Q_0(y)=\eta.\tag{23}$$

Equation (22) implies that $Q_1(y)$ is a constant. Let $Q_1(y)=\alpha$, then it follows from (23) that

$$Q_0(y+y_0)-Q_0(y)=\eta-\alpha x_0.$$

Since $Q_0(y)$ is a polynomial in y, it yields that $Q_0(y)$ is a polynomial in y with the degree $\leqslant 1$, that is, $Q_0(y)=\beta y+b_0$, where $\beta=\frac{\eta-\alpha x_0}{y_0}$. Hence, we have that $g(u)=\alpha x+\beta y+H(s)$, where $H(s)=b_0$.

If $n\geqslant 2$, by analyzing the coefficients of x^n, x^{n-1} in both sides of (21), we have

$$Q_n(y+y_0)-Q_n(y)\equiv 0,\tag{24}$$
$$Q_n(y+y_0)C_n^1x_0+Q_{n-1}(y+y_0)-Q_{n-1}(y)\equiv 0.\tag{25}$$

Equation (24) implies that $Q_n(y)$ is a constant, let $Q_n(y)=a_n^0$. Thus, it follows from (25) that $Q_{n-1}(y)$ is a polynomial in y with degree $\leqslant 1$, let $Q_{n-1}(y)=a_{n-1}^0y+a_{n-1}^1$, where a_{n-1}^0, a_{n-1}^1 are two constants satisfying

$$na_n^0x_0=-a_{n-1}^0y_0,$$

that is,

$$\frac{a_n^0}{a_{n-1}^0}=-\frac{1}{n}\frac{y_0}{x_0}.\tag{26}$$

Now, we continue to analyze the coefficient of x^{n-2} in both sides of (21) and obtain

$$C_n^2a_n^0(y_0)^2+Q_{n-1}(y+y_0)(n-1)x_0+Q_{n-2}(y+y_0)-Q_{n-2}(y)\equiv 0,$$

which implies that $Q_{n-2}(y)$ is a polynomial in y with degree $\leqslant 2$, let

$$Q_{n-2}(y)=a_{n-2}^0y^2+a_{n-2}^1y+a_{n-2}^2,$$

where a_{n-2}^0, a_{n-2}^1, a_{n-2}^2 are constants. Substituting the above into (25), we have

$$2y_0a_{n-2}^0=-a_{n-1}^0x_0(n-1),$$

that is,

$$\frac{a_{n-1}^0}{a_{n-2}^0}=-\frac{2}{n-1}\frac{y_0}{x_0}.\tag{27}$$

Similar to the same argument as in the above, we have that $Q_j(y)$ is a polynomial in y with degree $\leqslant n-j$ for $j=1,\dots,n$. Let

$$Q_j(y)=a_j^0y^{n-j}+a_j^1y^{n-j-1}+\dots+a_j^{n-j-1}y+a_j^{n-j},$$

where a_j^0, a_j^1, ..., a_j^{n-j} are constants. Thus, we have

$$\begin{aligned}\frac{a_{j+1}^0}{a_j^0}&=-\frac{C_n^{n-j-1}(x_0)^{n-j-1}(y_0)^{j+1}}{C_n^{n-j}(x_0)^{n-j}(y_0)^j}\\&=-\frac{n-j}{j+1}\frac{y_0}{x_0},\qquad j=1,2\dots,n.\end{aligned}\tag{28}$$

Hence, $g(x,y)$ can be represented as the following form

$$\begin{aligned}g(x,y)&=Q_n(y)x^n+Q_{n-1}(y)x^{n-1}+\dots+Q_1(y)x+Q_0(y)\\&=a_n^0x^n+(a_{n-1}^0y+a_{n-1}^1)x^{n-1}+(a_{n-2}^0y^2+a_{n-2}^1y+a_{n-2}^2)x^{n-2}\\&\quad+\dots+(a_1^0y^{n-1}+a_1^1y^{n-2}+\dots+a_1^{n-1})x+Q_0(y)\\&=a_n^0x^n+a_{n-1}^0yx^{n-1}+a_{n-2}^0y^2x^{n-2}+\dots+a_1^0y^{n-1}x+a_0^0y^n\\&\quad-a_0^0y^n+Q'_{n-1}(y)x^{n-1}+Q'_{n-2}(y)x^{n-2}+\dots+Q_0(y),\end{aligned}$$

where $Q'_j(y)=Q_j(y)-a_j^0y^{n-j}$, $j=1,2,\dots,n-1$, and a_0^0 is a constant satisfying

$$\frac{a_1^0}{a_0^0}=-n\frac{y_0}{x_0}.\tag{29}$$

Denote

$$P_n(x,y)=a_n^0x^n+a_{n-1}^0yx^{n-1}+a_{n-2}^0y^2x^{n-2}+\cdots+a_1^0y^{n-1}x+a_0^0y^n,\tag{30}$$

in view of (26)–(29), by a simple calculation, we can deduce that

$$\begin{aligned}P_n(x,y)&=a_n^0x^n+a_{n-1}^0yx^{n-1}+a_{n-2}^0y^2x^{n-2}+\cdots\\&\quad+a_1^0y^{n-1}x+a_0^0y^n\\&=b_0(y_0x-x_0y)^n,\end{aligned}\tag{31}$$

where

$$b_0 = \frac{a^0_{n-j}}{C^j_n(-x_0)^j y_0^{n-j}}, \qquad j = 0,1,\dots,n.$$

Thus, we have

$$\begin{aligned} g(x,y) &= P_n(x,y) + g'(x,y) \\ &= b_0(y_0x - x_0y)^n + g'(x,y), \end{aligned} \tag{32}$$

where

$$g'(x,y) = Q'_{n-1}(y)x^{n-1} + Q'_{n-2}(y)x^{n-2} + \dots + Q_0(y) - a^0_0 y^n.$$

Noting that $P_n(x+x_0, y+y_0) - P_n(x,y) \equiv 0$, we have from (16) that

$$\eta = g'(x+x_0, y+y_0) - g_1(x,y). \tag{33}$$

Similar to the above discussion for $g_1(x,y)$, we can get that

$$\begin{aligned} g(x,y) &= P_n(x,y) + P_{n-1}(x,y) + g_2(x,y) \\ &= b_0(y_0x - x_0y)^n + b_1(y_0x - x_0y)^{n-1} + g_2(x,y), \end{aligned}$$

where

$$\begin{aligned} g_2(x,y) = Q''_{n-2}(y)x^{n-2} + Q''_{n-3}(y)x^{n-3} + \cdots \\ + Q_0(y) - a^0_0 y^n - a^0_1 y^{n-1}. \end{aligned}$$

Repeat the above discussion several times, we have

$$\begin{aligned} g(x,y) &= P_n(x,y) + P_{n-1}(x,y) + \dots + P_2(x,y) + g_{n-1}(x,y) \\ &= b_0(y_0x - x_0y)^n + b_1(y_0x - x_0y)^{n-1} + \cdots \\ &\quad + b_2(y_0x - x_0y)^2 + g_{n-1}(x,y), \end{aligned} \tag{34}$$

where

$$\begin{aligned} g_{n-1}(x,y) &= a^{n-1}_1 x + Q_0(y) - a^0_0 y^n - a^0_1 y^{n-1} - \dots - a^0_{n-2} y^2 \\ &= a^{n-1}_1 x + Q'_0(y). \end{aligned} \tag{35}$$

Noting that $g_{n-1}(x+x_0, y+y_0) - g_{n-1}(x,y)$ is a constant, we have that $Q'_0(y)$ is a polynomial in y with degree $\leqslant 1$. Thus, we can denote that $Q'_0(y) = b^{n-1}_1 y + b_0$. Hence, we can deduce that

$$\begin{aligned} g(x,y) &= b_0(y_0x - x_0y)^n + b_1(y_0x - x_0y)^{n-1} \\ &\quad + b_2(y_0x - x_0y)^2 + \dots + b_{n-1}(y_0x - x_0y) + b_n. \end{aligned}$$

Therefore, this completes the proof of Lemma 3. □

PROOFS OF THEOREMS 1–2

The Proof of Theorem 1

Firstly, let (f,g) be a pair of finite order transcendental entire solutions of the system (8). Then we have that $f(z+c)$, $g(z+c)$, $f_{z_1} + f_{z_1z_1}$ and $g_{z_1} + g_{z_1z_1}$ have no any zero and pole. Otherwise, we can obtain a contradiction with f,g being entire functions. Then there exist two polynomials $\alpha, \beta \in \mathbb{C}^2$ such that

$$f(z+c) = e^{\alpha}, \quad g_{z_1} + g_{z_1z_1} = e^{-\alpha}, \tag{36}$$

and

$$g(z+c) = e^{\beta}, \quad f_{z_1} + f_{z_1z_1} = e^{-\beta}. \tag{37}$$

These yield that

$$\left[\alpha_{z_1} + \alpha_{z_1z_1} + (\alpha_{z_1})^2\right] e^{\alpha} = e^{-\beta(z+c)}, \tag{38}$$

and

$$\left[\beta_{z_1} + \beta_{z_1z_1} + (\beta_{z_1})^2\right] e^{\beta} = e^{-\alpha(z+c)}. \tag{39}$$

Equations (38) and (39) can lead to

$$\alpha(z) + \beta(z+c) = \eta_1, \quad \beta(z) + \alpha(z+c) = \eta_2, \tag{40}$$

where $\eta_1, \eta_2 \in \mathbb{C} - \{0\}$. It follows from (40) that

$$\begin{aligned} \alpha(z+2c) - \alpha(z) = \eta_2 - \eta_1, \\ \beta(z+2c) - \beta(z) = \eta_1 - \eta_2. \end{aligned} \tag{41}$$

By Lemma 3 and (40), we have

$$\begin{aligned} \alpha = L(z) + B_1 + H(c_2z_1 - c_1z_2), \\ \beta = -L(z) + B_2 - H(c_2z_1 - c_1z_2), \end{aligned} \tag{42}$$

where $H(s)$ is a polynomial in $s := c_2z_1 - c_1z_2$ in $\mathbb{C}^2$, $L(z) = A_1z_1 + A_2z_2$, A_1, A_2, B_1,$B_2 \in \mathbb{C}$. It follows from (42) that

$$\begin{aligned} \alpha_{z_1} = A_1 + c_2H', \quad \alpha_{z_1z_1} = c_2^2 H'', \\ \beta_{z_1} = -A_1 - c_2H', \quad \beta_{z_1z_1} = -c_2^2 H''. \end{aligned} \tag{43}$$

Substituting (43) into (38) and (39), we have

$$\begin{cases} (A_1 + c_2H')^2 = e^{-L(c) - B_2 - B_1} + A_1 + c_2H' + c_2^2H'', \\ (A_1 + c_2H')^2 = e^{L(c) - B_2 - B_1} - A_1 - c_2H' - c_2^2H''. \end{cases} \tag{44}$$

In view of $c_2 \neq 0$, this means that $\deg_s H \leqslant 1$. In fact, let $\deg_s H = n$. If $n \geqslant 2$, by comparing the exponent of s on both sides of the first equation or the second equation of (44), we have $2(n-1) = n-1$, which is a contradiction with $n-1 \neq 0$. Hence, $\deg_s H = n \leqslant 1$. Thus, we can still denote that

$$\alpha = L(z) + B_1, \quad \beta = -L(z) + B_2. \tag{45}$$

In view of (38), (39) and (45), we have

$$\begin{aligned} A_1(A_1+1)\, e^{-L(c) + B_1 + B_2} = 1, \\ A_1(A_1-1)\, e^{L(c) + B_1 + B_2} = 1, \end{aligned} \tag{46}$$

which implies that $A_1 \neq 0$, $A_1 \neq \pm 1$ and

$$e^{2L(c)} = \frac{A_1+1}{A_1-1}, \quad e^{2(B_1+B_2)} = \frac{1}{A_1^2(A_1^2-1)}. \tag{47}$$

And in view of (36),(37),(45) and (46), we can deduce that

$$\begin{cases} f = e^{\alpha(z-c)} = e^{L(z) + B_1 - L(c)} = \frac{1}{A_1(A_1+1)} e^{L(z) - B_2}, \\ g = e^{\beta(z-c)} = e^{-L(z) + L(c) + B_2} = \frac{1}{A_1(A_1-1)} e^{-L(z) - B_1}. \end{cases} \tag{48}$$

Therefore, this completes the proof of Theorem 1. □

The Proof of Theorem 2

Similar to the argument as in the proof of Theorem 1, there exist two polynomials α, β in $\mathbb{C}^2$ such that

$$f(z+c)=e^{\alpha}, \quad g_{z_1}+f_{z_1z_1}=e^{-\alpha}, \tag{49}$$

and

$$g(z+c)=e^{\beta}, \quad f_{z_1}+g_{z_1z_1}=e^{-\beta}. \tag{50}$$

Obviously, α,β are not constants, otherwise, we can obtain that f,g are constants, this is a contradiction with f,g being transcendental entire functions. Thus, it follows from (49) and (50) that

$$\beta_{z_1}e^{\beta}+\left[\alpha_{z_1z_1}+(\alpha_{z_1})^2\right]e^{\alpha}\equiv e^{-\alpha(z+c)},$$
$$\alpha_{z_1}e^{\alpha}+\left[\beta_{z_1z_1}+(\beta_{z_1})^2\right]e^{\beta}\equiv e^{-\beta(z+c)}.$$

These lead to

$$\beta_{z_1}e^{\beta+\alpha(z+c)}+\left[\alpha_{z_1z_1}+(\alpha_{z_1})^2\right]e^{\alpha+\alpha(z+c)}\equiv 1, \tag{51}$$
$$\alpha_{z_1}e^{\alpha+\beta(z+c)}+\left[\beta_{z_1z_1}+(\beta_{z_1})^2\right]e^{\beta+\beta(z+c)}\equiv 1. \tag{52}$$

Now, we will consider two cases below.

Case 1. If $\alpha_{z_1z_1}+(\alpha_{z_1})^2\equiv 0$. Set $X=\alpha_{z_1}$, we thus have $X_{z_1}+X^2\equiv 0$. If $X\not\equiv 0$, solving this equation, we have $\alpha_{z_1}=X=\frac{1}{z_1+\varphi_1(z_2)}$, where $\varphi_1(z_2)$ is a function in z_2. Then $\alpha=\log[z_1+\varphi_1(z_2)]+\varphi_2(z_2)$, where $\varphi_2(z_2)$ is a function in z_2. Thus, we can get a contradiction with α being a polynomial in $\mathbb{C}^2$. If $X=0$, then $\alpha=\phi(z_2)$, where $\phi(z_2)$ is a polynomial in z_2. By (52), we have

$$\left[\beta_{z_1z_1}+(\beta_{z_1})^2\right]e^{\beta+\beta(z+c)}\equiv 1,$$

which is impossible because β is a nonconstant polynomial. Similarly, we can get a contradiction if $\beta_{z_1z_1}+(\beta_{z_1})^2\equiv 0$, or $\alpha_{z_1}\equiv 0$ or $\beta_{z_1}\equiv 0$.

Case 2. If $\alpha_{z_1z_1}+(\alpha_{z_1})^2\not\equiv 0$. Noting that the fact that $\alpha+\alpha(z+c)\not\equiv 0$, using the Nevanlinna second fundamental for $G=[\alpha_{z_1z_1}+(\alpha_{z_1})^2]e^{\alpha+\alpha(z+c)}$, we have from (51) that

$$\begin{aligned}T(r,G)&\leqslant N(r,G)+N(r,\tfrac{1}{G})+N(r,\tfrac{1}{G-1})+S(r,G)\\&\leqslant N\left(r,\frac{1}{[\alpha_{z_1z_1}+(\alpha_{z_1})^2]e^{\alpha+\alpha(z+c)}}\right)\\&\quad+N\left(r,\frac{1}{\beta_{z_1}e^{\beta+\alpha(z+c)}}\right)+S(r,G)\\&\leqslant O(\log r)+S(r,G),\end{aligned}$$

which is a contradiction with α,β being nonconstant polynomials and $\alpha_{z_1z_1}+(\alpha_{z_1})^2\equiv 0$ and $\beta_{z_1}\equiv 0$.

This completes the proof of Theorem 2. □

PROOFS OF THEOREMS 3–5

The Proof of Theorem 3

Assume that (f,g) is a pair of finite order transcendental entire solutions of system (11). Then we have that $f(z+c)$, $f_{z_1}+f_{z_1z_2}$, $g(z+c)$ and $g_{z_1}+g_{z_1z_2}$ have no any zero and pole. Otherwise, we can obtain a contradiction with f,g being entire functions. Then there exist two polynomials α, $\beta\in\mathbb{C}^2$ such that

$$f(z+c)=e^{\alpha}, \quad g_{z_1}+g_{z_1z_2}=e^{-\alpha}, \tag{53}$$

and

$$g(z+c)=e^{\beta}, \quad f_{z_1}+f_{z_1z_2}=e^{-\beta}. \tag{54}$$

These yield that

$$\left[\alpha_{z_1}+\alpha_{z_1z_2}+\alpha_{z_1}\alpha_{z_2}\right]e^{\alpha}=e^{-\beta(z+c)}, \tag{55}$$

and

$$\left[\beta_{z_1}+\beta_{z_1z_2}+\beta_{z_1}\beta_{z_2}\right]e^{\beta}=e^{-\alpha(z+c)}. \tag{56}$$

Similar to the argument as in the proof of Theorem 1, and by Lemma 3 and (40), we have

$$\begin{aligned}\alpha&=L(z)+B_1+H(c_2z_1-c_1z_2),\\ \beta&=-L(z)+B_2-H(c_2z_1-c_1z_2),\end{aligned} \tag{57}$$

where $H(s)$ is a polynomial in $s:=c_2z_1-c_1z_2$ in $\mathbb{C}^2$, $L(z)=A_1z_1+A_2z_2$, A_1, A_2, B_1, $B_2\in\mathbb{C}$. It follows from (57) that

$$\alpha_{z_1}=A_1+c_2H',\ \alpha_{z_2}=A_2-c_1H',\ \alpha_{z_1z_2}=-c_1c_2H'', \tag{58}$$

$$\beta_{z_1}=-A_1-c_2H',\ \beta_{z_2}=-A_2+c_1H',\ \beta_{z_1z_2}=c_1c_2H''. \tag{59}$$

Substituting (58), (59) into (55) and (56), we have

$$\begin{aligned}(A_1+c_2H')(A_2-c_1H')&=e^{L(c)-B_2-B_1}-A_1-c_2H'+c_1c_2H'',\\(A_1+c_2H')(A_2-c_1H')&=e^{-L(c)-B_2-B_1}+A_1+c_2H'-c_1c_2H''.\end{aligned} \tag{60}$$

If $c_1=c_2=0$, then it follows from (60) that $A_1(A_2+1)=e^{-B_1-B_2}$ and $A_1(A_2-1)=e^{-B_1-B_2}$, which leads to $-1=1$. This is a contradiction.

If $c_1=0$, $c_2\neq 0$. Then it follows from (60) that $A_2(A_1+c_2H')=e^{L(c)-B_2-B_1}-A_1-c_2H'$ and $A_2(A_1+c_2H')=e^{L(c)-B_2-B_1}+A_1+c_2H'$. This leads to $H'\equiv$ Const. If $A_2\neq 0$, we have $A_2c_2H'=-c_2H'$ and $A_2c_2H'=c_2H'$. Noting that $c_2\neq 0$, we have $H'\equiv 0$. If $A_2\equiv 0$, then it follows $e^{L(c)-B_2-B_1}=A_1+c_2H'$ which means that $H'\equiv$ Const.

If $c_1\neq 0$, $c_2=0$. Then it follows from (60) that $A_1(A_2-c_1H')=e^{L(c)-B_2-B_1}-A_1$ and $A_1(A_2+c_1H')=e^{L(c)-B_2-B_1}+A_1$. Obviously, this leads to $H'\equiv$ Const.

If $c_1\neq 0$, $c_2\neq 0$, similar to the argument as in the proof of Theorem 1, we can deduce $H'\equiv$ Const. Hence, we have $\deg_s H=n\leqslant 1$. Thus, we can still denote that

$$\alpha=L(z)+B_1, \quad \beta=-L(z)+B_2. \tag{61}$$

In view of (55), (56) and (61), we have

$$\begin{aligned}A_1(A_2+1)e^{-L(c)+B_1+B_2}&=1,\\A_1(A_2-1)e^{L(c)+B_1+B_2}&=1,\end{aligned} \tag{62}$$

which implies that $A_1 \neq 0, A_2 \neq \pm 1$ and

$$e^{2L(c)} = \frac{A_2+1}{A_2-1}, \quad e^{2(B_1+B_2)} = \frac{1}{A_1^2(A_2^2-1)}. \tag{63}$$

And in view of (53),(54),(61) and (62), we can deduce that

$$\begin{aligned} f &= e^{\alpha(z-c)} = e^{L(z)+B_1-L(c)} = \tfrac{1}{A_1(A_2+1)} e^{L(z)-B_2}, \\ g &= e^{\beta(z-c)} = e^{-L(z)+L(c)+B_2} = \tfrac{1}{A_1(A_2-1)} e^{-L(z)-B_1}. \end{aligned} \tag{64}$$

This completes the proof of Theorem 3. □

Proofs of Theorems 4 and 5

We only give the details of the proof of Theorem 4 because the proof of Theorem 4 is similar with the proof of Theorem 5. Similar to the argument as in the proof of Theorem 2, there exist two polynomials α, β in $\mathbb{C}^2$ such that

$$f(z+c) = e^{\alpha}, \quad f_{z_1} + g_{z_1z_2} = e^{-\alpha}, \tag{65}$$

and

$$g(z+c) = e^{\beta}, \quad g_{z_1} + f_{z_1z_2} = e^{-\beta}. \tag{66}$$

Obviously, α, β are not constants, otherwise, we can obtain that f, g are constants, this is a contradiction with f, g being transcendental entire functions. Thus, it follows from (65) and (66) that

$$\begin{aligned} &\alpha_{z_1} e^{\alpha} + \left(\beta_{z_1z_2} + \beta_{z_1}\beta_{z_2}\right) e^{\beta} \equiv e^{-\alpha(z+c)}, \\ &\beta_{z_1} e^{\beta} + \left(\alpha_{z_1z_1} + \alpha_{z_1}\alpha_{z_2}\right) e^{\alpha} \equiv e^{-\beta(z+c)}. \end{aligned}$$

These lead to

$$\alpha_{z_1} e^{\alpha+\alpha(z+c)} + \left(\beta_{z_1z_2} + \beta_{z_1}\beta_{z_2}\right) e^{\beta+\alpha(z+c)} \equiv 1, \tag{67}$$

$$\beta_{z_1} e^{\beta+\beta(z+c)} + \left(\alpha_{z_1z_2} + \alpha_{z_1}\alpha_{z_2}\right) e^{\alpha+\beta(z+c)} \equiv 1. \tag{68}$$

Now, we will consider two cases below.

Case 1. If $\alpha_{z_1} \equiv 0$, then $\alpha = \phi(z_2)$ where $\phi(z_2)$ is a polynomial in z_2, and $(\beta_{z_1z_2} + \beta_{z_1}\beta_{z_2}) e^{\beta+\alpha(z+c)} \equiv 1$, which implies that $\beta + \alpha(z+c) \equiv \eta$, where η is a constant. Thus, it follows that $\beta = \eta - \phi(z_2+c_2)$. This leads to $\beta_{z_1} \equiv 0$ and $\beta_{z_1z_2} \equiv 0$. In view of (67), we can deduce a contradiction. If $\beta_{z_1} \equiv 0$, we can get a contradiction in view of (68).

Case 2. If $\alpha_{z_1} \not\equiv 0$, then $\beta_{z_1z_2} + \beta_{z_1}\beta_{z_2} \not\equiv 0$. Otherwise, it follows from (67) that $\alpha_{z_1} e^{\alpha+\alpha(z+c)} \equiv 1$, which implies that $\alpha+\alpha(z+c)$ is a constant, this is impossible. By using the Nevanlinna second fundamental for $F = \alpha_{z_1} e^{\alpha+\alpha(z+c)}$, we have from (67) that

$$\begin{aligned} T(r,F) &\leqslant N(r,F) + N(r,\tfrac{1}{F}) + N(r,\tfrac{1}{F-1}) + S(r,F) \\ &\leqslant N\left(r, \frac{1}{\alpha_{z_1} e^{\alpha+\alpha(z+c)}}\right) \\ &\quad + N\left(r, \frac{1}{\left(\beta_{z_1z_2} + \beta_{z_1}\beta_{z_2}\right) e^{\beta+\alpha(z+c)}}\right) + S(r,F) \\ &\leqslant O(\log r) + S(r,F), \end{aligned}$$

which is a contradiction with α, β being nonconstant polynomials and $\alpha_{z_1} \not\equiv 0$, $\beta_{z_1z_2} + \beta_{z_1}\beta_{z_2} \not\equiv 0$.

This completes the proof of Theorem 4. □

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