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Cubic symmetric graphs of order 24p

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ABSTRACT: A graph is symmetric if its automorphism group acts transitively on the set of arcs of the graph. In this paper, we classify connected cubic symmetric graphs of order 24p for each prime p. As a result, there are ten sporadic graphs and two infinite families of one-regular graphs: one is the $\mathbb{Z}_{3p}$-cover of the three dimensional hypercube Q_3 and the other is the normal bi-Cayley graph on the group $(\mathbb{Z}_p \rtimes \mathbb{Z}_3) \times \mathbb{Z}_2^2$. In particular, we use the normal bi-Cayley graphs rather than the voltage graph techniques to determine the latter graph.

KEYWORDS: symmetric graph, s-regular graph, bi-Cayley graph, voltage graph, covering graph

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INTRODUCTION

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [1, 2] or [3, 4], respectively. Let G be a permutation group on a set Ω and $v \in \Omega$. Denote by G_v the stabilizer of v in G, that is, the subgroup of G fixing the point v. We say that G is *semiregular* on Ω if $G_v = 1$ for every $v \in \Omega$ and *regular* if G is transitive and semiregular.

For a graph X, denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph X is said to be *G-vertex-transitive* if $G \leqslant \mathrm{Aut}(X)$ acts transitively on $V(X)$. X is simply called *vertex-transitive* if it is $\mathrm{Aut}(X)$-vertex-transitive. An *s-arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leqslant i \leqslant s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leqslant i \leqslant s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leqslant \mathrm{Aut}(X)$, a graph X is said to be *(G,s)-arc-transitive* or *(G,s)-regular* if G is transitive or regular on the set of s-arcs in X, respectively. A (G,s)-arc-transitive graph is said to be *(G,s)-transitive* if it is not $(G,s+1)$-arc-transitive. In particular, a $(G,1)$-arc-transitive graph is called *G-symmetric*. A graph X is simply called *s-arc-transitive*, *s-regular* or *s-transitive* if it is $(\mathrm{Aut}(X),s)$-arc-transitive, $(\mathrm{Aut}(X),s)$-regular or $(\mathrm{Aut}(X),s)$-transitive, respectively.

Tutte [5, 6] showed that every finite cubic symmetric graph is s-regular for some $s \geqslant 1$, and this s is at most five. Djoković and Miller [7] constructed an infinite family of cubic 2-regular graphs, and Conder and Praeger [8] constructed two infinite families of cubic s-regular graphs for $s = 2$ or 4. The first cubic 1-regular graph was constructed by Frucht [9] and later Miller [10] constructed an infinite family of cubic 1-regular graphs of order 2p, where $p \geqslant 13$ is a prime congruent to 1 modulo 3. By Cheng and Oxley's classification of symmetric graphs of order 2p [11], Miller's construction actually comprises all cubic 1-regular graphs of order 2p. Marušič and Xu [12] showed a way to construct a cubic 1-regular graph Y from a tetravalent half-arc-transitive graph X with girth 3 by letting the triangles of X be the vertices in Y with two triangles being adjacent when they share a common vertex in X. Using Marušič and Xu's result above, Miller's construction can be generalized to graphs of order 2n, where $n \geqslant 13$ is odd such that 3 divides $\phi(n)$, the Euler function (see [13]). It may be shown that all cubic 1-regular Cayley graphs on the dihedral groups are exactly the graphs obtained by the generalized Miller's construction. Also, as shown in [14] or [13], we can see an importance of a study for cubic 1-regular graphs in connection with chiral (that is regular and irreflexible) maps on a surface by means of tetravalent half-arc-transitive graphs or in connection with symmetries of hexagonal molecular graphs on the torus. Conder and Docsányi [15, 16] classified the cubic s-regular graphs of order up to 2048 with the help of Magma [17] system. Kutnar and Marušič [18] gave a complete classification of cubic symmetric graphs of girth 6.

Covering techniques have long been known as a powerful tool in topology and graph theory. Regular covering of a graph is currently an active topic in algebraic graph theory. By using voltage graph techniques, arc-transitive covers of many cubic symmetric graphs have been determined, see [19, 20] and references therein. In particular, arc-transitive cyclic covers have been determined in [19] for the complete graph K_4, [21] for the complete bipartite graph $K_{3,3}$, [22] for the three dimensional hypercube Q_3. As the results above, a complete classification of cubic symmetric graphs of order 4p, 6p, 8p and et al can be obtained, see [19, 22]. Along with the increase of the order of base graphs, the voltage graphs may be more complex. Thus, in this paper, in order to avoid using the voltage graphs, we use the automorphisms and normalities of bi-Cayley graphs to construct an infinite family of cubic 1-regular normal bi-Cayley graphs on the non-abelian group $(\mathbb{Z}_p \rtimes \mathbb{Z}_3) \times \mathbb{Z}_2^2$, and as an application, classify cubic symmetric graphs of order $24p$ for each prime p.

PRELIMINARY RESULTS

Let X be a connected G-symmetric graph with $G \leqslant \mathrm{Aut}(X)$, and let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [23, Theorem 9], we have the following:

Proposition 1 *Let X be a connected cubic G-symmetric graph with $G \leqslant \mathrm{Aut}(X)$, and let N be a normal subgroup of G. Then one of the following holds:*
(1) *N is transitive on $V(X)$;*
(2) *X is bipartite and N is transitive on each part of the bipartition;*
(3) *N has $r \geqslant 3$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected cubic G/N-symmetric graph.*

Djoković and Miller [10, Propositions 2–5] gave the structure of the vertex stabilizer of any group acting regularly on the s-arcs of a connected cubic graph.

Proposition 2 *Let X be a connected cubic (G,s)-transitive graph for some $G \leqslant \mathrm{Aut}(X)$ and $s \geqslant 1$. Let $v \in V(X)$. Then $s \leqslant 5$, and $G_v \cong \mathbb{Z}_3$ for $s=1$; $G_v \cong \mathrm{S}_3$ for $s=2$; $G_v \cong \mathrm{S}_3 \times \mathbb{Z}_2$ for $s=3$; $G_v \cong \mathrm{S}_4$ for $s=4$; $G_v \cong \mathrm{S}_4 \times \mathbb{Z}_2$ for $s=5$. In particular, $|G_v| \,\big|\, 48$.*

From [24, p. 12–14], we can deduce the non-abelian simple groups whose orders have at most three different prime divisors.

Proposition 3 *Let G be a non-abelian simple group. If the order $|G|$ has at most three different prime divisors, then G is called K_3-simple group and isomorphic to one of the following groups.*

Table 1 Non-abelian simple $\{2,3,p\}$-groups.

Group	Order	Group	Order
A_5	$2^2 \cdot 3 \cdot 5$	$\mathrm{PSL}(2,17)$	$2^4 \cdot 3^2 \cdot 17$
A_6	$2^3 \cdot 3^2 \cdot 5$	$\mathrm{PSL}(3,3)$	$2^4 \cdot 3^3 \cdot 13$
$\mathrm{PSL}(2,7)$	$2^3 \cdot 3 \cdot 7$	$\mathrm{PSU}(3,3)$	$2^5 \cdot 3^3 \cdot 7$
$\mathrm{PSL}(2,8)$	$2^3 \cdot 3^2 \cdot 7$	$\mathrm{PSU}(4,2)$	$2^6 \cdot 3^4 \cdot 5$

The following proposition is the famous "N/C-Theorem", see for example [25, Chapter I, Theorem 4.5]).

Proposition 4 *The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(H)$ of H.*

In view of [19, Theorem 5.2] and [18, Theorem 1.2], the classification of connected cubic symmetric graphs of order $6p$ are given. With the same notations as that in [19], we have that following proposition.

Proposition 5 *Let p be a prime and X a connected cubic symmetric graph of order $6p$. Then one of the following holds*
(1) *X is 1-regular, $X \cong CB_p$ and $\mathrm{Aut}(X) \cong D_{6p} \rtimes \mathbb{Z}_3$ with $3 \,\big|\, (p-1)$;*
(2) *X is 3-regular and isomorphic to Pappus graph with $p=3$;*
(3) *X is 4-regular and isomorphic to Smith-Biggs graph with $p=17$;*
(4) *X is 5-regular and isomorphic to Levi graph with $p=5$.*

With the same notation as that in [26], the classification of cubic symmetric graphs of order $8p$ can be obtained from [26, Theorem 5.1].

Proposition 6 *Let p be a prime and X a connected cubic symmetric graph of order $8p$. Then one of the following holds*
(1) *X is 1-regular, $X \cong CQ_p$ and $\mathrm{Aut}(X) \cong \mathbb{Z}_p \rtimes (\mathbb{Z}_2^3 \rtimes \mathbb{Z}_3)$ with $3 \,\big|\, (p-1)$;*
(2) *X is 2-regular and isomorphic to CQ_2, CQ_3 or Lorimer graph with $p=2$, 3 or 7;*
(3) *X is 3-regular and isomorphic to canonical double covering $D_{20}^{(2)}$ or $C_{28}^{(2)}$ with $p=5$ or 7.*

From [27, Theorem 1.1], we have that classification of the cubic symmetric graphs of order $12p$.

Proposition 7 *Let p be a prime and X a connected cubic symmetric graph of order $12p$. Then X is isomorphic to F_{24}, F_{60}, F_{84} or F_{204} with $p=2$, 5, 7 or 17.*

GRAPH CONSTRUCTIONS

In this section, we construct two infinite families of cubic 1-regular graphs. First, we introduce the bi-Cayley graph $X = \mathrm{BiCay}(H,R,L,S)$ over a group H. Here, we remind the reader that R, L and S are subsets of H such that $1 \notin R = R^{-1}$ and $1 \notin L = L^{-1}$, and

$V(X) = H_0 \cup H_1$, with $H_0 = \{h_0 \mid h \in H\}$ and $H_1 = \{h_1 \mid h \in H\}$,

$E(X) = \{h_0, g_0 \mid gh^{-1} \in R\} \cup \{h_1, g_1 \mid gh^{-1} \in L\} \cup \{h_0, g_1 \mid gh^{-1} \in S\}$.

If $R = L = \varnothing$, then $\mathrm{BiCay}(H,\varnothing,\varnothing,S)$ will be called 0-*type* bi-Cayley graph. It is easy to prove some basic facts for bi-Cayley graphs, see [28, Lemma 3.1].

Proposition 8 *The following hold for any connected bi-Cayley graph* $\mathrm{BiCay}(H,R,L,S)$.

(1) *H is generated by $R\cup L\cup S$.*
(2) *Up to graph isomorphism, S can be chosen to contain the identity of H.*
(3) *For any automorphism α of H,* $\mathrm{BiCay}(H,R,L,S)\cong\mathrm{BiCay}(H,R^\alpha,L^\alpha,S^\alpha)$.

Below, we collect several results about the automorphisms of $X=\mathrm{BiCay}(H,R,L,S)$. For each $g\in H$, we define a permutation $\mathscr{R}(g)$ on $V(X)$ as follows:

$$\mathscr{R}(g): h_i\mapsto (hg)_i, \forall i\in\mathbb{Z}_2, h\in H.$$

Set $\mathscr{R}(H)=\{\mathscr{R}(g)\mid g\in H\}$. Then $\mathscr{R}(H)$ is a semiregular subgroup of $\mathrm{Aut}(X)$ with H_0 and H_1 as its two orbits.

Take $\alpha\in\mathrm{Aut}(H)$ and $x,y,g\in H$. Then we define two permutations $\delta_{\alpha,x,y}$ and $\sigma_{\alpha,g}$ on $V(X)$ as follows:

$$\begin{aligned}\delta_{\alpha,x,y}:\ & h_0\mapsto (xh^\alpha)_1, \quad h_1\mapsto (yh^\alpha)_0, \quad \forall h\in H,\\ \sigma_{\alpha,g}:\ & h_0\mapsto (h^\alpha)_0, \quad h_1\mapsto (gh^\alpha)_1, \quad \forall h\in H.\end{aligned}$$

Set

$$\begin{aligned}I=\big\{\delta_{\alpha,x,y}\mid \alpha\in\mathrm{Aut}(H),\ & R^\alpha=x^{-1}Lx,\\ & L^\alpha=y^{-1}Ry,\ S^\alpha=y^{-1}S^{-1}x\big\},\\ F=\{\sigma_{\alpha,g}\mid \alpha\in\mathrm{Aut}(H),\ & R^\alpha=R,\ L^\alpha=L,\ S^\alpha=g^{-1}S\}.\end{aligned}$$

Zhou and Feng [28] proved that both I and F are contained in $\mathrm{Aut}(X)$, and furthermore normalize $\mathscr{R}(H)$. The following proposition is about the normalizer of $\mathscr{R}(H)$, see [28, Theorem 1.1].

Proposition 9 *Let $X=\mathrm{BiCay}(H,R,L,S)$ be a connected bi-Cayley graph over the group H. Then $N_{\mathrm{Aut}(X)}(\mathscr{R}(H))=\mathscr{R}(H)\rtimes F$ if $I=\varnothing$ and $N_{\mathrm{Aut}(X)}(\mathscr{R}(H))=\mathscr{R}(H)\langle F,\delta_{\alpha,x,y}\rangle$ if $I\neq\varnothing$ and $\delta_{\alpha,x,y}\in I$.*

When $\mathscr{R}(H)$ is normal in $\mathrm{Aut}(X)$, the bi-Cayley graph $X=\mathrm{BiCay}(H,R,L,S)$ will be called a *normal bi-Cayley graph* over H.

Construction 1 *Let p be prime with $3\,\big|\,(p-1)$ and $H\cong\mathbb{Z}_2^2\times F_{3p}$ with the following relations:*

$$\begin{aligned}H=\langle a,b,c,d\mid a^p=b^3=c^2=d^2=[a,c]=[a,d]\\ =[b,c]=[b,d]=[c,d]=1, a^b=a^r\rangle,\end{aligned}$$

where $r\in\mathbb{Z}_p^*$ is of order 3. Take $S=\{1,bc,ab^{-1}d\}$. Define the 0-type bi-Cayley graph as follows:

$$\mathscr{BC}_{24p}(\mathbb{Z}_2^2\times F_{3p})=\mathrm{BiCay}(H,\varnothing,\varnothing,S).$$

Then $\mathscr{BC}_{24p}(\mathbb{Z}_2^2\times F_{3p})$ is a connected cubic graph of order $24p$.

Lemma 1 *Let X be a connected cubic symmetric normal bi-Cayley graph over the group H. Then $X\cong\mathscr{BC}_{24p}(\mathbb{Z}_2^2\times F_{3p})$ and*

$$\begin{aligned}\mathrm{Aut}(X)&=N_{\mathrm{Aut}(\mathscr{BC}_{24p}(\mathbb{Z}_2^2\times F_{3p}))}(\mathscr{R}(H))\\ &\cong((\mathbb{Z}_2^2\times F_{3p}).\mathbb{Z}_2)\rtimes\mathbb{Z}_3.\end{aligned}$$

In particular, $F\cong\mathbb{Z}_3$ and the girth of $\mathscr{BC}_{24p}(\mathbb{Z}_2^2\times F_{3p})$ is 12.

Proof: Since X is a connected cubic symmetric normal bi-Cayley graph over the group H, X is a 0-type bi-Cayley graph and we can assume that $X=\mathrm{BiCay}(H,\varnothing,\varnothing,S)$. By Proposition 8, we may assume that $1\in S$. Clearly, $|S|=3$ and $\langle S\rangle=G$. Since X is a normal bi-Cayley graph, by Proposition 9, we have there is $\delta_{\alpha,x,y}\in I$ such that

$$\mathrm{Aut}(X)=N_{\mathrm{Aut}(X)}(\mathscr{R}(H))=\mathscr{R}(H)\langle F,\delta_{\alpha,x,y}\rangle.$$

Since X is cubic and symmetric, we have that $3\cdot 24p\,\Big|\,|\mathrm{Aut}(X)|$ and hence $6\,\Big|\,|\langle F,\delta_{\alpha,x,y}\rangle|$. On the other hand, the definition of I implies that any non-identity element of I interchanges H_0 and H_1 of X. Thus, $3\,\Big|\,|F|$. In the following we determine the structure of F and $\delta_{\alpha,x,y}\in I$. First, Each automorphism θ in $\mathrm{Aut}(H)$ can be written as follows:

$$\theta:\begin{cases}a\mapsto a^i, & 1\leqslant i\leqslant p-1,\\ b\mapsto a^jb, & 0\leqslant j\leqslant p-1,\\ c\mapsto c^kd^m, & k,l,m,n\in\mathbb{Z}_2;\ k^2+m^2\neq 0,\\ d\mapsto c^ld^n, & k\neq l \text{ or } m\neq n;\ l^2+n^2\neq 0.\end{cases}$$

Note that $H\cong\mathbb{Z}_2^2\times(\mathbb{Z}_p\rtimes\mathbb{Z}_3)$. It is easy to calculate that $\mathrm{Aut}(H)\cong\mathrm{S}_3\times(\mathbb{Z}_p\rtimes\mathbb{Z}_{p-1})$, and any automorphism of H cannot map b to its inverse b^{-1}.

Since the Sylow 2-subgroup of H is isomorphic to $\mathbb{Z}_2^2$ and has at least two generators and 2-elements commutes with p-elements and 3-elements in H, we can deduce that S has an element of order 6. Since $\mathrm{Aut}(H)$ acting on the elements of order 6 has two orbits, their representatives are bc and $b^{-1}c$. Thus, we may assume that $S=\{1,bc,sd\}$ or $\{1,b^{-1}c,sd\}$ where $s\in\langle a,b\rangle$. Take an automorphism $\eta\in\mathrm{Aut}(H)$: $a\mapsto a$, $b\mapsto b$, $c\mapsto c$, $d\mapsto cd$. Then, $(bc\{1,b^{-1}c,sd\})^\eta=\{bc,1,bscd\}^\eta=\{1,bc,bsd\}$ with $bs\in\langle a,b\rangle$. This implies that $\{1,b^{-1}c,sd\}\equiv\{1,bc,sd\}$. Since $\mathrm{Aut}(H)$ acting on $\{a,a^2,\cdots,a^{p-1}\}$ is transitive, we can choose $s=a$, ab or ab^{-1}. Thus, $S=\{1,bc,ad\}$, $\{1,bc,abd\}$ or $\{1,bc,ab^{-1}d\}$. Now we can use S to determine the normalizer of $\mathscr{R}(H)$ by the structure of F and I defined above.

Case 1. Let $S=\{1,bc,ad\}$.

Note that $F=\{\sigma_{\alpha,g}\mid\alpha\in\mathrm{Aut}(H), S^\alpha=g^{-1}S\}$. Thus,

$$S^\alpha=\{1,b^\alpha c^\alpha,a^\alpha d^\alpha\}=g^{-1}S=\{g^{-1},g^{-1}bc,g^{-1}ad\}.$$

It forces that $1 \in g^{-1}S$ and $g = 1$, bc or ad.

Let $g = 1$. Then $\{1, bc, ad\} = \{1, b^\alpha c^\alpha, a^\alpha d^\alpha\}$. Since bc has order 6 and ad has order $2p$ with $p \geqslant 7$, we have $b^\alpha c^\alpha = bc$ and $a^\alpha d^\alpha = ad$. It follows that $a^\alpha = a$, $b^\alpha = b$, $c^\alpha = c$ and $d^\alpha = d$. This forces that $\alpha = 1$ and hence $\sigma_{\alpha,g} = 1$.

Let $g = bc$. Then $\{1, b^{-1}c, a^r b^{-1}cd\} = \{1, b^\alpha c^\alpha, a^\alpha d^\alpha\}$. Since the order of ad is $2p$ with $p \geqslant 7$, we have that α cannot map ad to any element in $\{1, b^{-1}c, a^r b^{-1}cd\}$ because both $b^{-1}c$ and $a^r b^{-1}cd$ have orders 6, a contradiction.

Let $g = ad$. Then $\{1, a^{-1}d, a^{-1}bcd\} = \{1, b^\alpha c^\alpha, a^\alpha d^\alpha\}$. Since the orders $o(a^{-1}d) = 2p$ and $o(a^{-1}bcd) = 6$, we have that $a^\alpha d^\alpha = a^{-1}d$ and $b^\alpha c^\alpha = a^{-1}bcd$. Thus, $a^\alpha = a^{-1}$, $b^\alpha = a^{-1}b$, $c^\alpha = cd$ and $d^\alpha = d$. An easy calculation implies that the order $o(\alpha) = 2$ and

$$\sigma_{\alpha,ad}: \ h_0 \mapsto (h^\alpha)_0, \quad h_1 \mapsto (adh^\alpha)_1,$$
$$\sigma^2_{\alpha,ad}: \ h_0 \mapsto (h^{\alpha^2})_0 = h_0, \quad h_1 \mapsto (ad(adh^\alpha)^\alpha)_1 = h_1.$$

Thus, $o(\sigma_{\alpha,ad}) = 2$. By the definition of F, we can deduce that $F = \langle \sigma_{\alpha,ad} \rangle \cong \mathbb{Z}_2$, where $\alpha \in \mathrm{Aut}(H) : a^\alpha = a^{-1}$, $b^\alpha = a^{-1}b$, $c^\alpha = cd$ and $d^\alpha = d$. By Proposition 9, $N_{\mathrm{Aut}(X)}(\mathscr{R}(H))$ cannot act arc-transitively on X, a contradiction.

Case 2. Let $S = \{1, bc, abd\}$.

Note that $F = \{\sigma_{\alpha,g} \mid \alpha \in \mathrm{Aut}(H), S^\alpha = g^{-1}S\}$. Thus,

$$S^\alpha = \{1, b^\alpha c^\alpha, a^\alpha b^\alpha d^\alpha\} = g^{-1}S = \{g^{-1}, g^{-1}bc, g^{-1}abd\}.$$

It follows that $1 \in g^{-1}S$ and $g = 1$, bc or abd.

Let $g = 1$. Then $\{1, bc, abd\} = \{1, b^\alpha c^\alpha, a^\alpha b^\alpha d^\alpha\}$. If $b^\alpha c^\alpha = bc$, then $b^\alpha = b$, $c^\alpha = c$, and $a^\alpha b^\alpha d^\alpha = abd$. It follows that $a^\alpha = a$ and $d^\alpha = d$. This implies that $\alpha = 1$. Note that $g = 1$. Thus, $\sigma_{\alpha,g} = 1$. If $b^\alpha c^\alpha = abd$, then $b^\alpha = ab$ and $c^\alpha = d$. This forces that $a^\alpha b^\alpha d^\alpha = bc$. An easy calculation implies that $a^\alpha = a^{-1}$ and $d^\alpha = c$. Clearly, $o(\alpha) = 2$ and so $o(\sigma_{\alpha,g}) = 2$.

Let $g = bc$. Then $\{1, b^{-1}c, a^r cd\} = \{1, (bc)^\alpha, (ab)^\alpha d^\alpha\}$. It is easy to see that the orders of the elements in the left set are 1, 6 and $2p$, and the orders of the elements in the right set are 1, 6 and 6. Clearly, this is impossible.

Let $g = abd$. Then $\{1, a^{-1}b^{-1}d, a^{-r}cd\} = \{1, (bc)^\alpha, (ab)^\alpha d^\alpha\}$. A similar argument as the above paragraph we can also deduce a contradiction.

Thus, $F = \langle \sigma_{\alpha,1} \rangle \cong \mathbb{Z}_2$, where $\alpha \in \mathrm{Aut}(H) : a^\alpha = a^{-1}$, $b^\alpha = ab$, $c^\alpha = c$ and $d^\alpha = d$. By Proposition 9, $N_{\mathrm{Aut}(X)}(\mathscr{R}(H))$ cannot act arc-transitively on X, a contradiction.

Case 3. Let $S = \{1, bc, ab^{-1}d\}$.

With the similar argument as above we have

$$S^\alpha = \{1, b^\alpha c^\alpha, a^\alpha b^{-\alpha} d^\alpha\} = g^{-1}S = \{g^{-1}, g^{-1}bc, g^{-1}ab^{-1}d\}.$$

It follows that $1 \in g^{-1}S$ and $g = 1$, bc or $ab^{-1}d$.

Let $g = 1$. Then $\{1, bc, ab^{-1}d\} = \{1, b^\alpha c^\alpha, a^\alpha b^{-\alpha} d^\alpha\}$. If $b^\alpha c^\alpha = bc$, then $a^\alpha b^{-\alpha} d^\alpha = ab^{-1}d$. It forces that $\alpha = 1$ and hence $\sigma_{\alpha,g} = 1$. If $b^\alpha c^\alpha = ab^{-1}d$, then $b^\alpha = ab^{-1}$. Since $\alpha \in \mathrm{Aut}(H)$, this is impossible.

Let $g = bc$. Then $\{1, b^{-1}c, a^r bcd\} = \{1, b^\alpha c^\alpha, a^\alpha b^{-\alpha} d^\alpha\}$. Note that $\alpha \in \mathrm{Aut}(H)$ cannot map b to $a^i b^{-1}$. Thus, $b^\alpha c^\alpha = a^r bcd$ and $(ab^{-1})^\alpha d^\alpha = b^{-1}c$. It follows that $a^\alpha = a^{r^2}$, $b^\alpha = a^r b$, $c^\alpha = cd$, $d^\alpha = c$ and the order $o(\alpha) = 3$.

$$\sigma_{\alpha,bc} : h_0 \mapsto (h^\alpha)_0, \ h_1 \mapsto (bch^\alpha)_1,$$
$$\sigma^2_{\alpha,bc} : h_0 \mapsto (h^{\alpha^2})_0, \ h_1 \mapsto (bc(bch^\alpha)^\alpha)_1 = (ab^{-1}dh^{\alpha^2})_1,$$
$$\sigma^3_{\alpha,bc} : h_0 \mapsto (h^{\alpha^3})_0 = h_0, \ h_1 \mapsto (ab^{-1}dh^{\alpha^2})^\alpha)_1 = h_1.$$

This calculation implies that the order $o(\sigma_{\alpha,bc}) = 3$.

Let $g = ab^{-1}d$. Then $\{1, a^{-r^2}bd, a^{-r^2}b^{-1}cd\} = \{1, b^\alpha c^\alpha, a^\alpha b^{-\alpha} d^\alpha\}$. Then $b^\alpha c^\alpha = a^{-r^2}bd$ and $a^\alpha b^{-\alpha} d^\alpha = a^{-r^2}b^{-1}cd$. It follows that $b^\alpha = a^{-r^2}b$, $c^\alpha = d$, $d^\alpha = cd$, $a^\alpha = a^{-r^2-1}$, and the order $o(\alpha) = 3$. For convenience, we denote this automorphism of H by α'. An easy calculation implies that $\alpha'^2 = \alpha$ and

$$\sigma_{\alpha',ab^{-1}d}: \ h_0 \mapsto (h^{\alpha'})_0, \ h_1 \mapsto (ab^{-1}dh^{\alpha'})_1,$$
$$\sigma^2_{\alpha',ab^{-1}d}: \ h_0 \mapsto (h^{\alpha'^2})_0 = (h^\alpha)_0,$$
$$h_1 \mapsto (ab^{-1}d(ab^{-1}dh^{\alpha'})^{\alpha'})_1 = (bch^{\alpha'^2})_1 = (bch^\alpha)_1,$$
$$\sigma^3_{\alpha',ab^{-1}d}: \ h_0 \mapsto (h^{\alpha'^3})_0 = h_0, \ h_1 \mapsto (ab^{-1}dh^{\alpha'^2})^{\alpha'})_1 = h_1.$$

This forces that $o(\sigma_{\alpha',ab^{-1}d}) = 3$ and $\sigma^2_{\alpha',ab^{-1}d} = \sigma_{\alpha,bc}$. Thus, we have that $F = \langle \sigma_{\alpha,bc} \rangle \cong \mathbb{Z}_3$. Clearly, $\mathscr{R}(H) \rtimes F$ acts transitively on $E(X)$.

Take $\beta \in \mathrm{Aut}(H)$:

$$\beta : \begin{cases} a \mapsto a^{-1}, \\ b \mapsto a^{-r^2}b, \\ c \mapsto d, \\ d \mapsto c. \end{cases}$$

Then the order $o(\beta) = 2$. On the other hand,

$$S^\beta = \{1, bc, ab^{-1}d\}^\beta = \{1, a^{-r^2}bd, a^{-1}b^{-1}a^{r^2}c\}$$
$$= \{1, a^{-r^2}bd, b^{-1}c\} = S^{-1}.$$

By the definition of I, we have that $\delta_{\beta,1,1} \in I$ and

$$\delta_{\beta,1,1} : h_0 \mapsto (h^\beta)_1, h_1 \mapsto (h^\beta)_0, \forall h \in H.$$

Since $o(\beta) = 2$, we have that $\delta_{\beta,1,1}$ has order 2 and interchanges the two biparts of X. By Proposition 9, $N_{\mathrm{Aut}(X)}(\mathscr{R}(H)) = \mathscr{R}(H)\langle F, \delta_{\beta,1,1} \rangle$ is arc-transitive on X and hence $X \cong \mathscr{B}\mathscr{C}_{24p}(\mathbb{Z}_2^2 \times F_{3p})$. On the other hand, $\mathscr{R}(H) \cong \mathbb{Z}_2^2 \times F_{3p}$ has a normal Sylow 2-subgroup

$K \cong \mathbb{Z}_2^2$ and hence K is characteristic in $\mathscr{R}(H)$. The normality of $\mathscr{R}(H)$ in $N_{\mathrm{Aut}(X)}(\mathscr{R}(H))$ forces that K is also normal in $N_{\mathrm{Aut}(X)}(\mathscr{R}(H))$. Then the quotient graph X_K has order $6p$ and valence 3 with $3 \,\Big|\, (p-1)$. By Proposition 5, $X_K \cong CB_p$ and $N_{\mathrm{Aut}(X)}(\mathscr{R}(H))/K \leqslant \mathrm{Aut}(X_K) \cong D_{6p} \rtimes \mathbb{Z}_3$. However, $N_{\mathrm{Aut}(X)}(\mathscr{R}(H))/K \cong F_{3p}\langle F, \delta_{\beta,1,1}\rangle$ with $F \cong \mathbb{Z}_3$ and $o(\delta_{\beta,1,1}) = 2$. This forces that $N_{\mathrm{Aut}(X)}(\mathscr{R}(H))/K = \mathrm{Aut}(X_K) \cong D_{6p} \rtimes \mathbb{Z}_3$ and $N_{\mathrm{Aut}(X)}(\mathscr{R}(H)) \cong ((\mathbb{Z}_2^2 \times F_{3p}).\mathbb{Z}_2) \rtimes \mathbb{Z}_3$ is 1-regular on X. Furthermore, an easy calculation follows that the girth of $\mathscr{BC}_{24p}(\mathbb{Z}_2^2 \times F_{3p})$ is 12. □

For a finite group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G,S)$ on G with respect to S is defined to have vertex set $V(\mathrm{Cay}(G,S)) = G$ and edge set $E(\mathrm{Cay}(G,S)) = \{\{g, sg\} \mid g \in G, s \in S\}$. Clearly, a Cayley graph $\mathrm{Cay}(G,S)$ is connected if and only if S generates G. Furthermore, $\mathrm{Aut}(G,S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of the automorphism group $\mathrm{Aut}(\mathrm{Cay}(G,S))$. Given a $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg$, $x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$, called the *right regular representation* of G, is a permutation group isomorphic to G. The Cayley graph is vertex-transitive because it admits the right regular representation $R(G)$ of G as a regular group of automorphisms of $\mathrm{Cay}(G,S)$. A Cayley graph $\mathrm{Cay}(G,S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G,S))$. A graph X is isomorphic to a Cayley graph on G if and only if $\mathrm{Aut}(X)$ has a subgroup isomorphic to G, acting regularly on vertices (see [29]). For two subsets S and T of G not containing the identity 1, if there is an $\alpha \in \mathrm{Aut}(G)$ such that $S^\alpha = T$ then S and T are said to be *equivalent*, denoted by $S \equiv T$. We may easily show that if $S \equiv T$ then $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$ and $\mathrm{Cay}(G,S)$ is normal if and only if $\mathrm{Cay}(G,T)$ is normal.

To state the cyclic covers of the three dimensional hypercube Q_3, the so-called $CQ(k,r)$ graphs introduced in [22]. Let $V(Q_3) = u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8$. For any two non-negative integer k and r with $1 \leqslant k \leqslant r-1$ and $(k,r) = 1$, the graph $CQ(k,r)$ is defined to have vertex set $V(CQ(k,r)) = V(Q_3) \times \mathbb{Z}_r$ and edge set

$$\begin{aligned}E(CQ(k,r)) = \Big\{&\{(u_1,i),(u_6,i)\}, \{(u_1,i),(u_7,i)\},\\ &\{(u_1,i),(u_8,i)\}, \{(u_2,i),(u_5,i)\}, \{(u_2,i),(u_8,i)\},\\ &\{(u_3,i),(u_8,i)\}, \{(u_4,i),(u_7,i)\}, \{(u_2,i),(u_7,i+1)\},\\ &\{(u_3,i),(u_5,i+k)\}, \{(u_3,i),(u_6,i-k^{-1})\},\\ &\{(u_4,i),(u_5,i-k^{-1}-1)\}, \{(u_4,i),(u_6,i+k)\} \mid i \in \mathbb{Z}_r\Big\}.\end{aligned}$$

Let X be a connected $\mathbb{Z}_r$-cover of Q_3. Then by [22, Theorem 1.1] and [18, Proposition 5.6], X is 1-regular if and only if $X \cong CQ(k,r)$ where $r \geqslant 7$ divides k^2+k+1. Let p be a prime and $r = 3p$. Then $3p \,\Big|\, (k^2+k+1)$ forces that $3 \,\Big|\, (p-1)$ and k is an element in $\mathbb{Z}_{3p}^*$ of order 3. Clearly, $\mathbb{Z}_{3p}^* \cong \mathbb{Z}_2 \times \mathbb{Z}_{p-1}$ has two elements of order 3, say k and k^2. By [18, Theorem 1.2], $CQ(k,3p)$ is a normal Cayley graph on generalized dihedral group $G = (\mathbb{Z}_{6p} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ and $\mathrm{Aut}(CQ(k,3p)) \cong G \rtimes \mathbb{Z}_3 \cong (\mathbb{Z}_{6p} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_6$. Take

$$\begin{aligned}G &= \langle a,b,c \mid a^{6p} = b^2 = c^2 = [a,b] = [b,c] = 1,\ a^c = a^{-1}\rangle\\ &\cong (\mathbb{Z}_{6p} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_6.\end{aligned}$$

By [18, Proposition 5.3] or [30, Proposition 2.2], $CQ(k,3p) = \mathrm{Cay}(G,S)$ and $CQ(k^2,3p) = \mathrm{Cay}(G,T)$ with $S = \{c, ac, a^{-k}bc\}$ and $T = \{c, ac, a^{-k^2}bc\}$. Set

$$\alpha : \begin{cases} a \mapsto a^{-1},\\ b \mapsto b,\\ c \mapsto ac.\end{cases}$$

Then $\alpha \in \mathrm{Aut}(G)$ and $S^\alpha = T$. This implies that $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$. Thus, the graphs $CQ(k,3p)$ is independent of the choice k. We denote $CQ(k,3p)$ by CQ_{3p} and have the following lemma.

Lemma 2 *Let p be a prime and X a connected $\mathbb{Z}_{3p}$-cover of Q_3. Then $3 \,\Big|\, (p-1)$, $X \cong CQ_{3p}$ has girth 6 and* $\mathrm{Aut}(X) \cong (\mathbb{Z}_{6p} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_6$.

CLASSIFICATION

This section is devoted to classifying cubic symmetric graphs of order $24p$ for each prime p.

Theorem 2 *Let X be a connected cubic symmetric graph of order $24p$ with p a prime. Then one of the following holds:*

(1) *X is 1-regular and isomorphic to CQ_{3p} or $\mathscr{BC}_{24p}(\mathbb{Z}_2^2 \times F_{3p})$;*
(2) *X is 2-regular and isomorphic to F_{48}, F_{72}, F_{120A}, F_{120B}, F_{168B}, F_{168C}, F_{168D}, F_{168F} or F_{408A};*
(3) *X is 3-regular and isomorphic to F_{408B}.*

Remark 1 By Magma [17] and [15, 16], $CQ_{21} \cong F_{168A}$ and $\mathscr{BC}_{24\cdot 7}(\mathbb{Z}_2^2 \times F_{21})) \cong F_{168E}$.

Proof: By Lemmas 1 and 2, CQ_{3p} and $\mathscr{BC}_{24p}(\mathbb{Z}_2^2 \times F_{3p})$ have different girths and hence are non-isomorphic. Let $A = \mathrm{Aut}(X)$ and $v \in V(X)$. Then by Proposition 2, $|A_v| \,\Big|\, 48$ and hence $|A| \,\Big|\, 2^7 \cdot 3^2 \cdot p$. By [15, 16], for $p \leqslant 17$, the statements in Theorem 2 hold. In particular, with the calculation of Magma [17], for $p = 7$, we have that $CQ_{21} \cong F_{168A}$, and $\mathscr{BC}_{168}(\mathbb{Z}_2^2 \times F_{21})) \cong F_{168E}$. For $p = 13$, we have $CQ_{39} \cong F_{312A}$, and $\mathscr{BC}_{312}(\mathbb{Z}_2^2 \times F_{39})) \cong F_{312B}$. In what follows, we assume that $p \geqslant 19$.

Suppose that A is non-solvable. Then A has a composition factor isomorphic to a non-abelian simple group. Note that $|A| \,\Big|\, 2^7 \cdot 3^2 \cdot p$. Thus, by Proposition 3, this composition factor is isomorphic to a K_3-simple group with $p = 5$, 7, 13 or 17, contrary to our assumption. This implies that A is solvable. By

[31, Corollary 1.2], X is s-transitive with $s \leqslant 3$ and $|A_v| \,\Big|\, 12$. Thus, $|A| \,\Big|\, 2^5 \cdot 3^2 \cdot p$. Let N be a minimal normal subgroup of A. Then $|N| \,\Big|\, |A| \,\Big|\, 2^5 \cdot 3^2 \cdot p$ and hence $N \cong \mathbb{Z}_2^k (1 \leqslant k \leqslant 5)$, $\mathbb{Z}_3$, $\mathbb{Z}_3^2$ or $\mathbb{Z}_p$. If $N \cong \mathbb{Z}_2^k$ with $3 \leqslant k \leqslant 5$ or $\mathbb{Z}_3^2$, then by Proposition 1, N acting on $V(X)$ has more than $p \geqslant 19$ orbits and hence N is semiregular. This forces that $N \cong \mathbb{Z}_2^3$. However, X_N has order $3p$ and valency 3, this is impossible because there is no regular graph of odd order and odd valency. If $N \cong \mathbb{Z}_2$, then X_N is a cubic symmetric graph of order $12p$. By Proposition 7, $X_N \cong F_{24}$, F_{60}, F_{84} or F_{204} with $p = 2$, 5, 7 or 17. This is contrary to our assumption that $p \geqslant 19$. Thus, $N \cong \mathbb{Z}_2^2$, $\mathbb{Z}_3$ or $\mathbb{Z}_p$. In the following, we will prove that for each case, A has a normal subgroup isomorphic to $\mathbb{Z}_p$, so we only deal with the case $N \cong \mathbb{Z}_p$.

Let $N \cong \mathbb{Z}_2^2$. Then X_N is a cubic symmetric graph of order $6p$. By our assumption $p \geqslant 19$ and Proposition 5, $X_N \cong CB_p$ and $A/N \leqslant \mathrm{Aut}(X_N) \cong D_{6p} \rtimes \mathbb{Z}_3$. Since CB_p is 1-regular, we have that $A/N \cong D_{6p} \rtimes \mathbb{Z}_3$. Let P be a Sylow p-subgroup of A. Then PN/N is normal in A/N and so PN/N is the unique Sylow p-subgroup of A/N. On the other hand, by Proposition 4, $A/C_A(N) \lesssim \mathrm{Aut}(N) \cong \mathrm{S}_3$ and hence $p \,\Big|\, |C_A(N)|$. It follows that $P \leqslant C_A(N)$ and P is also the unique Sylow p-subgroup of A. Thus, A has a normal subgroup isomorphic to $\mathbb{Z}_p$.

Let $N \cong \mathbb{Z}_3$. Then X_N is a cubic symmetric graph of order $8p$. Note that $p \geqslant 19$. By Proposition 6, $X_N \cong CQ_p$ with $3 \,\Big|\, (p-1)$. Since CQ_p is 1-regular and a $\mathbb{Z}_p$-cover of three dimensional hypercube Q_3, we have A/N has a normal subgroup $M/N \cong \mathbb{Z}_p$. By Proposition 4, $M/C_M(N) \lesssim \mathrm{Aut}(N) \cong \mathbb{Z}_2$. It forces that $C_M(N) = M$ and $M = N \times P$ with $P \cong \mathbb{Z}_p$. Since $p \geqslant 19$, we have that P is characteristic in M. The normality of M in A implies that P is also normal in A, that is, A has a normal subgroup isomorphic to $\mathbb{Z}_p$.

Let $N \cong \mathbb{Z}_p$. Then X_N is a cubic symmetric graph of order 24 and by [16], X_N is isomorphic to Nauru graph, which is 2-regular. By Magma [17], $A/N \leqslant \mathrm{Aut}(X_N) \cong \mathrm{S}_4 \times \mathrm{S}_3$ and $\mathrm{Aut}(X_N)$ has two minimal arc-transitive subgroups, that is, $\mathrm{S}_4 \times \mathbb{Z}_3$ and $\mathrm{A}_4 \times \mathrm{S}_3$, which have index 2 in $\mathrm{S}_4 \times \mathrm{S}_3$. It follows that $A/N \cong \mathrm{S}_4 \times \mathbb{Z}_3$, $\mathrm{A}_4 \times \mathrm{S}_3$ or $\mathrm{S}_4 \times \mathrm{S}_3$. It is easy to see that these three groups both have a normal subgroup isomorphic to $\mathbb{Z}_2^2 \times \mathbb{Z}_3$. This forces that A/N has a normal subgroup $M/N \cong \mathbb{Z}_2^2 \times \mathbb{Z}_3$ and hence A has a normal subgroup K of order $3p$. Note that $|V(X)| = 24p$. Thus, K acting on $V(X)$ has at least 8 orbits and by Proposition 1, K is semiregular. Since $p \geqslant 19$, we have that $K \cong \mathbb{Z}_{3p}$ or F_{3p}.

Suppose that $K \cong \mathbb{Z}_{3p}$. Then X_K has order 8 and is isomorphic to the three dimensional hypercube Q_3. This implies that X is a $\mathbb{Z}_{3p}$-cover of Q_3. By Lemma 2, $X \cong CQ_{3p}$.

Suppose that $K \cong F_{3p}$. Similarly, $X_K \cong Q_3$ and $A/K \leqslant \mathrm{Aut}(Q_3) \cong \mathrm{S}_4 \times \mathbb{Z}_2$. Since K is semiregular, $M \cong F_{3p} \rtimes \mathbb{Z}_2^2$ acting on $V(X)$ has at least 6 orbits. By Proposition 1, M is also semiregular and hence X is a normal bi-Cayley graph over the group M. Since $N \cong \mathbb{Z}_p$ is normal in M, we have that $M/C_M(N) \lesssim \mathrm{Aut}(N) \cong \mathbb{Z}_{p-1}$ by Proposition 4. It follows that $M \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_6) \times \mathbb{Z}_2$ or $F_{3p} \times \mathbb{Z}_2^2$. If $M \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_6) \times \mathbb{Z}_2$, then M has a characteristic subgroup $L \cong \mathbb{Z}_p \times \mathbb{Z}_2$. The normality of M in A forces that L is also normal in A. Thus, X_L is a cubic symmetric graph of order 12. However, by [15], there is no cubic symmetric graph of order 12, a contradiction. If $M \cong F_{3p} \times \mathbb{Z}_2^2$, then by Construction 1 and Lemma 1, $X \cong \mathcal{BC}_{24p}(\mathbb{Z}_2^2 \times F_{3p})$, as required. □

CONCLUSION

As far as we know, the covering technique is very useful in the classification of cubic symmetric graphs whose quotient graphs have small orders. The main aim of this paper is to determine the classification of cubic symmetric graphs of order $24p$ for each prime p. However, one case of these graphs has a quotient graph isomorphic to Nauru graph of order 24, it is more complicated to determine these graph by using covering graphs. We use the structure of full automorphism groups and find a semiregular subgroup $\mathbb{Z}_2^2 \times F_{3p}$. Thus, by using the bi-Cayley graph and its automorphisms, we give the construction of the one-regular normal bi-Cayley graphs on $\mathbb{Z}_2^2 \times F_{3p}$, and then obtain the classification of cubic symmetric graphs of order $24p$. As a natural continuation, could we construct some one-regular normal bi-Cayley graphs of more general prime valencies on the group $\mathbb{Z}_2^2 \times F_{3p}$, and then classify symmetric graphs of order $24p$ and any prime valency?

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