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Green's relations and natural partial order on Baer-Levi semigroups of partial transformations with restricted range

Boorapa Singha

Department of Mathematics and Statistics, Faculty of Science and Technology, Chiang Mai Rajabhat University, Chiang Mai 50180 Thailand

e-mail: boorapa_sin@cmru.ac.th

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ABSTRACT: Let X be an infinite set and $I(X)$ the symmetric inverse semigroup on X. For a nonempty subset Y of X and an infinite cardinal q such that $|X| \geqslant q$, let $PS(X,Y,q) = \{\alpha \in I(X) : |X \setminus X\alpha| = q \text{ and } X\alpha \subseteq Y\}$. Then $PS(X,Y,q)$ is a generalization of the partial Baer-Levi semigroup $PS(X,q) = \{\alpha \in I(X) : |X \setminus X\alpha| = q\}$ which has been studying since 1975. In this paper, we describe the Green's relations and characterize the natural partial order on $PS(X,Y,q)$. With respect to this partial order, we determine when two elements are related, find all the maximum, minimum, maximal, minimal, lower cover and upper cover elements. Also, we describe elements which are compatible and we investigate the greatest lower bound and the least upper bound of two elements in $PS(X,Y,q)$.

KEYWORDS: Baer-Levi semigroup, natural partial order, Green's relations, transformation semigroup

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INTRODUCTION

Let X be a nonempty set and let $P(X)$ denote the set of all partial transformations of X, i.e., all transformations α whose domain, dom α, and range, $X\alpha$ are subsets of X. Let $T(X)$ denote the subsemigroup of $P(X)$ consisting of all $\alpha \in P(X)$ with dom $\alpha = X$, which is called the full transformation semigroup. Also, let $I(X)$ denote the symmetric inverse semigroup on X: that is, the set of all injective mappings in $P(X)$. When X is an infinite set and q is a fixed cardinal such that $|X| \geqslant q \geqslant \aleph_0$, we write

$$BL(X,q) = \{\alpha \in T(X) \cap I(X) : d(\alpha) = q\},$$

where $d(\alpha) = |X \setminus X\alpha|$ is called the *defect* of α. Then $BL(X,q)$ is called the Baer-Levi semigroup of type $(|X|, q)$. It is known that $BL(X,q)$ is a right cancellative, right simple semigroup without idempotents. Moreover, for any semigroup S satisfying these three properties, S can be embedded in a Baer-Levi semigroup of type (p,p), where $p = |S|$ (see [1, Section 8.1]).

In 1975, Sullivan [2] introduced and studied a semigroup containing $BL(X,q)$, namely

$$PS(X,q) = \{\alpha \in I(X) : d(\alpha) = q\},$$

and call this the partial Baer-Levi semigroup on X. He showed that, when $p = q$, every automorphism of $PS(X,q)$ is inner and the set of all automorphisms of $PS(X,q)$ is isomorphic to $G(X)$, the permutation group on X. Later, in 2004, Pinto and Sullivan [3] showed that this is also true when $p > q$. Also, a characterization of the Green's relations, regular elements and ideals of $PS(X,q)$ have been provided in this paper. In contrast with $BL(X,q)$, the semigroup $PS(X,q)$ is neither right simple nor right cancellative (see [3, Example 1]). Moreover, this semigroup always cantains idempotents (see [3, p 89]).

In this paper, we introduce a family of subsets of $PS(X,q)$ defined by

$$PS(X,Y,q) = \{\alpha \in I(X) : d(\alpha) = q \text{ and } X\alpha \subseteq Y\},$$

where Y is a fixed nonempty subset of X. Since $PS(X,q)$ is closed under composition of functions and if $X\alpha \subseteq Y, X\beta \subseteq Y$, then $X\alpha\beta \subseteq X\beta \subseteq Y$, thus $PS(X,Y,q)$ is a subsemigroup of $PS(X,q)$. We also observe that $|X \setminus Y| \leqslant |X \setminus X\alpha| = q$ for any $\alpha \in PS(X,Y,q)$, therefore $PS(X,Y,q) \neq \varnothing$ only when $|X \setminus Y| \leqslant q$. Moreover, when $X = Y$, we obtain that $PS(X,Y,q) = PS(X,q)$. Thus, we may regard $PS(X,Y,q)$ as a generalization of $PS(X,q)$.

The natural partial order on regular semigroups was first defined in 1980 independently by Hartwig [4] and Nambooripad [5]. The most recognized and widely used definition is the following: $a \leqslant b$ if and only if $a = eb = bf$ for some idempotents $e, f \in S$. Later, in 1986, Mitsch [6] generalized the definition of the above partial order on regular semigroups to arbitrary semigroup S by: $a \leqslant b$ if and only if $a = xb = by$ and $a = ay$ for some $x, y \in S^1$, where the notation S^1 means S itself if S contains the identity element, otherwise S^1 denotes the semigroup obtained from S by adjoining an extra identity element 1. However, when S is regular the Mitsch's order coincides with the Hartwig-Nambooripad's order. A significant amount of research has been done studying the natural partial order on various transformation semigroups on the nonempty set X. In [7], Kowol and Mitsch characterized the natural partial order on $T(X)$ in terms of images and kernels. In 2003, Marques-Smith and Sullivan [8] studied and compared various properties of the natural partial order $\leqslant$ and the another partial order $\subseteq$ on $P(X)$, namely the containment order defined by : $\alpha \subseteq \beta$ if and only ifdom $\alpha \subseteq \operatorname{dom}\beta$ and $x\alpha = x\beta$ for all $x \in \operatorname{dom}\alpha$. Later, Singha, Sanwong and Sullivan [9, 10] investigated various properties of $\leqslant$ and $\subseteq$ on $I(X), PS(X,q)$ and its largest regular subsemigroup. The natural partial order has also been studied in many other recent papers on several transformation semigroups, see for example [11–14]. For the description for the natural partial order on $BL(X,q)$, as far as we know, it were not characterized before. But we observe that, if $\alpha \leqslant \beta$ in $BL(X,q)$, then by the definition of $\leqslant$, we have $\alpha\mu = \beta\mu$ for some $\mu \in BL(X,q)^1$, so $\alpha = \beta$ since $BL(X,q)$ is right cancellative. Therefore, the natural partial order on $BL(X,q)$ is just the identity relation on $BL(X,q)$. Although $PS(X,Y,q)$ is a generalization of $PS(X,q)$, in general, when $X \neq Y$ the natural partial order on $PS(X,Y,q)$ is not the restriction of the natural partial order on $PS(X,q)$ to $PS(X,Y,q)$. In other words, for $\alpha,\beta \in PS(X,Y,q)$ such that $\alpha \leqslant \beta$ in $PS(X,q)$, it does not necessarily follow that $\alpha \leqslant \beta$ in $PS(X,Y,q)$. For example, let $X = \mathbb{N}$ be the set of all positive integers, let Y be the set of all positive even integers and let $q = \aleph_0$. Let α,β,λ,μ be defined as follows:

$$\alpha = \begin{pmatrix} 4 & 5 \\ 2 & 4 \end{pmatrix}, \quad \beta = \begin{pmatrix} 4 & 5 & 6 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix},$$
$$\lambda = \begin{pmatrix} 4 & 5 \\ 4 & 5 \end{pmatrix} \text{ and } \mu = \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}.$$

Then α, β, $\mu \in PS(X,Y,q)$ and $\lambda \in PS(X,q)\setminus PS(X,Y,q)$. We see that $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$, so $\alpha \leqslant \beta$ in $PS(X,q)$. But there is no $\lambda' \in PS(X,Y,q)$ such that $\alpha = \lambda'\beta$, so $\alpha \nleqslant \beta$ in $PS(X,Y,q)$. It is therefore of interest to characterize the natural partial order on $PS(X,Y,q)$.

The main objective of this paper is to study the semigroup $PS(X,Y,q)$. To achieve this aim, we first investigate some elementary results of $PS(X,Y,q)$. In the following section, we give descriptions of the Green's relations and describe the natural partial order on this semigroup. The results for $PS(X,Y,q)$ obtained in this paper extend and generalize the corresponding results for $PS(X,q)$ obtained in [3, 9, 10].

PRELIMINARY NOTATION AND RESULTS

Throughout this paper, unless otherwise specified, we suppose that X is an infinite set with $|X| = p$, q is an infinite cardinal such that $q \leqslant p$ and Y is a nonempty subset of X such that $|X\setminus Y| \leqslant q$. For each mapping $\alpha \in PS(X,Y,q)$, we write

$$\alpha = \begin{pmatrix} a_i \\ y_i \end{pmatrix},$$

where the subscript i belongs to some unmentioned index set I, the abbreviation $\{y_i\}$ denotes $\{y_i : i \in I\}$, $X\alpha = \{y_i\} \subseteq Y$, $\operatorname{dom}\alpha = \{a_i\}$ and $a_i\alpha = y_i$. We also write $g(\alpha) = |X\setminus \operatorname{dom}\alpha|$ and $r(\alpha) = |X\alpha|$, and refer to these cardinals as the *gap* and the *rank* of α, respectively. For a subset A of X, we denote by $\alpha|_A$ the restriction of α to A. Also, denote by id_A the identity function on A and we write $A = B \,\dot{\cup}\, C$ to denote A is a disjoint union of B and C. As usual, $\varnothing$ denotes the emptyset, but in some contexts, $\varnothing$ is used to refer to the empty (one-to-one) transformation which is the zero element in $P(X)$.

We begin with some basic results on $PS(X,Y,q)$ which analogous to those obtained for $PS(X,q)$ in [3].

Proposition 1 *The semigroup $PS(X,Y,q)$ contains zero element precisely when $|X| = q$. Moreover, $PS(X,Y,q)$ has no identity element.*

Proof: Sine every mapping in $PS(X,Y,q)$ has defect q and $d(\varnothing) = p$, so $\varnothing \in PS(X,Y,q)$ precisely when $p = q$.Next, to show that $PS(X,Y,q)$ has no identity element, we first observe that if γ is the identity element in $PS(X,Y,q)$, then for all $\alpha \in PS(X,Y,q)$, $\alpha = \gamma\alpha$. So $\operatorname{dom}\alpha \subseteq \operatorname{dom}\gamma$ and $\gamma|_{\operatorname{dom}\alpha} = \mathrm{id}_{\operatorname{dom}\alpha}$. If $|Y| = p$, then we can write $Y = A \,\dot{\cup}\, B \,\dot{\cup}\, C$, where $|A| = p$ and $|B| = |C| = q$. As $|X\setminus Y| \leqslant q$, we have $|A\cup B\cup(X\setminus Y)| = |A\cup C\cup(X\setminus Y)| = p$. Thus, there exist $\theta : A\cup B\cup(X\setminus Y) \to A$ and $\varepsilon : A\cup C\cup(X\setminus Y) \to A$, where θ and ε are bijections. We have $X\theta = X\varepsilon = A \subseteq Y$ and $d(\theta) = d(\varepsilon) = |B| + |C| + |X\setminus Y| = q$, whence $\theta,\varepsilon \in PS(X,Y,q)$. As γ is the identity, we have $\gamma|_{\operatorname{dom}\theta} = \mathrm{id}_{A\cup B\cup(X\setminus Y)}$ and $\gamma|_{\operatorname{dom}\varepsilon} = \mathrm{id}_{A\cup C\cup(X\setminus Y)}$, that is $\gamma = \mathrm{id}_X$, contradicting the fact that $|X\setminus X\gamma| = q$. On the other hand, if $|Y| < p$, then $p = |X\setminus Y| \leqslant q$ and so $p = q$. In this case, all mappings whose domain is a singleton and range is a subset of Y belong to $PS(X,Y,q)$. Fix $y \in Y$ and for any $x \in X$, we let $\alpha_x = \begin{pmatrix} x \\ y \end{pmatrix} \in PS(X,Y,q)$. Again, as γ is the identity, we have $\alpha_x = \gamma\alpha_x$ and so $x\gamma = x$ for all $x \in X$. Then we obtain that $\gamma = \mathrm{id}_X$ and this leads to a contradiction again. Hence $PS(X,Y,q)$ has no identity element. □

Proposition 2 *The semigroup $PS(X,Y,q)$ is neither right cancellative nor right simple. Furthermore, $PS(X,Y,q)$ always contains idempotents and*

$$E(PS(X,Y,q)) = \{\mathrm{id}_A : A \subseteq Y \text{ and } |X\setminus A| = q\}$$

is the set of all idempotents in $PS(X,Y,q)$.

Proof: If $|X| = q$, then we let $y \in Y$ and $t,u,v \in X\setminus\{y\}$, where t,u and v are all distinct. We define

$$\alpha = \begin{pmatrix} t \\ y \end{pmatrix}, \quad \beta = \begin{pmatrix} u \\ y \end{pmatrix}, \quad \gamma = \begin{pmatrix} v \\ y \end{pmatrix}.$$

As $|X\setminus\{y\}| = q$, we have that $\alpha,\beta,\gamma \in PS(X,Y,q)$ and $\alpha\gamma = \varnothing = \beta\gamma$ but $\alpha \neq \beta$. Therefore $PS(X,Y,q)$ is not a right cancellative semigroup. Moreover, for any $\lambda \in PS(X,Y,q)$, we see that $t \notin \operatorname{dom} \beta\lambda$. So $\alpha \neq \beta\lambda$, that is $PS(X,Y,q)$ is not right simple. On the other hand, suppose that $|X| = p > q$. Since $|X\backslash Y| \leqslant q$, we have $|Y| = p$. We write $Y = A \dot\cup B$, where $A = \{a_i\}$, $|A| = p$ and $|B| = q$. Choose $b, c \in B$ with $b \neq c$ and define

$$\alpha = \begin{pmatrix} a_i & b \\ a_i & b \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} a_i & c \\ a_i & b \end{pmatrix},$$

it is easy to see that $\alpha, \beta, \mathrm{id}_A \in PS(X,Y,q)$ and $\alpha \cdot \mathrm{id}_A = \mathrm{id}_A = \beta \cdot \mathrm{id}_A$ but $\alpha \neq \beta$. Thus, $PS(X,Y,q)$ is not a right cancellative semigroup. Furthermore, for any $\lambda \in PS(X,Y,q)$, we see that $b \notin \operatorname{dom} \beta\lambda$. Then $\alpha \neq \beta\lambda$, this again implies $PS(X,Y,q)$ is not a right simple semigroup.

Next, we characterize all idempotents in $PS(X,Y,q)$. It is clear that for any $A \subseteq Y$ such that $|X\backslash A| = q$, we have $\mathrm{id}_A \in PS(X,Y,q)$ and $\mathrm{id}_A \cdot \mathrm{id}_A = \mathrm{id}_A$, whence id_A is an idempotent. Conversely, if α is an idempotent in $PS(X,Y,q)$, then $\alpha^2 = \alpha$. So $(x\alpha)\alpha = x\alpha$ for all $x \in \operatorname{dom} \alpha$. Since α is injective, we have $x\alpha = x$ and thus $\alpha = \mathrm{id}_A$, where $A = \operatorname{dom} \alpha = X\alpha \subseteq Y$. Hence, $|X\backslash A| = |X\backslash X\alpha| = q$ as required. □

Proposition 3 *The semigroup $PS(X,Y,q)$ is not a regular semigroup.*

Proof: If $Y = X$, then $PS(X,Y,q) = PS(X,q)$ which was shown in [3, Theorem 4], that it is not a regular semigroup. Otherwise, if $X\backslash Y \neq \varnothing$, then we let $\alpha \in PS(X,Y,q)$ be such that $\operatorname{dom} \alpha \cap (X\backslash Y) \neq \varnothing$. Let $x \in \operatorname{dom} \alpha \cap (X\backslash Y)$ and suppose that $x\alpha = y$. If α is regular, then there exists $\beta \in PS(X,Y,q)$ such that $\alpha = \alpha\beta\alpha$, so $y\beta = x \notin Y$, this contradicts to that $X\beta \subseteq Y$. Hence, α is not a regular element in $PS(X,Y,q)$. □

GREEN'S RELATIONS

In this section, we characterize the Green's relations on $PS(X,Y,q)$ by using some ideas of the proof for $PS(X,q)$ in [3] with the idea of restricted range concerned. For the definition of Green's relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$, and $\mathscr{J}$ on a semigroup, see [15, Chapter 2]. We also recall from Proposition 1 that $PS(X,Y,q)$ has no the identity element, so $PS(X,Y,q)^1 \neq PS(X,Y,q)$.

For comparison with what follows, we quote the descriptions for Green's relations on $PS(X,q)$ from [3, Theorems 7–10 and Remark 2].

Theorem 1 *Let $\alpha, \beta \in PS(X,q)$. Then the following statements hold.*

(a) $\alpha\mathscr{R}\beta$ *if and only if* $\operatorname{dom} \alpha = \operatorname{dom} \beta$.

(b) $\alpha\mathscr{L}\beta$ *if and only if* $(X\alpha = X\beta$ *and* $q \leqslant g(\alpha) = g(\beta))$ *or* $(\alpha = \beta$ *and* $g(\alpha) < q)$.

(c) $\alpha\mathscr{H}\beta$ *if and only if* $(X\alpha = X\beta, \operatorname{dom} \alpha = \operatorname{dom} \beta$ *and* $q \leqslant g(\alpha))$ *or* $(\alpha = \beta$ *and* $g(\alpha) < q)$.

(d) $\alpha\mathscr{D}\beta$ *if and only if* $(g(\alpha) < q$ *and* $\operatorname{dom} \alpha = \operatorname{dom} \beta)$ *or* $(r(\alpha) = r(\beta)$ *and* $q \leqslant g(\alpha) = g(\beta))$.

(e) $\alpha\mathscr{J}\beta$ *if and only if* $(\max\{g(\alpha), g(\beta)\} \leqslant q$ *and* $r(\alpha) = r(\beta))$ *or* $(q < g(\alpha) = g(\beta))$.

We begin by characterizing the relation $\mathscr{R}$ on $PS(X,Y,q)$. This finding appears to coincide with the results in [3, Theorem 7], when $\alpha\mathscr{R}\beta$ in $PS(X,q)$.

Theorem 2 *Let $\alpha, \beta \in PS(X,Y,q)$. Then $\alpha = \beta\mu$ for some $\mu \in PS(X,Y,q)$ if and only if $\operatorname{dom} \alpha \subseteq \operatorname{dom} \beta$. In other word, $\alpha\mathscr{R}\beta$ in $PS(X,Y,q)$ if and only if $\operatorname{dom} \alpha = \operatorname{dom} \beta$.*

Proof: It is clear that, if $\alpha = \beta\mu$ for some $\mu \in PS(X,Y,q)$, then $\operatorname{dom} \alpha \subseteq \operatorname{dom} \beta$. For the converse, we suppose that $\operatorname{dom} \alpha \subseteq \operatorname{dom} \beta$. We can write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} a_i & x_j \\ c_i & y_j \end{pmatrix},$$

where $\operatorname{dom} \alpha = \{a_i\} \subseteq \operatorname{dom} \beta$. We define $\mu = \begin{pmatrix} c_i \\ b_i \end{pmatrix}$. Then $X\mu = X\alpha \subseteq Y$ and $d(\mu) = d(\alpha) = q$, whence $\mu \in PS(X,Y,q)$ and $\alpha = \beta\mu$ as required. □

In order to characterize the $\mathscr{L}$-relation on $PS(X,Y,q)$, the following lemma is needed.

Lemma 1 *Let $\alpha, \beta \in PS(X,Y,q)$. Then $\alpha = \lambda\beta$ for some $\lambda \in PS(X,Y,q)$ if and only if the following conditions hold.*

(a) $X\alpha \subseteq X\beta$.

(b) $(X\alpha)\beta^{-1} \subseteq Y$.

(c) $q \leqslant \max\{g(\beta), |X\beta\backslash X\alpha|\} \leqslant \max\{g(\alpha), q\}$.

Proof: Suppose that $\alpha = \lambda\beta$ for some $\lambda \in PS(X,Y,q)$. Then $X\alpha \subseteq X\beta$ and we may write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} x_i & x_k \\ b_i & b_k \end{pmatrix},$$

where $X\alpha = \{b_i\} \subseteq X\beta$ and $\{x_i\} = (X\alpha)\beta^{-1}$. Thus $a_i\lambda\beta = a_i\alpha = b_i = x_i\beta$. Since β is injective, we have that $x_i = a_i\lambda \in Y$, that is $(X\alpha)\beta^{-1} \subseteq Y$. Observe that

$$X\backslash X\lambda = ((X\backslash X\lambda) \cap \operatorname{dom} \beta) \dot\cup ((X\backslash X\lambda) \cap (X\backslash \operatorname{dom} \beta)).$$

In addition, as β is injective, we have that

$$\begin{aligned} q &= |X\backslash X\lambda| \\ &= |(X\backslash X\lambda) \cap \operatorname{dom} \beta| + |(X\backslash X\lambda) \cap (X\backslash \operatorname{dom} \beta)| \\ &= |(X\backslash X\lambda)\beta| + |(X\backslash X\lambda) \cap (X\backslash \operatorname{dom} \beta)| \\ &= |X\beta\backslash X\alpha| + |(X\backslash X\lambda) \cap (X\backslash \operatorname{dom} \beta)| \\ &\leqslant |X\beta\backslash X\alpha| + |X\backslash \operatorname{dom} \beta| \\ &= \max\{g(\beta), |X\beta\backslash X\alpha|\}. \end{aligned} \tag{1}$$

Next, we see that $\operatorname{dom} \alpha = \operatorname{dom} \lambda\beta = (X\lambda \cap \operatorname{dom} \beta)\lambda^{-1}$. Then $(X\lambda \cap (X\backslash \operatorname{dom} \beta))\lambda^{-1} \subseteq X\backslash \operatorname{dom} \alpha$, whence $|X\lambda \cap (X\backslash \operatorname{dom} \beta)| = |(X\lambda \cap (X\backslash \operatorname{dom} \beta))\lambda^{-1}| \leqslant$

$|X\backslash \mathrm{dom}\,\alpha|$. This implies that

$$\begin{aligned} g(\beta) &= |X\backslash \mathrm{dom}\,\beta| \\ &= |(X\backslash \mathrm{dom}\,\beta)\cap X\lambda| + |(X\backslash \mathrm{dom}\,\beta)\cap(X\backslash X\lambda)| \\ &\leqslant |X\backslash \mathrm{dom}\,\alpha| + |X\backslash X\lambda| \\ &= \max\{g(\alpha), q\}. \end{aligned} \quad (2)$$

As $|X\beta\backslash X\alpha| \leqslant |X\backslash X\alpha| = q$ and from (1) and (2), we have that $q \leqslant \max\{g(\beta), |X\beta\backslash X\alpha|\} \leqslant \max\{g(\alpha), q\}$ as required.

Conversely, suppose that the conditions (a), (b) and (c) hold. From (a) and (b), we can write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} x_i & x_k \\ b_i & b_k \end{pmatrix},$$

where $X\alpha = \{b_i\} \subseteq X\beta$, $\{x_i\} = (X\alpha)\beta^{-1} \subseteq Y$ and $X\beta\backslash X\alpha = \{b_k\}$. We aim to define $\lambda \in PS(X,Y,q)$ such that $\alpha = \lambda\beta$. We consider two cases.

Case 1: $g(\alpha) < q$ or $g(\beta) \leqslant q$.

If $g(\alpha) < q$ then $\max\{g(\alpha), q\} = q$. So, the condition (c) implies $\max\{g(\beta), |X\beta\backslash X\alpha|\} = q$. On the other hand, if $g(\beta) \leqslant q$, then, as $|X\beta\backslash X\alpha| \leqslant |X\backslash X\alpha| = q$, the condition (c) implies $\max\{g(\beta), |X\beta\backslash X\alpha|\} = q$ again. Now, define $\lambda = \begin{pmatrix} a_i \\ x_i \end{pmatrix}$. Then $\alpha = \lambda\beta$, $X\lambda \subseteq Y$ and $d(\lambda) = g(\beta) + |\{x_k\}| = \max\{g(\beta), |X\beta\backslash X\alpha|\} = q$, whence $\lambda \in PS(X,Y,q)$.

Case 2: $g(\alpha) \geqslant q$ and $g(\beta) > q$.

From (c), we have that $q < g(\beta) \leqslant g(\alpha)$. Then we may write $X\backslash \mathrm{dom}\,\alpha = \{u_m\} \,\dot\cup\, T$, where $g(\beta) = |\{u_m\}|$ and $g(\alpha) = |T|$. We see that

$$X\backslash \mathrm{dom}\,\beta = (Y\backslash \mathrm{dom}\,\beta) \,\dot\cup\, ((X\backslash \mathrm{dom}\,\beta)\cap(X\backslash Y)), \quad (3)$$

where $|X\backslash \mathrm{dom}\,\beta| = g(\beta) > q$ and $|(X\backslash \mathrm{dom}\,\beta)\cap(X\backslash Y)| \leqslant |X\backslash Y| \leqslant q$. Then, from (3), we have $|Y\backslash \mathrm{dom}\,\beta| = |X\backslash \mathrm{dom}\,\beta| > q$. Now, write $Y\backslash \mathrm{dom}\,\beta = \{v_m\} \,\dot\cup\, U$, where $g(\beta) = |\{v_m\}|$ and $|U| = q$. In this case, we define $\lambda = \begin{pmatrix} a_i & u_m \\ x_i & v_m \end{pmatrix}$. Then λ is injective, $X\lambda \subseteq Y$ and $\alpha = \lambda\beta$. Moreover, since $|\{x_k\}| = |X\beta\backslash X\alpha| \leqslant |X\backslash X\alpha| = q$, $|U| = q$ and $|(X\backslash \mathrm{dom}\,\beta)\cap(X\backslash Y)| \leqslant |X\backslash Y| \leqslant q$, we have

$$\begin{aligned} d(\lambda) &= |X\backslash(\{x_i\}\cup\{v_m\})| \\ &= |\{x_k\}| + |U| + |(X\backslash \mathrm{dom}\,\beta)\cap(X\backslash Y)| = q, \end{aligned}$$

so $\lambda \in PS(X,Y,q)$ as required. □

Now, we can present our description of the relation $\mathscr{L}$ on $PS(X,Y,q)$.

Theorem 3 *Let $\alpha, \beta \in PS(X,Y,q)$. Then $\alpha\mathscr{L}\beta$ if and only if*

$$\alpha = \beta \;\; or \;\; (X\alpha = X\beta,\ \mathrm{dom}\,\alpha \subseteq Y,\ \mathrm{dom}\,\beta \subseteq Y \;\; and \;\; q \leqslant g(\alpha) = g(\beta)).$$

Proof: Suppose that $\alpha\mathscr{L}\beta$ in $PS(X,Y,q)$. Then $\alpha = \lambda\beta$ and $\beta = \mu\alpha$ for some $\lambda, \mu \in PS(X,Y,q)^1$. If $\alpha \neq \beta$, then $\lambda, \mu \in PS(X,Y,q)$. Thus, Lemma 1 implies that

(a_1) $X\alpha \subseteq X\beta$,
(a_2) $(X\alpha)\beta^{-1} \subseteq Y$,
(a_3) $q \leqslant \max\{g(\beta), |X\beta\backslash X\alpha|\} \leqslant \max\{g(\alpha), q\}$,
(b_1) $X\beta \subseteq X\alpha$,
(b_2) $(X\beta)\alpha^{-1} \subseteq Y$,
(b_3) $q \leqslant \max\{g(\alpha), |X\alpha\backslash X\beta|\} \leqslant \max\{g(\beta), q\}$.

Then (a_1) and (b_1) imply $X\alpha = X\beta$. Consequently, $|X\alpha\backslash X\beta| = 0 = |X\beta\backslash X\alpha|$. Thus, (a_3) implies $q \leqslant g(\beta) \leqslant \max\{g(\alpha), q\}$ and (b_3) implies $q \leqslant g(\alpha) \leqslant \max\{g(\beta), q\}$. As $q \leqslant g(\alpha)$ and $q \leqslant g(\beta)$, we have $\max\{g(\alpha), q\} = g(\alpha)$ and $\max\{g(\beta), q\} = g(\beta)$, so we obtain that $q \leqslant g(\beta) \leqslant g(\alpha)$ and $q \leqslant g(\alpha) \leqslant g(\beta)$, whence $q \leqslant g(\alpha) = g(\beta)$. Finally, as $X\alpha = X\beta$, we see that (a_2) implies $\mathrm{dom}\,\beta = (X\beta)\beta^{-1} = (X\alpha)\beta^{-1} \subseteq Y$. Similarly, (b_2) implies $\mathrm{dom}\,\alpha \subseteq Y$ as required.

Conversely, it is clear that if $\alpha = \beta$, then $\alpha\mathscr{L}\beta$. We suppose that $X\alpha = X\beta$, $\mathrm{dom}\,\alpha \subseteq Y$, $\mathrm{dom}\,\beta \subseteq Y$ and $q \leqslant g(\alpha) = g(\beta)$. Then $|X\beta\backslash X\alpha| = 0$, $\max\{g(\alpha), q\} = g(\alpha)$ and $\max\{g(\beta), |X\beta\backslash X\alpha|\} = g(\beta)$. Consequently, as $q \leqslant g(\alpha) = g(\beta)$, we obtain that $q \leqslant \max\{g(\beta), |X\beta\backslash X\alpha|\} = \max\{g(\alpha), q\}$. Moreover, the conditions $X\alpha = X\beta$ and $\mathrm{dom}\,\beta \subseteq Y$ imply $(X\alpha)\beta^{-1} = (X\beta)\beta^{-1} = \mathrm{dom}\,\beta \subseteq Y$. Thus, by Lemma 1, $\alpha = \lambda\beta$ for some $\lambda \in PS(X,Y,q)$. Analogously, we can prove that $\beta = \mu\alpha$ for some $\mu \in PS(X,Y,q)$. Hence, $\alpha\mathscr{L}\beta$ as required. □

According to Theorem 2 and Theorem 3, we have the following conclusion readily for $\mathscr{H}$-relation on $PS(X,Y,q)$.

Corollary 1 *Let $\alpha, \beta \in PS(X,Y,q)$. Then $\alpha\mathscr{H}\beta$ in $PS(X,Y,q)$ if and only if*

$$\alpha = \beta \;\; or \;\; (X\alpha = X\beta,\ \mathrm{dom}\,\alpha = \mathrm{dom}\,\beta \subseteq Y \; and \; q \leqslant g(\alpha)).$$

In what follows we describe the relation $\mathscr{D}$.

Theorem 4 *Let $\alpha, \beta \in PS(X,Y,q)$. Then $\alpha\mathscr{D}\beta$ if and only if*

$$\mathrm{dom}\,\alpha = \mathrm{dom}\,\beta \;\; or \;\; (r(\alpha) = r(\beta),\ \mathrm{dom}\,\alpha \subseteq Y,\ \mathrm{dom}\,\beta \subseteq Y \; and \; q \leqslant g(\alpha) = g(\beta)).$$

Proof: Suppose that $\alpha\mathscr{D}\beta$. Then there exists $\gamma \in PS(X,Y,q)$ such that $\alpha\mathscr{L}\gamma$ and $\gamma\mathscr{R}\beta$. Then by Theorem 2 and Theorem 3, $\mathrm{dom}\,\gamma = \mathrm{dom}\,\beta$ and (a) $\alpha = \gamma$ or (b) $X\alpha = X\gamma$, $\mathrm{dom}\,\alpha \subseteq Y$, $\mathrm{dom}\,\gamma \subseteq Y$ and $q \leqslant g(\alpha) = g(\gamma)$.

If (a) holds, then $\mathrm{dom}\,\alpha = \mathrm{dom}\,\gamma = \mathrm{dom}\,\beta$. Otherwise, if (b) holds, then $\mathrm{dom}\,\beta = \mathrm{dom}\,\gamma \subseteq Y$, $\mathrm{dom}\,\alpha \subseteq Y$, $g(\beta) = g(\gamma) = g(\alpha) \geqslant q$ and $|X\alpha| = |X\gamma| =$

$|\text{dom}\,\gamma| = |\text{dom}\,\beta| = |X\beta|$, that is $r(\alpha) = r(\beta)$ as required.

Conversely, if $\text{dom}\,\alpha = \text{dom}\,\beta$, then $\alpha\mathscr{R}\beta$ in $PS(X,Y,q)$. As $\mathscr{D}$ is an equivalence relation containing $\mathscr{R}$, we have that $\alpha\mathscr{D}\beta$ as required. Next, we assume $r(\alpha) = r(\beta)$, $\text{dom}\,\alpha \subseteq Y$, $\text{dom}\,\beta \subseteq Y$ and $q \leqslant g(\alpha) = g(\beta)$. As $r(\alpha) = r(\beta)$, we may write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \text{ and } \beta = \begin{pmatrix} c_i \\ d_i \end{pmatrix}.$$

We define $\gamma = \begin{pmatrix} c_i \\ b_i \end{pmatrix}$. Then $X\gamma = X\alpha \subseteq Y$ and $d(\gamma) = d(\alpha) = q$, whence $\gamma \in PS(X,Y,q)$. Moreover, $\text{dom}\,\gamma = \text{dom}\,\beta \subseteq Y$ and $g(\gamma) = g(\beta) = g(\alpha) \geqslant q$. So, by Theorem 2 and Theorem 3, $\alpha\mathscr{L}\gamma$ and $\gamma\mathscr{R}\beta$. It follows that $\alpha\mathscr{D}\beta$ in $PS(X,Y,q)$. □

In order to describe the Green's relation $\mathscr{J}$ on $PS(X,Y,q)$ when $|X| = q$, we need the following lemma.

Lemma 2 *Suppose that $|X| = q$. Let $\alpha,\beta \in PS(X,Y,q)$. Then $\beta = \lambda\alpha\mu$ for some $\lambda,\mu \in PS(X,Y,q)$ if and only if $|\text{dom}\,\beta| \leqslant |\text{dom}\,\alpha \cap Y|$.*

Proof: Suppose that $\beta = \lambda\alpha\mu$ for some $\lambda,\mu \in PS(X,Y,q)$. Then

$$|X\beta| = |X\lambda\alpha\mu| = |X\lambda\alpha \cap \text{dom}\,\mu| \leqslant |X\lambda\alpha| = |X\lambda \cap \text{dom}\,\alpha|.$$

Since $X\lambda \subseteq Y$, we have that $|X\lambda \cap \text{dom}\,\alpha| \leqslant |Y \cap \text{dom}\,\alpha|$. Thus, the above inequality implies that $|\text{dom}\,\beta| = |X\beta| \leqslant |\text{dom}\,\alpha \cap Y|$ as required.

Conversely, suppose that $\text{dom}\,\alpha \cap Y = \{a_i\}$, $i \in I$ and $\beta = \begin{pmatrix} c_j \\ d_j \end{pmatrix}$, $j \in J$ with $|J| \leqslant |I|$. If $|I|$ is finite, then $|J|$ is also finite, and we can write $\{a_i\} = \{x_j\} \,\dot\cup\, A$, for some finite set A with $|A| \leqslant |I|$. We note that the set A could be empty, and in the event that this occurs, it results in $|I| = |J|$ and $\{a_i\} = \{x_j\}$. We define

$$\lambda = \begin{pmatrix} c_j \\ x_j \end{pmatrix} \text{ and } \mu = \begin{pmatrix} x_j\alpha \\ d_j \end{pmatrix}.$$

Then $\beta = \lambda\alpha\mu$. Since $\{x_j\}$ is finite, we have $d(\lambda) = |X \setminus \{x_j\}| = q$. Moreover, $X\lambda \subseteq Y$, $X\mu = X\beta \subseteq Y$ and $d(\mu) = d(\beta) = q$, whence $\lambda,\mu \in PS(X,Y,q)$. On the other hand, if I is infinite, then we write $\{a_i\} = \{y_i\} \,\dot\cup\, \{y_j\}$ and define

$$\lambda' = \begin{pmatrix} c_j \\ y_j \end{pmatrix} \text{ and } \mu' = \begin{pmatrix} y_j\alpha \\ d_j \end{pmatrix}.$$

Clearly, $\beta = \lambda'\alpha\mu'$, $X\lambda' \subseteq Y$, $X\mu' = X\beta \subseteq Y$ and $d(\mu) = d(\beta) = q$, that is, $\mu' \in PS(X,Y,q)$. It remains to verify that $\lambda' \in PS(X,Y,q)$. We see that $|\text{dom}\,\alpha \cap Y| = |\{a_i\}| = |\{y_i\}|$, so

$$\begin{aligned} d(\lambda') &= |X \setminus \{y_j\}| = |X \setminus (\text{dom}\,\alpha \cap Y)| + |\{y_i\}| \\ &= |X \setminus (\text{dom}\,\alpha \cap Y)| + |\text{dom}\,\alpha \cap Y| = |X| = q. \end{aligned}$$

Therefore, $\lambda' \in PS(X,Y,q)$, which finishes the proof. □

The following theorem is a consequence of the above lemma.

Theorem 5 *Suppose that $|X| = q$. Let $\alpha,\beta \in PS(X,Y,q)$. Then $\alpha\mathscr{J}\beta$ if and only if $\text{dom}\,\alpha = \text{dom}\,\beta$ or $|\text{dom}\,\alpha| = |\text{dom}\,\alpha \cap Y| = |\text{dom}\,\beta| = |\text{dom}\,\beta \cap Y|$.*

Proof: Suppose that $\alpha\mathscr{J}\beta$ in $PS(X,Y,q)$. Then, there exist $\sigma,\delta,\sigma',\delta' \in PS(X,Y,q)^1$ such that $\alpha = \sigma\beta\delta$ and $\beta = \sigma'\alpha\delta'$. If $\sigma = 1 = \sigma'$, then $\alpha = \beta\delta$ and $\beta = \alpha\delta'$. So $\alpha\mathscr{R}\beta$, whence $\text{dom}\,\alpha = \text{dom}\,\beta$ by Theorem 2. If $\delta = 1 = \delta'$, then $\alpha = \sigma\beta$ and $\beta = \sigma'\alpha$, which imply $\alpha\mathscr{L}\beta$. Then, by Theorem 3, we have $\alpha = \beta$ or ($X\alpha = X\beta, \text{dom}\,\alpha \subseteq Y, \text{dom}\,\beta \subseteq Y$ and $q \leqslant g(\alpha) = g(\beta)$).

Here, if $\alpha = \beta$, then we obtain that $\text{dom}\,\alpha = \text{dom}\,\beta$. Otherwise, if the latter holds, then $\text{dom}\,\alpha \cap Y = \text{dom}\,\alpha$ and $\text{dom}\,\beta \cap Y = \text{dom}\,\beta$. Moreover, as $X\alpha = X\beta$, we obtain that $|\text{dom}\,\alpha \cap Y| = |\text{dom}\,\alpha| = |X\alpha| = |X\beta| = |\text{dom}\,\beta| = |\text{dom}\,\beta \cap Y|$.

In other cases, it is a routine to check that $\alpha = \lambda\beta\mu$ and $\beta = \lambda'\alpha\mu'$ for some $\lambda,\lambda',\mu,\mu' \in PS(X,Y,q)$ (for example, if $\sigma = 1$ and $\delta,\delta',\sigma' \in PS(X,Y,q)$, then $\alpha = \beta\delta$ and $\beta = \sigma'\alpha\delta'$. So $\alpha = \beta\delta = (\sigma'\alpha\delta')\delta = \sigma'(\beta\delta)\delta'\delta = \sigma'\beta(\delta\delta'\delta)$, where $\delta\delta'\delta \in PS(X,Y,q)$). Thus, by Lemma 2, $|\text{dom}\,\beta| \leqslant |\text{dom}\,\alpha \cap Y| \leqslant |\text{dom}\,\alpha| \leqslant |\text{dom}\,\beta \cap Y| \leqslant |\text{dom}\,\beta|$. Hence, $|\text{dom}\,\alpha| = |\text{dom}\,\alpha \cap Y| = |\text{dom}\,\beta| = |\text{dom}\,\beta \cap Y|$ as required.

Conversely, if $\text{dom}\,\alpha = \text{dom}\,\beta$, then $\alpha\mathscr{R}\beta$ and so $\alpha\mathscr{J}\beta$ in $PS(X,Y,q)$. Now, we assume that $|\text{dom}\,\alpha| = |\text{dom}\,\alpha \cap Y| = |\text{dom}\,\beta| = |\text{dom}\,\beta \cap Y|$. Then Lemma 2 implies $\alpha = \lambda\beta\mu$ and $\beta = \lambda'\alpha\mu'$ for some $\lambda,\lambda',\mu,\mu' \in PS(X,Y,q)$. Therefore, $\alpha\mathscr{J}\beta$ and the proof is complete. □

To finish the study of the Green's relations in $PS(X,Y,q)$, we give the following description of the $\mathscr{J}$- relation when $|X| > q$.

Theorem 6 *Suppose that $|X| = p > q$. Let $\alpha,\beta \in PS(X,Y,q)$. Then, $\beta = \lambda\alpha\mu$ for some $\lambda,\mu \in PS(X,Y,q)$ if and only if $g(\alpha) \leqslant q$ or $q < g(\alpha) \leqslant g(\beta)$. In other word, $\alpha\mathscr{J}\beta$ if and only if $\max\{g(\alpha), g(\beta)\} \leqslant q$ or $q < g(\alpha) = g(\beta)$.*

Proof: Suppose that $\beta = \lambda\alpha\mu$ for some $\lambda,\mu \in PS(X,Y,q)$. Since $p > q$, we have $r(\alpha) = r(\beta) = p$. If $g(\alpha) = t$ for some infinite cardinal t greater than q, then we have $t = |X \setminus \text{dom}\,\alpha| = |(X \setminus \text{dom}\,\alpha) \cap X\lambda| + |(X \setminus \text{dom}\,\alpha) \cap (X \setminus X\lambda)|$, where $|(X \setminus \text{dom}\,\alpha) \cap (X \setminus X\lambda)| \leqslant |X \setminus X\lambda| = q < t$. So, the above equation implies $|(X \setminus \text{dom}\,\alpha) \cap X\lambda| = t$. Next, suppose that $(X \setminus \text{dom}\,\alpha) \cap X\lambda = \{v_i\}$. Then, $v_i = u_i\lambda$ for some $u_i \in \text{dom}\,\lambda$ and $u_i\lambda \notin \text{dom}\,\alpha$. So $u_i\lambda\alpha\mu$ is not defined. Consequently, as $\beta = \lambda\alpha\mu$, we have that $u_i \notin \text{dom}\,\beta$ for all i. Therefore, $\{u_i\} \subseteq X \setminus \text{dom}\,\beta$, where $|\{u_i\}| = t$. It follows that $g(\beta) \geqslant t = g(\alpha) > q$ as required.

Conversely, notice that, since $p > q$, we have

$|\text{dom }\alpha| = |\text{dom }\beta| = p$. We may write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \text{ and } \beta = \begin{pmatrix} c_i \\ d_i \end{pmatrix},$$

where $i \in I$ and $|I| = p$. We also see that $\text{dom }\alpha = (\text{dom }\alpha \cap Y) \dot\cup (\text{dom }\alpha \backslash Y)$ and $|\text{dom }\alpha \backslash Y| \leqslant |X \backslash Y| \leqslant q$. Therefore, $|\text{dom }\alpha \cap Y| = p$. We write $\text{dom }\alpha \cap Y = \{x_i\} \dot\cup A$, where $|A| = q$ and define $\mu = \begin{pmatrix} x_i\alpha \\ d_i \end{pmatrix}$. Then $X\mu = X\beta \subseteq Y$ and $d(\mu) = d(\beta) = q$, whence $\mu \in PS(X,Y,q)$. Now, if $g(\alpha) \leqslant q$, then we define $\lambda = \begin{pmatrix} c_i \\ x_i \end{pmatrix}$. Clearly, $\beta = \lambda\alpha\mu$ and $X\lambda \subseteq Y$. Moreover, since $|\text{dom }\alpha \backslash Y| \leqslant q$ and $|A| = q$, we have $|X \backslash X\lambda| = |X \backslash \{x_i\}| = |X \backslash \text{dom }\alpha| + |\text{dom }\alpha \backslash Y| + |A| = q$, that is, $\lambda \in PS(X,Y,q)$. Finally, if $q < g(\alpha) = t \leqslant g(\beta)$, then we consider $X \backslash \text{dom }\alpha = ((X \backslash \text{dom }\alpha) \cap Y) \dot\cup ((X \backslash \text{dom }\alpha) \backslash Y)$. Since $|(X \backslash \text{dom }\alpha) \backslash Y| \leqslant |X \backslash Y| \leqslant q < t$, we have $|(X \backslash \text{dom }\alpha) \cap Y| = t$. We write $(X \backslash \text{dom }\alpha) \cap Y = B \dot\cup C$, where $|B| = t$ and $|C| = q$. Since $q < g(\alpha) = t \leqslant g(\beta)$, there exists a subset D of $X \backslash \text{dom }\beta$ such that $|D| = t$. Now, define $\lambda' = \begin{pmatrix} c_i & D \\ x_i & B \end{pmatrix}$, where $\lambda'|_D$ is a bijection from D onto B. We see that $\beta = \lambda'\alpha\mu$ and $X\lambda' = \{x_i\} \cup B \subseteq Y$. Moreover, $d(\lambda') = |X \backslash X\lambda'| = |X \backslash Y| + |A| + |C| = q$, whence $\lambda' \in PS(X,Y,q)$. This completes the proof. □

It is known that $\mathscr{D} \subseteq \mathscr{J}$ on any semigroup and $\mathscr{D} = \mathscr{J}$ on some well known transformation semigroups, for example, on $P(X), T(X)$ and $I(X)$ see [15, p. 63 and p. 211]. However, this is not always true for $PS(X,Y,q)$ as shown in the following example.

Example 1 Let $X = \mathbb{N}$ denote the set of all positive integers. Let Y be the set of all positive even integers and let $q = \aleph_0$. We define $\alpha = \begin{pmatrix} 3n \\ 2n \end{pmatrix}$, where $n \in \mathbb{N}$, and $\beta = \text{id}_Y$. It can be verified that $\alpha, \beta \in PS(X,Y,q)$. We see that $|\text{dom }\alpha \cap Y| = |\{6n : n \in \mathbb{N}\}| = \aleph_0 = |\text{dom }\alpha|$ and $|\text{dom }\beta \cap Y| = |\text{dom }\beta| = |Y| = \aleph_0$. So $\alpha \mathscr{J} \beta$ in $PS(X,Y,q)$ by Theorem 5. But α and β are not $\mathscr{D}$-related in $PS(X,Y,q)$ by Theorem 4 since $\text{dom }\alpha \neq \text{dom }\beta$ and $\text{dom }\alpha \nsubseteq Y$.

To close this section, it is worth noticing that, unlike the $\mathscr{R}$-relation, when $X \neq Y$ the relations $\mathscr{L}$, $\mathscr{H}$, $\mathscr{D}$ and $\mathscr{J}$ on $PS(X,Y,q)$ are not the restriction of the corresponding relations from $PS(X,q)$ to $PS(X,Y,q)$. We provide some examples below.

Example 2 Let X, Y and q be as in Example 1.

(i) Define $\alpha = \begin{pmatrix} 5 & 4 \\ 2 & 4 \end{pmatrix}$ and $\beta = \begin{pmatrix} 4 & 5 \\ 2 & 4 \end{pmatrix}$. We can verify that $\alpha, \beta \in PS(X,Y,q) \subseteq PS(X,q)$ and $X\alpha = X\beta$, $\text{dom }\alpha = \text{dom }\beta$ and $g(\alpha) = g(\beta) = \aleph_0$. So $\alpha \mathscr{H} \beta$ in $PS(X,q)$ by Theorem 1. Since $\mathscr{H} \subseteq \mathscr{L}$, we obtain that $\alpha \mathscr{L} \beta$ in $PS(X,q)$. But α and β are not $\mathscr{L}$-related in $PS(X,Y,q)$ by Theorem 3 since $\alpha \neq \beta$ and $\text{dom }\alpha \nsubseteq Y$. Consequently, they are not $\mathscr{H}$-related in $PS(X,Y,q)$.

(ii) Define $\gamma = \begin{pmatrix} 3 & 4 \\ 2 & 4 \end{pmatrix}$ and $\mu = \begin{pmatrix} 5 & 6 \\ 2 & 4 \end{pmatrix}$. Then $\gamma, \mu \in PS(X,Y,q) \subseteq PS(X,q)$, $r(\gamma) = 2 = r(\mu)$ and $g(\gamma) = g(\mu) = \aleph_0$. So $\gamma \mathscr{D} \mu$ in $PS(X,q)$ by Theorem 1. Since $\mathscr{D} \subseteq \mathscr{J}$, we obtain that $\gamma \mathscr{J} \mu$ in $PS(X,q)$. But γ and μ are not $\mathscr{J}$-related in $PS(X,Y,q)$ by Theorem 5 since $\text{dom }\gamma \neq \text{dom }\mu$ and $|\text{dom }\gamma| = 2$ whereas $|\text{dom }\gamma \cap Y| = 1$. Consequently, they are not $\mathscr{D}$-related in $PS(X,Y,q)$.

NATURAL PARTIAL ORDER

In this section, we investigate various properties of the natural partial order on $PS(X,Y,q)$. First, we recall that by Proposition 3, $PS(X,Y,q)$ is not a regular semigroup, so the definition of the natural partial order that is used in this paper is the Mitsch's order, that is, for $\alpha, \beta \in PS(X,Y,q)$, $\alpha \leqslant \beta$ if and only if $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$ for some $\lambda, \mu \in PS(X,Y,q)^1$.

We also notice that, for $\alpha, \beta \in PS(X,Y,q)$ with $\alpha \subseteq \beta$, since they are injective, we obtain the following results which will be used throughout this section.

(i) $|\text{dom }\beta \backslash \text{dom }\alpha| = |X\beta \backslash X\alpha|$.
(ii) $(X\alpha)\alpha^{-1} = (X\alpha)\beta^{-1}$.
(iii) If $\text{dom }\alpha = \text{dom }\beta$ or $X\alpha = X\beta$, then $\alpha = \beta$.

We denote by $\alpha \subset \beta$ when $\alpha \subseteq \beta$ and $\alpha \neq \beta$. Similarly, we write $\alpha < \beta$ when $\alpha \leqslant \beta$ and $\alpha \neq \beta$.

We begin with describing the conditions for $\alpha, \beta \in PS(X,Y,q)$ are related under the natural partial order.

Theorem 7 *Let $\alpha, \beta \in PS(X,Y,q)$. Then $\alpha \leqslant \beta$ if and only if $\alpha = \beta$ or $(\alpha \subseteq \beta, \text{dom }\alpha \subseteq Y$ and $q \leqslant \max\{g(\beta), |X\beta \backslash X\alpha|\})$.*

Proof: Suppose that $\alpha \leqslant \beta$ in $PS(X,Y,q)$. Then $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$ for some $\lambda, \mu \in PS(X,Y,q)^1$, which imply $X\alpha \subseteq X\beta$ and $\text{dom }\alpha \subseteq \text{dom }\beta$. If $\alpha \neq \beta$, then $\lambda, \mu \in PS(X,Y,q)$. So, by Lemma 1, we have $q \leqslant \max\{g(\beta), |X\beta \backslash X\alpha|\}$ and $(X\alpha)\beta^{-1} \subseteq Y$. Next, as $\alpha = \beta\mu = \alpha\mu$, we have that $x\alpha\mu = x\beta\mu$ for all $x \in \text{dom }\alpha$. Thus, $x\alpha = x\beta$ as μ is injective. Therefore, $\alpha \subseteq \beta$ and so $\text{dom }\alpha = (X\alpha)\alpha^{-1} = (X\alpha)\beta^{-1} \subseteq Y$ as required.

For the converse, if $\alpha = \beta$, then clearly, $\alpha \leqslant \beta$. So we assume that $\alpha \subseteq \beta$, $\text{dom }\alpha \subseteq Y$ and $q \leqslant \max\{g(\beta), |X\beta \backslash X\alpha|\}$. We may write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix} \text{ and } \beta = \begin{pmatrix} a_i & a_j \\ x_i & x_j \end{pmatrix}.$$

Let $\mu = \begin{pmatrix} x_i \\ x_i \end{pmatrix}$, clearly $\alpha = \beta\mu = \alpha\mu$, where $X\mu = X\alpha \subseteq Y$ and $d(\mu) = d(\alpha) = q$, whence $\mu \in PS(X,Y,q)$. We also see that the condition $\alpha \subseteq \beta$ implies $X\alpha \subseteq X\beta$ and $\text{dom }\alpha \subseteq \text{dom }\beta$, so $g(\beta) \leqslant g(\alpha)$. In addition, since $|X\beta \backslash X\alpha| \leqslant |X \backslash X\alpha| = q$, the condition $q \leqslant \max\{g(\beta), |X\beta \backslash X\alpha|\}$ implies $q \leqslant \max\{g(\beta), |X\beta \backslash X\alpha|\} \leqslant \max\{g(\alpha), q\}$. Moreover, as $\alpha \subseteq \beta$, we also obtain that $(X\alpha)\beta^{-1} = (X\alpha)\alpha^{-1} = \operatorname{dom}\alpha \subseteq Y$. Thus, by Lemma 1, $\alpha = \lambda\beta$ for some $\lambda \in PS(X, Y, q)$. Hence, $\alpha \leqslant \beta$ as required. □

From Theorem 7, we see that the natural partial order $\leqslant$ is contained in $\subseteq$ on $PS(X, Y, q)$. We will subsequently use this fact without further mention.

Theorem 8 *$PS(X, Y, q)$ has no maximum element with respect to $\leqslant$.*

Proof: For a contradiction, suppose $\gamma \in PS(X, Y, q)$ is the maximum under $\leqslant$. If $|X| = q$, then we choose $a, b \in X$ and $c \in Y$ with $a \neq b$. Define

$$\alpha = \begin{pmatrix} a \\ c \end{pmatrix} \text{ and } \beta = \begin{pmatrix} b \\ c \end{pmatrix}.$$

It can be verified that $\alpha, \beta \in PS(X, Y, q)$. Then, $\alpha, \beta \leqslant \gamma$ and so $\alpha, \beta \subseteq \gamma$. This implies $a\alpha = a\gamma$ and $b\beta = b\gamma$. Since $a\alpha = c = b\beta$, we have that $a\gamma = b\gamma$ and thus $a = b$ (as γ is injective), a contradiction. On the other hand, assume that $|X| = p > q$. In this case, we choose $u, v \in \operatorname{dom}\gamma$ with $u \neq v$ (possible since $|\operatorname{dom}\gamma| = p$ when $p > q$), then $u\gamma \neq v\gamma$. We define

$$\mu = \begin{pmatrix} \operatorname{dom}\gamma \backslash \{u, v\} & u & v \\ X\gamma \backslash \{u\gamma, v\gamma\} & v\gamma & u\gamma \end{pmatrix},$$

where $x\mu = x\gamma$ for all $x \in \operatorname{dom}\gamma \backslash \{u, v\}$, then $X\mu = X\gamma \subseteq Y$ and $d(\mu) = d(\gamma) = q$, whence $\mu \in PS(X, Y, q)$. Since γ is the maximum, we have $\mu \leqslant \gamma$, which implies $\mu \subseteq \gamma$ and so $u\mu = u\gamma$. Thus, $u\gamma = v\gamma$, a contradiction again. In all cases, we deduce that $PS(X, Y, q)$ has no the maximum element under $\leqslant$. □

Theorem 9 *The following statements hold for the minimum element with respect to $\leqslant$ in $PS(X, Y, q)$.*

(a) If $|X| = q$, then $\varnothing$ is the minimum element in $PS(X, Y, q)$.

(b) If $|X| > q$, then $PS(X, Y, q)$ has no minimum element.

Proof: In order to prove (a), suppose that $|X| = q$ and let $\alpha \in PS(X, Y, q)$. It is clear that $\varnothing \subseteq \alpha$ and $\operatorname{dom}\varnothing = \varnothing \subseteq Y$. If $q \leqslant g(\alpha)$, then $q \leqslant \max\{g(\alpha), |X\alpha \backslash X\varnothing|\}$ and so $\varnothing \leqslant \alpha$ by Theorem 7. Otherwise, if $g(\alpha) < q$, then $|X\alpha \backslash X\varnothing| = |X\alpha| = q$. Thus, $q \leqslant \max\{g(\alpha), |X\alpha \backslash X\varnothing|\}$ and so $\varnothing \leqslant \alpha$ again. Hence, $\varnothing$ is the the minimum element under $\leqslant$.

To prove (b), we suppose that $|X| = p > q$ and let $\alpha \in PS(X, Y, q)$. Then, $|\operatorname{dom}\alpha| = p$. We choose $a \in \operatorname{dom}\alpha$ and define $\beta \in PS(X, Y, q)$ by $\operatorname{dom}\beta = \operatorname{dom}\alpha \backslash \{a\}$ and $x\beta = x\alpha$ for all $x \in \operatorname{dom}\alpha \backslash \{a\}$. Then, $X\beta \subseteq X\alpha \subseteq Y$ and $d(\beta) = d(\alpha) + 1 = q$, whence $\beta \in PS(X, Y, q)$. We also see that $\beta \subset \alpha$, so $PS(X, Y, q)$ has no the minimum element under $\subseteq$. As the relation $\leqslant$ implies $\subseteq$ on $PS(X, Y, q)$, it can be verified that $PS(X, Y, q)$ has no the minimum element under $\leqslant$, and the proof is complete. □

Theorem 10 *Let $\alpha \in PS(X, Y, q)$. Then, α is a maximal element in $PS(X, Y, q)$ with respect to $\leqslant$ if and only if*

$$g(\alpha) < q \text{ or } X\alpha = Y \text{ or } \operatorname{dom}\alpha \nsubseteq Y.$$

Proof: For the first part, we will prove the contrapositive version. Suppose that $g(\alpha) \geqslant q$, $X\alpha \subsetneq Y$ and $\operatorname{dom}\alpha \subseteq Y$. We choose $a \in X \backslash \operatorname{dom}\alpha$, $b \in Y \backslash X\alpha$ and define $\beta : \operatorname{dom}\alpha \cup \{a\} \to Y$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \in \operatorname{dom}\alpha, \\ b & \text{if } x = a. \end{cases}$$

It is clear that $X\beta \subseteq Y$ and $d(\beta) = d(\alpha) - 1 = q$, whence $\beta \in PS(X, Y, q)$. In addition, $\alpha \subset \beta$ and $g(\beta) = g(\alpha) - 1 \geqslant q$, so $q \leqslant \max\{g(\beta), |X\beta \backslash X\alpha|\}$. Then, by Theorem 7, $\alpha < \beta$. Hence, α is not a maximal element.

To prove the converse, assume that the conditions hold and suppose $\alpha \leqslant \beta$, where $\beta \in PS(X, Y, q)$. We aim to show that $\alpha = \beta$. Since $\alpha \leqslant \beta$, we have $\alpha \subseteq \beta$ and so $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$. It follows that $X \backslash \operatorname{dom}\alpha = (X \backslash \operatorname{dom}\beta) \mathbin{\dot\cup} (\operatorname{dom}\beta \backslash \operatorname{dom}\alpha)$. Therefore,

$$g(\alpha) = g(\beta) + |\operatorname{dom}\beta \backslash \operatorname{dom}\alpha| = g(\beta) + |X\beta \backslash X\alpha|. \quad (4)$$

If $g(\alpha) < q$, then the sum on the right of (4) is also less than q, whence $\max\{g(\beta), |X\beta \backslash X\alpha|\} < q$. Thus, as $\alpha \leqslant \beta$, we can deduce from Theorem 7 that it is possible only when $\alpha = \beta$. Similarly, if $\operatorname{dom}\alpha \nsubseteq Y$, then $\alpha = \beta$ by Theorem 7 again. Finally, by using the fact that $\alpha \subseteq \beta$, if $X\alpha = Y$, then $Y = X\alpha \subseteq X\beta \subseteq Y$ and so $X\alpha = X\beta$, whence $\alpha = \beta$. In all cases, we deduce that α is maximal under $\leqslant$. This completes the proof. □

In order to describe all minimal elements in $PS(X, Y, q)$, we need the following lemma.

Lemma 3 *Suppose that $|X| = q$. Let $\alpha \in PS(X, Y, q)$ be such that $\alpha \neq \varnothing$. If α is a minimal element with respect to $\leqslant$ in $PS(X, Y, q)$, then either* $\operatorname{dom}\alpha \subseteq Y$ *or* $\operatorname{dom}\alpha \subseteq X \backslash Y$.

Proof: Let α be a non-zero minimal element under $\leqslant$ in $PS(X, Y, q)$ and suppose that $\operatorname{dom}\alpha \nsubseteq X \backslash Y$, so $\operatorname{dom}\alpha \cap Y \neq \varnothing$. First, if $|\operatorname{dom}\alpha \cap Y| = q$, then we write $\operatorname{dom}\alpha \cap Y = A \mathbin{\dot\cup} B$, where $|A| = |B| = q$. Let $\gamma = \alpha|_A$, clearly $\varnothing \neq \gamma \subset \alpha$, $X\gamma \subseteq X\alpha \subseteq Y$ and $d(\gamma) = d(\alpha) + |X\alpha \backslash A\alpha| = q$, whence $\gamma \in PS(X, Y, q)$. Since $B \subseteq \operatorname{dom}\alpha \backslash \operatorname{dom}\gamma$, we obtain that $|X\alpha \backslash X\gamma| = |\operatorname{dom}\alpha \backslash \operatorname{dom}\gamma| = q$, and thus $\max\{g(\alpha), |X\alpha \backslash X\gamma|\} = q$. In addition, as $\operatorname{dom}\gamma = A \subseteq Y$, then $\varnothing \neq \gamma < \alpha$ by Theorem 7, this contradicts the minimality of α. Therefore, $|\operatorname{dom}\alpha \cap Y| < q$. Next, let $\beta = \alpha|_{\operatorname{dom}\alpha \cap Y}$. It is clear that $\varnothing \neq \beta \subseteq \alpha$, $\operatorname{dom}\beta = \operatorname{dom}\alpha \cap Y \subseteq Y$, $X\beta = Y\alpha \subseteq X\alpha \subseteq Y$ and $d(\beta) = d(\alpha) + |X\alpha \backslash Y\alpha| = q$, whence $\beta \in PS(X, Y, q)$. Now, if $g(\alpha) = q$, then $\max\{g(\alpha), |X\alpha \backslash X\beta|\} = q$. In this case, Theorem 7 implies $\varnothing \neq \beta \leqslant \alpha$, and so $\alpha = \beta$ since α is minimal. It follows that $\operatorname{dom}\alpha = \operatorname{dom}\alpha \cap Y$, whence $\operatorname{dom}\alpha \subseteq Y$. Otherwise, if $g(\alpha) < q$, then $|\operatorname{dom}\alpha| = q$. As $\operatorname{dom}\alpha$ is a disjoint union of $\operatorname{dom}\alpha \cap (X \backslash Y)$ and dom $\alpha \cap Y$, and we have shown that $|\mathrm{dom}\,\alpha \cap Y| < q$, hence, $|\mathrm{dom}\,\alpha \cap (X \setminus Y)| = q$. Consequently, $|X\alpha \setminus X\beta| = |\mathrm{dom}\,\alpha \setminus \mathrm{dom}\,\beta| = |\mathrm{dom}\,\alpha \cap (X \setminus Y)| = q$, which implies that $\max\{g(\alpha), |X\alpha \setminus X\beta|\} = q$. By Theorem 7 again, we have $\varnothing \neq \beta \leqslant \alpha$, so the result that $\mathrm{dom}\,\alpha \subseteq Y$ can be derived like before. □

The next result characterizes all the minimal elements under $\leqslant$ in $PS(X,Y,q)$.

Theorem 11 *Let $\alpha \in PS(X,Y,q)$. Then the following statements hold.*

(a) If $|X| = q$, then α is a non-zero minimal element with respect to $\leqslant$ in $PS(X,Y,q)$ if and only if $|\mathrm{dom}\,\alpha| = 1$ or $\mathrm{dom}\,\alpha \subseteq X \setminus Y$.

(b) If $|X| > q$, then $PS(X,Y,q)$ has no minimal element with respect to $\leqslant$.

Proof: To show (a), suppose that $|X| = q$ and let α be a non-zero minimal element under $\leqslant$. By Lemma 3, either $\mathrm{dom}\,\alpha \subseteq Y$ or $\mathrm{dom}\,\alpha \subseteq X \setminus Y$. If the latter holds, then the proof is complete. So, we suppose $\mathrm{dom}\,\alpha \subseteq Y$. In this case, if $|\mathrm{dom}\,\alpha| > 1$, then we choose $a \in \mathrm{dom}\,\alpha$ and let $\theta = \begin{pmatrix} a \\ a\alpha \end{pmatrix}$. It can be verified that $\theta \in PS(X,Y,q)$ and, as $|\mathrm{dom}\,\alpha| > 1$, we have $\varnothing \neq \theta \subset \alpha$. If $q \leqslant g(\alpha)$, then $q \leqslant \max\{g(\alpha), |X\alpha \setminus X\theta|\}$. On the other hand, if $g(\alpha) < q$, then $|X\alpha| = q$ and so $|X\alpha \setminus X\theta| = |X\alpha \setminus \{a\alpha\}| = q$. Thus, $q \leqslant \max\{g(\alpha), |X\alpha \setminus X\theta|\}$ again. Then, in both cases, $\varnothing \neq \theta < \alpha$ by Theorem 7, which contradicts to the minimality of α. Therefore, $|\mathrm{dom}\,\alpha| = 1$.

Conversely, suppose the conditions hold and let $\beta \in PS(X,Y,q)$ be such that $\varnothing \neq \beta \leqslant \alpha$. Then, $\varnothing \neq \beta \subseteq \alpha$ and so $0 < |\mathrm{dom}\,\beta| \leqslant |\mathrm{dom}\,\alpha|$. If $|\mathrm{dom}\,\alpha| = 1$ then $|\mathrm{dom}\,\beta| = 1$, whence $\mathrm{dom}\,\alpha = \mathrm{dom}\,\beta$ and so $\alpha = \beta$. If $\mathrm{dom}\,\alpha \subseteq X \setminus Y$, then $\mathrm{dom}\,\beta \subseteq \mathrm{dom}\,\alpha \subseteq X \setminus Y$. As $\beta \leqslant \alpha$, we obtain that $\alpha = \beta$ by Theorem 7. In both cases, we deduce that α is non-zero minimal under $\leqslant$.

In order to prove (b), suppose that $|X| = p > q$. In this case, for any $\alpha \in PS(X,Y,q)$, we see that $|\mathrm{dom}\,\alpha| = p$. As $\mathrm{dom}\,\alpha = (\mathrm{dom}\,\alpha \cap Y) \,\dot\cup\, (\mathrm{dom}\,\alpha \cap (X \setminus Y)$, where $|\mathrm{dom}\,\alpha \cap (X \setminus Y)| \leqslant |X \setminus Y| \leqslant q < p$, we have $|\mathrm{dom}\,\alpha \cap Y| = p$. We may write $\mathrm{dom}\,\alpha \cap Y = A \,\dot\cup\, B$, where $|A| = p, |B| = q$. Let $\gamma = \alpha|_A$, we have $\gamma \subset \alpha$, $\mathrm{dom}\,\gamma \subseteq Y$, $X\gamma \subseteq X\alpha \subseteq Y$ and $d(\gamma) = d(\alpha) + |B\alpha| + |C\alpha| = q$, where $C = \mathrm{dom}\,\alpha \cap (X \setminus Y)$, whence $\gamma \in PS(X,Y,q)$. Moreover, $|X\alpha \setminus X\gamma| = |X\alpha \setminus A\alpha| = |B\alpha| + |C\alpha| = q$, so $q \leqslant \max\{g(\alpha), |X\alpha \setminus X\gamma|\}$. Then by Theorem 7, $\gamma < \alpha$, which means α is not a minimal element. □

Next, we examine the compatibility of the natural partial order on $PS(X,Y,q)$. To do this, we first recall from [3, p. 104] that, the containment order $\subseteq$ is both left and right compatible on $P(X)$, in other words, if $\alpha \subseteq \beta$, then $\gamma\alpha \subseteq \gamma\beta$ and $\alpha\gamma \subseteq \beta\gamma$ for all $\alpha, \beta, \gamma \in P(X)$. Therefore, it is also left and right compatible on $PS(X,Y,q)$ since $PS(X,Y,q)$ is contained in $P(X)$.

Theorem 12 *The natural partial order is right compatible on $PS(X,Y,q)$.*

Proof: Let $\alpha, \beta, \gamma \in PS(X,Y,q)$ be such that $\alpha \leqslant \beta$. Clearly, if $\alpha = \beta$, then $\alpha\gamma = \beta\gamma$, whence $\alpha\gamma \leqslant \beta\gamma$. Similarly, if $X\alpha \cap \mathrm{dom}\,\gamma = \varnothing$, then $\alpha\gamma = \varnothing \leqslant \beta\gamma$ (this occurs only when $|X| = q$). In both cases, γ is a right compatible element in $PS(X,Y,q)$. Now, we suppose that $\alpha \neq \beta$ and $X\alpha \cap \mathrm{dom}\,\gamma \neq \varnothing$. Then by Theorem 7, we have $\alpha \subseteq \beta$, $\mathrm{dom}\,\alpha \subseteq Y$ and $q \leqslant \max\{g(\beta), |X\beta \setminus X\alpha|\}$. If $(X\beta \setminus X\alpha) \cap \mathrm{dom}\,\gamma = \varnothing$, then, as $\alpha \subseteq \beta$, we have $\alpha\gamma = \beta\gamma$, whence γ is right compatible. Now, we assume $(X\beta \setminus X\alpha) \cap \mathrm{dom}\,\gamma \neq \varnothing$. Since $\subseteq$ is right compatible on $PS(X,Y,q)$, we have $\alpha\gamma \subseteq \beta\gamma$. Moreover, $\mathrm{dom}\,\alpha\gamma \subseteq \mathrm{dom}\,\alpha \subseteq Y$. To verify that $\alpha\gamma \leqslant \beta\gamma$, by Theorem 7, it remains to prove that $q \leqslant \max\{g(\beta\gamma), |X\beta\gamma \setminus X\alpha\gamma|\}$. We consider two cases.

Case 1: $q \leqslant g(\beta)$. Since $\mathrm{dom}\,\beta\gamma \subseteq \mathrm{dom}\,\beta$, we get that $q \leqslant g(\beta) \leqslant g(\beta\gamma)$. Therefore, $q \leqslant \max\{g(\beta\gamma), |X\beta\gamma \setminus X\alpha\gamma|\}$ as required.

Case 2: $g(\beta) < q$. In this case, the condition $q \leqslant \max\{g(\beta), |X\beta \setminus X\alpha|\}$ implies $q \leqslant \max\{g(\beta), |X\beta \setminus X\alpha|\} = |X\beta \setminus X\alpha| \leqslant |X \setminus X\alpha| = q$, whence $|X\beta \setminus X\alpha| = q$. Next, since

$$X\beta \setminus X\alpha = ((X\beta \setminus X\alpha) \setminus \mathrm{dom}\,\gamma) \,\dot\cup\, ((X\beta \setminus X\alpha) \cap \mathrm{dom}\,\gamma), \quad (5)$$

we have that at least one set on the right of (5) has cardinality q. If $|(X\beta \setminus X\alpha) \setminus \mathrm{dom}\,\gamma| = q$, as $(X\beta \setminus X\alpha) \setminus \mathrm{dom}\,\gamma \subseteq X\beta \setminus \mathrm{dom}\,\gamma$, then

$$\begin{aligned} q &\leqslant |X\beta \setminus \mathrm{dom}\,\gamma| = |(X\beta \setminus \mathrm{dom}\,\gamma)\beta^{-1}| \\ &= |\mathrm{dom}\,\beta \setminus \mathrm{dom}\,\beta\gamma| \leqslant |X \setminus \mathrm{dom}\,\beta\gamma| = g(\beta\gamma), \end{aligned}$$

which implies that $q \leqslant \max\{g(\beta\gamma), |X\beta\gamma \setminus X\alpha\gamma|\}$. Otherwise, if $|(X\beta \setminus X\alpha) \cap \mathrm{dom}\,\gamma| = q$, then we have $|X\beta\gamma \setminus X\alpha\gamma| = |((X\beta \setminus X\alpha) \cap \mathrm{dom}\,\gamma)\gamma| = q$ and so again we obtain $q \leqslant \max\{g(\beta\gamma), |X\beta\gamma \setminus X\alpha\gamma|\}$.

Hence, by Theorem 7 we deduce that $\alpha\gamma \leqslant \beta\gamma$. Therefore, γ is right compatible as required. □

Theorem 13 *Let $\alpha \in PS(X,Y,q)$. Then, α is left compatible with respect to $\leqslant$ if and only if $|\mathrm{dom}\,\alpha| = 1$ or ($q \leqslant g(\alpha)$ and $\mathrm{dom}\,\alpha \subseteq Y$).*

Proof: Suppose that α is left compatible under $\leqslant$ and $|\mathrm{dom}\,\alpha| \neq 1$. If $|\mathrm{dom}\,\alpha| = 0$, then $\alpha = \varnothing$, and this situation arises only when $|X| = q$. So $g(\alpha) = |X| = q$ and $\mathrm{dom}\,\alpha = \varnothing \subseteq Y$. On the other hand, suppose $|\mathrm{dom}\,\alpha| > 1$. For any $x \in \mathrm{dom}\,\alpha$, we suppose $x\alpha = y \in Y$ and notice that $\mathrm{dom}\,\alpha = \bigcup_{x \in \mathrm{dom}\,\alpha} \mathrm{dom}\,\alpha \setminus \{x\}$. We define $\gamma = \mathrm{id}_{X\alpha \setminus \{y\}}$ and $\mu = \mathrm{id}_{X\alpha}$, then $\gamma \subset \mu$, $\mathrm{dom}\,\gamma \subseteq Y$, $X\gamma, X\mu \subseteq Y$, $d(\gamma) = d(\alpha) + 1 = q$ and $g(\mu) = d(\mu) = d(\alpha) = q$, whence $\gamma, \mu \in PS(X,Y,q)$. Since $g(\mu) = q$, we have $q \leqslant \max\{g(\mu), |X\mu \setminus X\gamma|\}$. Then by Theorem 7, $\gamma \leqslant \mu$. By the assumption that α is left compatible, we have $\alpha\gamma \leqslant \alpha\mu$. We also see that $\alpha\gamma \neq \alpha\mu = \alpha$, then by Theorem 7 again, we get the following three conditions:

$$\alpha\gamma \subseteq \alpha\mu,\ \mathrm{dom}\,\alpha\gamma \subseteq Y \quad \text{and} \quad q \leqslant \max\{g(\alpha\mu), |X\alpha\mu \setminus X\alpha\gamma|\}. \quad (6)$$

Since $|X\alpha\mu \backslash X\alpha\gamma| = |\{y\}| = 1$, we obtain by the last condition of (6) that $q \leqslant g(\alpha\mu) = g(\alpha)$. Moreover, the second condition of (6) implies $\operatorname{dom} \alpha\backslash\{x\} = \operatorname{dom} \alpha\gamma \subseteq Y$, whence $\operatorname{dom} \alpha = \bigcup_{x\in \operatorname{dom} \alpha} \operatorname{dom} \alpha\backslash\{x\} \subseteq Y$ as required.

Conversely, suppose that the conditions hold and let $\alpha, \gamma, \mu \in PS(X, Y, q)$ be such that $\gamma \leqslant \mu$, then $\gamma \subseteq \mu$. First, assume that $|\operatorname{dom} \alpha| = 1$, where $\operatorname{dom} \alpha = \{x\}$. If $x\alpha \notin \operatorname{dom} \gamma$, then $\alpha\gamma = \varnothing \leqslant \alpha\mu$. Otherwise, if $x\alpha \in \operatorname{dom} \gamma$, then, as $\gamma \subseteq \mu$, we have $(x\alpha)\gamma = (x\alpha)\mu$, where $\operatorname{dom} \alpha\gamma = \{x\} = \operatorname{dom} \alpha\mu$, whence $\alpha\gamma = \alpha\mu$. Therefore, α is left compatible. Finally, we assume that $q \leqslant g(\alpha)$ and $\operatorname{dom} \alpha \subseteq Y$. Then, $\operatorname{dom} \alpha\gamma \subseteq \operatorname{dom} \alpha \subseteq Y$ and $q \leqslant g(\alpha) \leqslant g(\alpha\mu)$, so $q \leqslant \max\{g(\alpha\mu), |X\alpha\mu\backslash X\alpha\gamma|\}$. As $\gamma \subseteq \mu$ and $PS(X, Y, q)$ is left compatible under $\subseteq$, we have that $\alpha\gamma \subseteq \alpha\mu$. By Theorem 7, we deduce that $\alpha\gamma \leqslant \alpha\mu$. In all cases, we have shown that α is left compatible with respect to $\leqslant$, and the proof is complete. □

Next, we consider the existence of the meet (or the greatest lower bound) and the join (or the least upper bound) under the natural partial order for any subset $\{\alpha, \beta\}$ of $PS(X, Y, q)$. We let $\alpha \wedge \beta$ and $\alpha \vee \beta$ denote the meet and the join of $\{\alpha, \beta\}$ respectively. We also note that, when α and β are comparable, the meet and the join always exists, that is, if $\alpha \leqslant \beta$, then $\alpha \wedge \beta = \alpha$ and $\alpha \vee \beta = \beta$. Therefore, in what follows we suppose that α and β are incomparable under $\leqslant$. For $\alpha, \beta \in PS(X, Y, q)$, we let $E(\alpha, \beta) = \{x \in \operatorname{dom} \alpha \cap \operatorname{dom} \beta : x\alpha = x\beta\}$ and, for convenience, we will denote $E(\alpha, \beta)$ by E. It is also clear that $\alpha|_E = \beta|_E$.

Lemma 4 *Let $\alpha, \beta \in PS(X, Y, q)$ which are incomparable with respect to $\leqslant$. If $\gamma \in PS(X, Y, q)$ is a lower bound of $\{\alpha, \beta\}$, then* $\operatorname{dom} \gamma \subseteq E \cap Y$ *and* $\gamma \subseteq \alpha|_{E\cap Y} = \beta|_{E\cap Y}$.

Proof: Suppose that $\gamma \in PS(X, Y, q)$ is a lower bound of $\{\alpha, \beta\}$ under $\leqslant$. If $\gamma = \varnothing$, then it is clear that $\operatorname{dom} \gamma = \varnothing \subseteq E \cap Y$ and $\gamma \subseteq \alpha|_{E\cap Y} = \beta|_{E\cap Y}$. If $\gamma \neq \varnothing$, then we let $x \in \operatorname{dom} \gamma$ and recall that $\gamma \leqslant \alpha$ and $\gamma \leqslant \beta$ imply $\gamma \subseteq \alpha$ and $\gamma \subseteq \beta$. So $\varnothing \neq \operatorname{dom} \gamma \subseteq \operatorname{dom} \alpha \cap \operatorname{dom} \beta$ and $x\alpha = x\gamma = x\beta$ for all $x \in \operatorname{dom} \gamma$, whence $\operatorname{dom} \gamma \subseteq E$. Moreover, since α and β are incomparable, we have that $\alpha \neq \gamma$. Then, by Theorem 7, as $\gamma \leqslant \alpha$, we have $\operatorname{dom} \gamma \subseteq Y$. Hence, $\operatorname{dom} \gamma \subseteq E \cap Y$ and $\gamma \subseteq \alpha|_{E\cap Y} = \beta|_{E\cap Y}$ as required. □

Theorem 14 *Suppose that $|X| = q$. Let $\alpha, \beta \in PS(X, Y, q)$ which are incomparable with respect to $\leqslant$. Then the following statements hold.*

(a) If $E \cap Y = \varnothing$, then $\alpha \wedge \beta = \varnothing$.

(b) If $E\cap Y \neq \varnothing$, then $\alpha\wedge\beta$ exists if and only if $(g(\alpha) = q$ or $g(\beta) = q)$ or $|X\alpha\backslash(E\cap Y)\alpha| = q = |X\beta\backslash(E\cap Y)\beta|$.

In this case, $\alpha \wedge \beta = \alpha|_{E\cap Y} = \beta|_{E\cap Y} \neq \varnothing$.

Proof: To prove (a), suppose that $E \cap Y = \varnothing$. Then by Lemma 4, if $\gamma \leqslant \alpha$ and $\gamma \leqslant \beta$, then $\operatorname{dom} \gamma \subseteq E \cap Y = \varnothing$, that is $\gamma = \varnothing$. Thus, the only lower bound of $\{\alpha, \beta\}$ is $\varnothing$, whence $\alpha \wedge \beta = \varnothing$.

To prove (b), we suppose that $E \cap Y \neq \varnothing$. Let $\gamma \in PS(X, Y, q)$ be such that $\alpha \wedge \beta = \gamma$ and suppose $g(\alpha) < q$ and $g(\beta) < q$. Then $|X\alpha| = q = |X\beta|$. For any $x \in E\cap Y$, define $\lambda_x = \begin{pmatrix} x \\ x\alpha \end{pmatrix} = \begin{pmatrix} x \\ x\beta \end{pmatrix}$, then $X\lambda_x \subseteq X\alpha \subseteq Y$, $d(\lambda_x) = |X\backslash\{x\alpha\}| = q$, so $\lambda_x \in PS(X, Y, q)$. Moreover, $\lambda_x \subseteq \alpha$, $\operatorname{dom} \lambda_x = \{x\} \subseteq Y$ and $|X\alpha\backslash X\lambda_x| = |X\alpha\backslash\{x\alpha\}| = q$, so $\max\{g(\alpha), |X\alpha\backslash X\lambda_x|\} = q$. Then, by Theorem 7, $\lambda_x \leqslant \alpha$. Similarly, we can verify that $\lambda_x \leqslant \beta$, whence λ_x is a lower bound under $\leqslant$ of $\{\alpha, \beta\}$. By supposition that $\alpha\wedge\beta = \gamma$, we have $\lambda_x \leqslant \gamma$, which implies $\lambda_x \subseteq \gamma$ and so $\{x\} = \operatorname{dom} \lambda_x \subseteq \operatorname{dom} \gamma$. Therefore, $E \cap Y \subseteq \operatorname{dom} \gamma$. Consequently, Lemma 4 implies that $\operatorname{dom} \gamma = E \cap Y$ and $\gamma = \alpha|_{E\cap Y} = \beta|_{E\cap Y}$. Since α and β are incomparable, we have that $\alpha \neq \gamma$, and as $\gamma \leqslant \alpha$, we obtain that $\max\{g(\alpha), |X\alpha\backslash X\gamma|\} = q$ by Theorem 7. Consequently, by the assumption that $g(\alpha) < q$, we have $|X\alpha\backslash(E \cap Y)\alpha| = |X\alpha\backslash X\gamma| = q$. Similarly, we may show that $|X\beta\backslash(E\cap Y)\beta| = q$.

Conversely, suppose that the conditions hold. We claim that $\alpha\wedge\beta = \alpha|_{E\cap Y} = \beta|_{E\cap Y}$. For convenience, we let $\gamma = \alpha|_{E\cap Y} = \beta|_{E\cap Y}$. Clearly, $X\gamma = (E\cap Y)\alpha \subseteq Y$, and since $|X\backslash X\gamma| = |X\backslash(E \cap Y)\alpha| \geqslant |X\backslash X\alpha| = q$, we have $|X\backslash X\gamma| = q$, whence $\gamma \in PS(X, Y, q)$. We also see that $\gamma \subseteq \alpha$, $\gamma \subseteq \beta$ and $\operatorname{dom} \gamma \subseteq Y$. Next, our goal is to show that γ is a lower bound under $\leqslant$ of $\{\alpha, \beta\}$, and finally, we will show that for any $\mu \in PS(X, Y, q)$ such that μ is a lower bound under $\leqslant$ of $\{\alpha, \beta\}$, $\mu \leqslant \gamma$. By the assumptions, we have two possible cases.

Case 1: $g(\alpha) = q$ or $g(\beta) = q$. If both $g(\alpha)$ and $g(\beta)$ have the same cardinality q, then it is clear that $\max\{g(\alpha), |X\alpha\backslash X\gamma|\} = q = \max\{g(\beta), |X\beta\backslash X\gamma|\}$. Otherwise, without loss of generality, we suppose $g(\alpha) = q$ and $g(\beta) < q$, then we have $\max\{g(\alpha), |X\alpha\backslash X\gamma|\} = q$. Next, we consider

$$X\backslash\operatorname{dom} \alpha = (\operatorname{dom} \beta\backslash\operatorname{dom} \alpha) \mathbin{\dot\cup} ((X\backslash\operatorname{dom} \alpha) \cap (X\backslash\operatorname{dom} \beta)). \quad (7)$$

As $g(\alpha) = q$, we obtain that at least one term on the right of (7) has carnality q. But we notice that $|(X\backslash\operatorname{dom} \alpha) \cap (X\backslash\operatorname{dom} \beta)| \leqslant |X\backslash\operatorname{dom} \beta| < q$, so $q = |\operatorname{dom} \beta\backslash\operatorname{dom} \alpha| \leqslant |\operatorname{dom} \beta\backslash(E \cap Y)|$, whence $|\operatorname{dom} \beta\backslash(E \cap Y)| = q$. Therefore, $|X\beta\backslash X\gamma| = |X\beta\backslash(E\cap Y)\beta| = |\operatorname{dom} \beta\backslash(E\cap Y)| = q$, which implies $\max\{g(\beta), |X\beta\backslash X\gamma|\} = q$. Then, by Theorem 7, γ is a lower bound of $\{\alpha, \beta\}$ under $\leqslant$. Let $\mu \in PS(X, Y, q)$ with $\mu \leqslant \alpha$ and $\mu \leqslant \beta$. Then, Lemma 4 implies that $\operatorname{dom} \mu \subseteq E \cap Y \subseteq Y$ and $\mu \subseteq \alpha|_{E\cap Y} = \gamma$. Since $\operatorname{dom} \gamma \subseteq \operatorname{dom} \alpha$ and $\operatorname{dom} \gamma \subseteq \operatorname{dom} \beta$, we have $g(\alpha) \leqslant g(\gamma)$ and $g(\beta) \leqslant g(\gamma)$. Consequently, by the assumption $g(\alpha) = q$ or $g(\beta) = q$, we can deduce that $g(\gamma) = q$. Hence, $\max\{g(\gamma), |X\gamma\backslash X\mu|\} = q$. It follows that $\mu \leqslant \gamma$ by Theorem 7 and so $\gamma = \alpha\wedge\beta$.

Case 2: $|X\alpha\backslash(E\cap Y)\alpha| = q = |X\beta\backslash(E\cap Y)\beta|$. In this case, as $X\gamma = (E\cap Y)\alpha = (E\cap Y)\beta$, we have $|X\alpha\backslash X\gamma| = q = |X\beta\backslash X\gamma|$ which implies $\max\{g(\alpha), |X\alpha\backslash X\gamma|\} = q = \max\{g(\beta), |X\beta\backslash X\gamma|\}$. Then, by Theorem 7, γ is a lower bound of $\{\alpha,\beta\}$ under $\leqslant$. Finally, let $\mu\in PS(X,Y,q)$ with $\mu\leqslant\alpha$ and $\mu\leqslant\beta$. Again, by Lemma 4, we have that $\operatorname{dom}\mu\subseteq E\cap Y\subseteq Y$ and $\mu\subseteq\alpha|_{E\cap Y}=\gamma$. Moreover, by the assumption $|X\alpha\backslash(E\cap Y)\alpha|=q$, we obtain that $q=|X\alpha\backslash(E\cap Y)\alpha|=|\operatorname{dom}\alpha\backslash(E\cap Y)|\leqslant|X\backslash(E\cap Y)|=g(\gamma)$, whence $\max\{g(\gamma),|X\gamma\backslash X\mu|\}=q$. Again, by Theorem 7, $\mu\leqslant\gamma$ and so $\alpha\wedge\beta=\gamma=\alpha|_{E\cap Y}=\beta|_{E\cap Y}$ as required. □

Theorem 15 *Suppose that $|X|=p>q$. Let $\alpha,\beta\in PS(X,Y,q)$ which are incomparable with respect to $\leqslant$. Then, $\alpha\wedge\beta$ exists if and only if the following conditions hold.*
(a) $E\cap Y\neq\varnothing$.
(b) $\max\{|X\alpha\backslash(E\cap Y)\alpha|,|X\beta\backslash(E\cap Y)\beta|\}\leqslant q$.
(c) $q\leqslant\max\{g(\alpha),|X\alpha\backslash(E\cap Y)\alpha|\}$ *and* $q\leqslant\max\{g(\beta),|X\beta\backslash(E\cap Y)\beta|\}$.
In this case, $\alpha\wedge\beta=\alpha|_{E\cap Y}=\beta|_{E\cap Y}$.

Proof: Suppose that $\alpha\wedge\beta=\gamma$, where $\gamma\in PS(X,Y,q)$. Since $|X|>q$, we have $\gamma\neq\varnothing$. Then, by Lemma 4, $\varnothing\neq\operatorname{dom}\gamma\subseteq E\cap Y$, so (a) holds. Moreover, $\gamma\subseteq\alpha|_{E\cap Y}=\beta|_{E\cap Y}$, which implies $X\gamma\subseteq(E\cap Y)\alpha\subseteq X\alpha$. It follows that $q=|X\backslash X\alpha|\leqslant|X\backslash(E\cap Y)\alpha|\leqslant|X\backslash X\gamma|=q$, whence $|X\backslash(E\cap Y)\alpha|=q$ and so $|X\alpha\backslash(E\cap Y)\alpha|\leqslant|X\backslash(E\cap Y)\alpha|=q$. Similarly, as $X\gamma\subseteq(E\cap Y)\beta\subseteq X\beta$, we can verify that $|X\beta\backslash(E\cap Y)\beta|\leqslant q$. Therefore, $\max\{|X\alpha\backslash(E\cap Y)\alpha|,|X\beta\backslash(E\cap Y)\beta|\}\leqslant q$, that is (b) holds. Next, we will prove (c). As $|X|=p>q$ and $|X\backslash(E\cap Y)\alpha|=q$, we have $p=|(E\cap Y)\alpha|=|E\cap Y|$. Then, we write $E\cap Y=A\dot\cup B\dot\cup C$, where $|A|=p$ and $|B|=|C|=q$. Let $\lambda=\alpha|_{A\cup B}$ and $\mu=\alpha|_{A\cup C}$. Then, $X\lambda\subseteq X\alpha\subseteq Y$, $X\mu\subseteq X\alpha\subseteq Y$, $d(\lambda)=|X\backslash(E\cap Y)\alpha|+|C\alpha|=q$ and $d(\mu)=|X\backslash(E\cap Y)\alpha|+|B\alpha|=q$, whence $\lambda,\mu\in PS(X,Y,q)$. It is also clear that $\operatorname{dom}\lambda\subseteq Y,\operatorname{dom}\mu\subseteq Y$, $\lambda\subseteq\alpha$ and $\mu\subseteq\alpha$. Moreover, $|X\alpha\backslash X\lambda|=|X\alpha\backslash(E\cap Y)\alpha|+|C\alpha|=q$ and $|X\alpha\backslash X\mu|=|X\alpha\backslash(E\cap Y)\alpha|+|B\alpha|=q$, so $q\leqslant\max\{g(\alpha),|X\alpha\backslash X\lambda|\}$ and $q\leqslant\max\{g(\alpha),|X\alpha\backslash X\mu|\}$. Then, by Theorem 7, we have $\lambda\leqslant\alpha$ and $\mu\leqslant\alpha$. As $(E\cap Y)\alpha=(E\cap Y)\beta$, we can show that $\lambda\leqslant\beta$ and $\mu\leqslant\beta$ in a similar way, so λ and μ are lower bounds of $\{\alpha,\beta\}$. By the supposition that γ is the greatest lower bound under $\leqslant$ of $\{\alpha,\beta\}$, we conclude that $\lambda\leqslant\gamma$ and $\mu\leqslant\gamma$, which imply $\lambda\subseteq\gamma$ and $\mu\subseteq\gamma$. Then, $E\cap Y=\operatorname{dom}\lambda\cup\operatorname{dom}\mu\subseteq\operatorname{dom}\gamma$. Hence, $\operatorname{dom}\gamma=E\cap Y$ and so $\gamma=\alpha|_{E\cap Y}=\beta|_{E\cap Y}$. We recall that α and β are incomparable, so $\alpha\neq\gamma\neq\beta$. Then, $\gamma<\alpha$ and $\gamma<\beta$. Thus, Theorem 7 implies that, $q\leqslant\max\{g(\alpha),|X\alpha\backslash X\gamma|\}$ and $q\leqslant\max\{g(\beta),|X\beta\backslash X\gamma|\}$. Consequently, as $X\gamma=(E\cap Y)\alpha=(E\cap Y)\beta$, we obtain that $q\leqslant\max\{g(\alpha),|X\alpha\backslash(E\cap Y)\alpha|\}$ and $q\leqslant\max\{g(\beta),|X\beta\backslash(E\cap Y)\beta|\}$ as required.

For the converse, suppose that the conditions (a), (b), and (c) hold. Take $\gamma=\alpha|_{E\cap Y}=\beta|_{E\cap Y}$ and let us prove that $\alpha\wedge\beta=\gamma$. It is clear that $\gamma\subseteq\alpha,\gamma\subseteq\beta$, $\operatorname{dom}\gamma\subseteq Y$ and $X\gamma=(E\cap Y)\alpha=(E\cap Y)\beta\subseteq Y$. We also see that

$$d(\gamma)=|X\backslash(E\cap Y)\alpha|=|X\backslash X\alpha|+|X\alpha\backslash(E\cap Y)\alpha|,$$

where $|X\backslash X\alpha|=q$ and from (b), $|X\alpha\backslash(E\cap Y)\alpha|\leqslant q$, so $d(\gamma)=q$, that is $\gamma\in PS(X,Y,q)$. Next, we take $(E\cap Y)\alpha=(E\cap Y)\beta=X\gamma$ in (c), we get that $q\leqslant\max\{g(\alpha),|X\alpha\backslash X\gamma|\}$ and $q\leqslant\max\{g(\beta),|X\beta\backslash X\gamma|\}$. So, by Theorem 7 we have $\gamma\leqslant\alpha$ and $\gamma\leqslant\beta$. Finally, let $\mu\in PS(X,Y,q)$ be a lower bound of $\{\alpha,\beta\}$ under $\leqslant$. We aim to show that $\mu\leqslant\gamma$. As α and β are incomparable, so $\alpha\neq\mu$. Then, $\mu<\alpha$ and by Theorem 7, we have $\operatorname{dom}\mu\subseteq Y$. Moreover, Lemma 4 implies that $\mu\subseteq\alpha|_{E\cap Y}=\gamma$. Now, if $g(\gamma)<q$, then Theorem 10 implies that γ is maximal. Then, as γ is a lower bound of $\{\alpha,\beta\}$, we have $\alpha=\gamma=\beta$, which contradicts to our assumption that α and β are incomparable. Thus, $q\leqslant g(\gamma)$ and hence $q\leqslant\max\{g(\gamma),|X\gamma\backslash X\mu|\}$. Again, by Theorem 7 we have $\mu\leqslant\gamma$. Therefore, $\alpha\wedge\beta=\gamma=\alpha|_{E\cap Y}=\beta|_{E\cap Y}$. This completes the proof. □

In what follows, for $\alpha,\beta\in I(X)$, we denote by $\alpha\cup\beta$ the mapping from $\operatorname{dom}\alpha\cup\operatorname{dom}\beta$ to $X\alpha\cup X\beta$ defined by

$$x(\alpha\cup\beta)=\begin{cases}x\alpha & \text{if } x\in\operatorname{dom}\alpha\backslash\operatorname{dom}\beta,\\ x\beta & \text{if } x\in\operatorname{dom}\beta\backslash\operatorname{dom}\alpha,\\ x\alpha=x\beta & \text{if } x\in\operatorname{dom}\alpha\cap\operatorname{dom}\beta.\end{cases}$$

Clearly, $\alpha\cup\beta$ is well defined if and only if $\operatorname{dom}\alpha\cap\operatorname{dom}\beta=\varnothing$ or $x\alpha=x\beta$ for all $x\in\operatorname{dom}\alpha\cap\operatorname{dom}\beta$, i.e., $\operatorname{dom}\alpha\cap\operatorname{dom}\beta=E$. In this case, $\alpha\cup\beta$ is injective only when the sets $(\operatorname{dom}\alpha\backslash\operatorname{dom}\beta)\alpha$ and $(\operatorname{dom}\beta\backslash\operatorname{dom}\alpha)\beta$ are disjoint.

Next, we give the existence of the least upper bound for $\alpha,\beta\in PS(X,Y,q)$.

Theorem 16 *Let $\alpha,\beta\in PS(X,Y,q)$ which are incomparable with respect to $\leqslant$. Then, $\alpha\vee\beta$ exists if and only if the following conditions hold.*
(a) $\operatorname{dom}\alpha\cap\operatorname{dom}\beta=E$.
(b) $(\operatorname{dom}\alpha\backslash\operatorname{dom}\beta)\alpha\cap(\operatorname{dom}\beta\backslash\operatorname{dom}\alpha)\beta=\varnothing$.
(c) $|X\backslash(X\alpha\cup X\beta)|=q$.
(d) $q\leqslant\min\{g(\alpha),g(\beta)\}$, $\operatorname{dom}\alpha\subseteq Y$ *and* $\operatorname{dom}\beta\subseteq Y$.
(e) $q\leqslant|X\backslash(\operatorname{dom}\alpha\cup\operatorname{dom}\beta)|$ *or* $\operatorname{dom}\alpha\cup\operatorname{dom}\beta=X$ *or* $X\alpha\cup X\beta=Y$.
In this case, $\alpha\vee\beta=\alpha\cup\beta$.

Proof: Suppose that $\alpha\vee\beta=\gamma$, where $\gamma\in PS(X,Y,q)$. As $\alpha\leqslant\gamma$, $\beta\leqslant\gamma$ and α and β are incomparable, we have that α is not maximal under $\leqslant$ (otherwise $\alpha\leqslant\gamma$ implies $\alpha=\gamma$ and so $\beta\leqslant\alpha$, a contradiction). Similarly, β is not maximal. Then, Theorem 10, implies that $q\leqslant g(\alpha)$, $q\leqslant g(\beta)$, $\operatorname{dom}\alpha\subseteq Y$ and $\operatorname{dom}\beta\subseteq Y$, that is (d) holds. Moreover, by Theorem 7 we also have that $\alpha\subseteq\gamma,\beta\subseteq\gamma$, $q\leqslant\max\{g(\gamma),|X\gamma\backslash X\alpha|\}$ and $q\leqslant\max\{g(\gamma),|X\gamma\backslash X\beta|\}$. To show (a), it is clear that $E\subseteq\operatorname{dom}\alpha\cap\operatorname{dom}\beta$. For the equality, let $x\in\operatorname{dom}\alpha\cap\operatorname{dom}\beta$. As $\alpha\subseteq\gamma,\beta\subseteq\gamma$, we have $x\alpha=x\gamma=x\beta$, which implies $x\in E$, whence $\operatorname{dom}\alpha\cap\operatorname{dom}\beta=E$, that is (a) holds. We also see that $X\alpha\cup X\beta\subseteq X\gamma$, which implies

$$q=|X\backslash X\gamma|\leqslant|X\backslash(X\alpha\cup X\beta)|\leqslant|X\backslash X\alpha|=q,$$

so $|X\backslash(X\alpha\cup X\beta)|=q$, and this proves (c). To show (b), suppose for a contradiction that there exists $y\in(\mathrm{dom}\,\alpha\backslash\mathrm{dom}\,\beta)\alpha\cap(\mathrm{dom}\,\beta\backslash\mathrm{dom}\,\alpha)\beta$. Then $x\alpha=y=z\beta$ for some $x\in\mathrm{dom}\,\alpha\backslash\mathrm{dom}\,\beta$, $z\in\mathrm{dom}\,\beta\backslash\mathrm{dom}\,\alpha$. As $\alpha\subseteq\gamma,\beta\subseteq\gamma$, we have that $x\alpha=x\gamma$ and $z\beta=z\gamma$, so $x\gamma=y=z\gamma$. It follows that $x=z$ since γ is injective, a contradiction. Thus $(\mathrm{dom}\,\alpha\backslash\mathrm{dom}\,\beta)\alpha\cap(\mathrm{dom}\,\beta\backslash\mathrm{dom}\,\alpha)\beta=\varnothing$, and this proves (b). Next, we define $\theta=\alpha\cup\beta$. Then, $X\theta=X\alpha\cup X\beta\subseteq Y$. Moreover, the conditions (a) and (b) imply that θ is an injective mapping, where the condition (c) implies $d(\theta)=q$, whence $\theta\in PS(X,Y,q)$. Next, to prove (e) by contradiction we first assume that $g(\theta)=|X\backslash(\mathrm{dom}\,\alpha\cup\mathrm{dom}\,\beta)|<q$, $\mathrm{dom}\,\alpha\cup\mathrm{dom}\,\beta\subsetneq X$ and $X\alpha\cup X\beta\subsetneq Y$. Let $a\in X\backslash(\mathrm{dom}\,\alpha\cup\mathrm{dom}\,\beta)$ and $b\in Y\backslash(X\alpha\cup X\beta)$ and define $\mu_{a,b}=\theta\cup\begin{pmatrix}a\\b\end{pmatrix}$. Then, $X\mu_{a,b}=X\theta\cup\{b\}\subseteq Y$ and $d(\mu_{a,b})=d(\theta)-1=q$, whence $\mu_{a,b}\in PS(X,Y,q)$. It is also clear that $\alpha\subseteq\theta\subseteq\mu_{a,b}$ and $\beta\subseteq\theta\subseteq\mu_{a,b}$. As $\alpha\subseteq\theta$, we have $X\backslash\mathrm{dom}\,\alpha=(X\backslash\mathrm{dom}\,\theta)\,\dot\cup\,(\mathrm{dom}\,\theta\backslash\mathrm{dom}\,\alpha)$. Therefore,

$$\begin{aligned}q\leqslant g(\alpha)&=|X\backslash\mathrm{dom}\,\alpha|\\&=|X\backslash\mathrm{dom}\,\theta|+|\mathrm{dom}\,\theta\backslash\mathrm{dom}\,\alpha|.\end{aligned}\tag{8}$$

As $|X\backslash\mathrm{dom}\,\theta|=|X\backslash(\mathrm{dom}\,\alpha\cup\mathrm{dom}\,\beta)|<q$, we obtain from (8) that $q\leqslant|\mathrm{dom}\,\theta\backslash\mathrm{dom}\,\alpha|=|X\theta\backslash X\alpha|=|X\beta\backslash X\alpha|\leqslant|X\backslash X\alpha|=q$, whence $|X\beta\backslash X\alpha|=q$. Thus, $|X\mu_{a,b}\backslash X\alpha|=|X\beta\backslash X\alpha|+|\{b\}|=q+1=q$. It follows that $q\leqslant\max\{g(\mu_{a,b}),|X\mu_{a,b}\backslash X\alpha|\}$, and then Theorem 7 implies $\alpha\leqslant\mu_{a,b}$. Similarly, we can verify that $\beta\leqslant\mu_{a,b}$. Thus, as $\alpha\vee\beta=\gamma$, we get that $\gamma\leqslant\mu_{a,b}$. Since $\alpha\subseteq\gamma$ and $\beta\subseteq\gamma$, so $\mathrm{dom}\,\theta=\mathrm{dom}\,\alpha\cup\mathrm{dom}\,\beta\subseteq\mathrm{dom}\,\gamma$, whence $g(\gamma)\leqslant g(\theta)<q$, which implies that γ is maximal by Theorem 10. Therefore, $\gamma=\mu_{a,b}$, which implies $\mathrm{dom}\,\gamma=\mathrm{dom}\,\mu_{a,b}$ and $X\gamma=X\mu_{a,b}$ for all $a\in X\backslash(\mathrm{dom}\,\alpha\cup\mathrm{dom}\,\beta)$ and $b\in Y\backslash(X\alpha\cup X\beta)$. Then, $X\backslash(\mathrm{dom}\,\alpha\cup\mathrm{dom}\,\beta)\subseteq\mathrm{dom}\,\gamma$ and $Y\backslash(X\alpha\cup X\beta)\subseteq X\gamma$. As as $\alpha\subseteq\gamma$ and $\beta\subseteq\gamma$, we deduce that $\mathrm{dom}\,\gamma=X$ and $X\gamma=Y$. Since $\mathrm{dom}\,\alpha\subseteq Y$ and $\mathrm{dom}\,\beta\subseteq Y$, so $q=|X\backslash X\gamma|=|X\backslash Y|\leqslant|X\backslash(\mathrm{dom}\,\alpha\cup\mathrm{dom}\,\beta)|=g(\theta)$, which contradicts to our assumption $|X\backslash(\mathrm{dom}\,\alpha\cup\mathrm{dom}\,\beta)|<q$. Hence, (e) holds.

For the converse, suppose that all of the conditions hold. Let $\gamma=\alpha\cup\beta$, we aim to show that $\alpha\vee\beta=\gamma$. We see that the conditions (a) and (b) imply that γ is is well defined injective mapping from $\mathrm{dom}\,\alpha\cup\mathrm{dom}\,\beta$ to Y. In addition, the condition (c) implies that $d(\gamma)=q$, that is $\gamma\in PS(X,Y,q)$. Firstly, we show that γ is an upper bound of $\{\alpha,\beta\}$ under $\leqslant$. It is clear that $\alpha\subseteq\gamma$ and by the condition (d), we have that $\mathrm{dom}\,\alpha\subseteq Y$ and $q\leqslant g(\alpha)$, so

$$\begin{aligned}q\leqslant g(\alpha)=|X\backslash\mathrm{dom}\,\alpha|&=|X\backslash\mathrm{dom}\,\gamma|+|\mathrm{dom}\,\gamma\backslash\mathrm{dom}\,\alpha|\\&=|X\backslash\mathrm{dom}\,\gamma|+|X\gamma\backslash X\alpha|.\end{aligned}\tag{9}$$

Thus, from (9), we get that $q\leqslant|X\backslash\mathrm{dom}\,\gamma|$ or $q\leqslant|X\gamma\backslash X\alpha|$, whence $q\leqslant\max\{g(\gamma),|X\gamma\backslash X\alpha|\}$. Then, by Theorem 7, $\alpha\leqslant\gamma$. In a similar way, we can verify that $\beta\leqslant\gamma$. So γ is an upper bound of $\{\alpha,\beta\}$. Finally, let $\mu\in PS(X,Y,q)$ be an upper bound under $\leqslant$ of $\{\alpha,\beta\}$, we aim to show that $\gamma\leqslant\mu$. As $\alpha\leqslant\mu$ and $\beta\leqslant\mu$, we have $\alpha\subseteq\mu$ and $\beta\subseteq\mu$. Then, $\gamma=\alpha\cup\beta\subseteq\mu$. By the condition (e), we consider three cases.

Case 1: $q\leqslant|X\backslash(\mathrm{dom}\,\alpha\cup\mathrm{dom}\,\beta)|$. In this case, we have that $q\leqslant g(\gamma)$. Since $\gamma\subseteq\mu$, we have that

$$q\leqslant g(\gamma)=|X\backslash\mathrm{dom}\,\gamma|=|X\backslash\mathrm{dom}\,\mu|+|X\mu\backslash X\gamma|.\tag{10}$$

Then, from (10), $q\leqslant|X\backslash\mathrm{dom}\,\mu|$ or $q\leqslant|X\mu\backslash X\gamma|$, that is, $q\leqslant\max\{g(\mu),|X\mu\backslash X\gamma|\}$. From (d), we have $\mathrm{dom}\,\gamma=\mathrm{dom}\,\alpha\cup\mathrm{dom}\,\beta\subseteq Y$, whence $\gamma\leqslant\mu$ by Theorem 7.

Case 2: $\mathrm{dom}\,\alpha\cup\mathrm{dom}\,\beta=X$. As $\gamma\subseteq\mu$, we have $X=\mathrm{dom}\,\alpha\cup\mathrm{dom}\,\beta=\mathrm{dom}\,\gamma\subseteq\mathrm{dom}\,\mu\subseteq X$. So $\mathrm{dom}\,\gamma=\mathrm{dom}\,\mu$, and hence $\gamma=\mu$.

Case 3: $X\alpha\cup X\beta=Y$. This case implies $Y=X\alpha\cup X\beta=X\gamma\subseteq X\mu\subseteq Y$. So $X\gamma=X\mu$. As $\gamma\subseteq\mu$, we obtain that $\gamma=\mu$.

In all cases, we deduce that $\alpha\vee\beta=\gamma=\alpha\cup\beta$ as required. □

Let $(X,\leqslant)$ be a partially ordered set. For any distinct $a,b\in X$, we call a a lower cover of b if $a<b$ and there is no $c\in S$ such that $a<c<b$. When this occurs, b is called an upper cover of a. The following result describes the existence of upper covers and lower covers of elements in $PS(X,Y,q)$.

Theorem 17 *Let $\alpha,\beta\in PS(X,Y,q)$ be such that $\alpha<\beta$. Then, β is an upper cover of α if and only if $|\mathrm{dom}\,\beta\backslash\mathrm{dom}\,\alpha|=1$ or $(\mathrm{dom}\,\beta\backslash\mathrm{dom}\,\alpha)\cap Y=\varnothing$. In other words, in the event that this occurs, α is a lower cover of β.*

Proof: Suppose that $\alpha<\beta$, where β is an upper cover of α. We suppose that $(\mathrm{dom}\,\beta\backslash\mathrm{dom}\,\alpha)\cap Y\neq\varnothing$, we aim to show that $|\mathrm{dom}\,\beta\backslash\mathrm{dom}\,\alpha|=1$. Let $a\in(\mathrm{dom}\,\beta\backslash\mathrm{dom}\,\alpha)\cap Y$ and define $\gamma=\alpha\cup\begin{pmatrix}a\\a\beta\end{pmatrix}$. Then, $X\gamma=X\alpha\cup\{a\beta\}\subseteq Y$ and $d(\gamma)=d(\alpha)-1=q$, whence $\gamma\in PS(X,Y,q)$. As $\alpha<\beta$, by Theorem 7 we have $\mathrm{dom}\,\alpha\subseteq Y$, $q\leqslant\max\{g(\beta),|X\beta\backslash X\alpha|\}$ and $\alpha\subset\beta$. It follows that $\mathrm{dom}\,\alpha\subset\mathrm{dom}\,\beta$ and so

$$\begin{aligned}|X\backslash\mathrm{dom}\,\alpha|&=|X\backslash\mathrm{dom}\,\beta|+|\mathrm{dom}\,\beta\backslash\mathrm{dom}\,\alpha|\\&=|X\backslash\mathrm{dom}\,\beta|+|X\beta\backslash X\alpha|.\end{aligned}$$

Clearly, $\alpha\subset\gamma\subseteq\beta$, so $\mathrm{dom}\,\alpha\subset\mathrm{dom}\,\gamma$ and then $|X\backslash\mathrm{dom}\,\alpha|=|X\backslash\mathrm{dom}\,\gamma|+|X\gamma\backslash X\alpha|$. Therefore, by the last two equations, we obtain that

$$|X\backslash\mathrm{dom}\,\beta|+|X\beta\backslash X\alpha|=|X\backslash\mathrm{dom}\,\gamma|+|X\gamma\backslash X\alpha|.\tag{11}$$

Observe that the sum on the left of (11) is equal to $\max\{g(\beta),|X\beta\backslash X\alpha|\}$, which has the carnality greater than or equal to q. This implies that the sum on the right of (11) which is $\max\{g(\gamma),|X\gamma\backslash X\alpha|\}$ has the same cardinality, whence $q \leqslant \max\{g(\gamma), |X\gamma \backslash X\alpha|\}$. Thus, by Theorem 7, $\alpha < \gamma$. Next, we aim to show that $q \leqslant \max\{g(\beta), |X\beta \backslash X\gamma|\}$. If $q \leqslant g(\beta)$, then we obtain $q \leqslant \max\{g(\beta), |X\beta \backslash X\gamma|\}$ as required. Otherwise, if $g(\beta) < q$, then by Theorem 7, as $\alpha < \beta$, we have that $q \leqslant \max\{g(\beta), |X\beta \backslash X\alpha|\} = |X\beta \backslash X\alpha|$. Therefore, $q \leqslant |X\beta \backslash X\alpha| \leqslant |X \backslash X\alpha| = q$, whence $|X\beta \backslash X\alpha| = q$. Consequently,

$$\begin{aligned}|X\beta \backslash X\gamma| &= |X\beta \backslash (X\alpha \cup \{a\beta\})| \\ &= |(X\beta \backslash X\alpha) \backslash \{a\beta\}| = |X\beta \backslash X\alpha| = q.\end{aligned}$$

This again implies $q \leqslant \max\{g(\beta), |X\beta \backslash X\gamma|\}$. Finally, as $\operatorname{dom}\gamma = \operatorname{dom}\alpha \cup \{a\} \subseteq Y$, then by Theorem 7 we have that $\gamma \leqslant \beta$ and so $\alpha < \gamma \leqslant \beta$. By the assumption that β is an upper cover of α, we deduce that $\gamma = \beta$. Therefore, $\operatorname{dom}\beta = \operatorname{dom}\gamma = \operatorname{dom}\alpha \cup \{a\}$. Hence, $|\operatorname{dom}\beta \backslash \operatorname{dom}\alpha| = |\{a\}| = 1$ as required.

Conversely, suppose that the conditions hold and there exists $\gamma \in PS(X, Y, q)$ such that $\alpha < \gamma \leqslant \beta$. Then, $\alpha \subset \gamma \subseteq \beta$ and so $\operatorname{dom}\alpha \subset \operatorname{dom}\gamma \subseteq \operatorname{dom}\beta$. It follows that

$$\operatorname{dom}\beta \backslash \operatorname{dom}\alpha = (\operatorname{dom}\beta \backslash \operatorname{dom}\gamma) \,\dot{\cup}\, (\operatorname{dom}\gamma \backslash \operatorname{dom}\alpha). \quad (12)$$

If $|\operatorname{dom}\beta \backslash \operatorname{dom}\alpha| = 1$, then from (12), we obtain that $|\operatorname{dom}\beta \backslash \operatorname{dom}\gamma| = 0$ and $|\operatorname{dom}\gamma \backslash \operatorname{dom}\alpha| = 1$. This implies $\operatorname{dom}\gamma = \operatorname{dom}\beta$, and thus $\gamma = \beta$. Otherwise, in the case that $(\operatorname{dom}\beta \backslash \operatorname{dom}\alpha) \cap Y = \varnothing$, we have $\varnothing \neq \operatorname{dom}\gamma \backslash \operatorname{dom}\alpha \subseteq \operatorname{dom}\beta \backslash \operatorname{dom}\alpha$. So $(\operatorname{dom}\gamma \backslash \operatorname{dom}\alpha) \cap Y \subseteq (\operatorname{dom}\beta \backslash \operatorname{dom}\alpha) \cap Y = \varnothing$, whence $(\operatorname{dom}\gamma \backslash \operatorname{dom}\alpha) \cap Y = \varnothing$, and therefore, $\operatorname{dom}\gamma \nsubseteq Y$. Then, by Theorem 10, γ is maximal under $\leqslant$. Consequently, the assumption $\gamma \leqslant \beta$ implies $\gamma = \beta$. In both cases, we deduce that β is an upper cover of α, which completes the proof. □

The descriptions of maximum, minimum, maximal, minimal, compatible elements, a meet $\alpha \wedge \beta$ and a join $\alpha \vee \beta$ in $PS(X, Y, q)$ presented in this section generalize the corresponding results for $PS(X, q)$ in [9, Theorems 3.3, 4.1, 4.3, 4.6, and 4.7] and [10, Theorems 6 and 10]. In special, by taking $X = Y$ in Theorem 17, we obtain descriptions for a lower cover and an upper cover in $PS(X, q)$, which surprisingly were not characterized before. We observe that, if $\alpha < \beta$, then $\alpha \subset \beta$ and so $\operatorname{dom}\beta \backslash \operatorname{dom}\alpha \neq \varnothing$. Therefore, the condition $(\operatorname{dom}\beta \backslash \operatorname{dom}\alpha) \cap X = \varnothing$ cannot occur. Hence, the final result is an immediate consequence of Theorem 17.

Corollary 2 *Let $\alpha, \beta \in PS(X, q)$ be such that $\alpha < \beta$. Then, β is an upper cover of α if and only if $|\operatorname{dom}\beta \backslash \operatorname{dom}\alpha| = 1$. In other words, in the event that this occurs, α is a lower cover of β.*

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