RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2024.010

ScienceAsia 50 (3): 2024: ID 2024010: **1–6**

A note on the parity of meromorphic functions

Changwen Peng[a,*], **Huawei Huang**[b], **Jianren Long**[b]

[a] School of Science, Guiyang University, Guiyang 550005 China
[b] School of Mathematical Sciences, Guizhou Normal University, Guiyang 550001 China

*Corresponding author, e-mail: pengcw716@126.com

Received 17 Mar 2023, Accepted 2 Sep 2023
Available online 23 Jan 2024

ABSTRACT: Parity is an important and easy to recognise property for meromorphic functions. On the parity of meromorphic functions, Liu, Liu and Korhonen [*J Math Anal Appl* **512**(2022):126129] obtained some meaningful results. In this paper, we investigate the parity of a meromorphic function $y(z)$ under the hypothesis that $y(z)^{2n}-2y(z)^n$ is even. In addition, we discuss the relationship on the parity of a meromorphic function with its q-difference polynomials and differential expressions. For instance, we consider the parity of a meromorphic function $y(z)$ under the assumption that $y'(z)/y(z)^n$ and $y(qz)/y(z)^n$ are odd or even functions, where n is a positive integer.

KEYWORDS: parity, even functions, odd functions, q-difference polynomials

MSC2020: 30D35 39B32

INTRODUCTION AND MAIN RESULTS

A function $y(z)$ is called meromorphic if it is analytic in the complex plane $\mathbb{C}$ except at isolated poles. In what follows, we use standard notations in the Nevanlinna theory of meromorphic functions. We also use the basic symbols such as $n(r,y)$, $T(r,y)$, etc., see [1–3]. We recall that the order of $y(z)$ is defined by

$$\sigma(y)=\limsup_{r\to\infty}\frac{\log T(r,y)}{\log r}$$

and the low order of $y(z)$ is defined by

$$\mu(y)=\liminf_{r\to\infty}\frac{\log T(r,y)}{\log r}.$$

Periodicity and parity are two important and easy to recognise properties for meromorphic functions. Recently, a number of papers focus on the periodicity of meromorphic functions, see [4–8]. There are also papers focusing on the parity of meromorphic functions, see [9–11].

In this paper, we mainly consider the relationship on the parity of meromorphic functions with their q-difference polynomials and differential expressions. Let us start by recalling a basic fact on the parity between $y(z)$ and $y'(z)$. Obviously, if $y(z)$ is odd, then $y'(z)$ is even. On the contrary, if $y(z)$ is even, then $y'(z)$ is odd. However, the converse is not true. For instance, $y'(z)=z\sin z$ is even, but $y(z)=\sin z-z\cos z+1$ has no parity.

Beardon [9] and Horwitz [10] have studied the parity of entire functions, respectively. Liu et al [11] considered the inverse problems on the parity of meromorphic functions. In particular, Liu et al [11, Theorem 1.3] considered that a meromorphic function $y(z)$ such that $P\circ y$ is an even function for $P(z)=z^4-2z^2$, and they obtained the following result.

Theorem 1 ([11]) *Let y be a meromorphic function. If $P(z)=z^4-2z^2$ and $P\circ y$ is even, then either $y(z)$ is even or odd or*

$$y(z)=\frac{h(z)+1/h(z)}{\sqrt{2}},$$

where $h(z)$ satisfies $h(-z)h(z)=\mathrm{i}$.

It inspires us to propose a related question which will be studied in the paper.

Question 1 Let $y(z)$ be a non-constant meromorphic function. If $P(z)=z^{2n}-2z^n$ and $P\circ y$ is even, does it follow that $y(z)$ has the same or opposite parity.

We begin to consider Question 1, and we obtain Theorem 2 as show below.

Theorem 2 *Let n be a non-zero integer, and let y be a non-constant meromorphic function. Suppose that $P(z)=z^{2n}-2z^n$ and $P\circ y$ is even.*

(i) *If $|n|\geqslant 4$, then $y(z)^n$ is even.*

(ii) *If $|n|=2$, then either $y(z)$ is even or odd or*

① $y(z)=\frac{h(z)+1/h(z)}{\sqrt{2}}$ *when $n=2$,*

② $y(z)=\frac{\sqrt{2}h(z)}{h(z)^2+1}$ *when $n=-2$,*

where $h(z)$ satisfies $h(-z)h(z)=\mathrm{i}$.

(iii) *If $|n|=1$, then $y(z)$ cannot be an odd function.*

Remark 1 If $n=3$ and $y(z)^6-2y(z)^3$ is even, then we have

$$[y(z)^3-y(-z)^3][y(z)^3+y(-z)^3]=2[y(z)^3-y(-z)^3]. \quad (1)$$

By (1), we have that either $y(z)^3=y(-z)^3$ or

$$y(z)^3+y(-z)^3=2. \qquad (2)$$

Obviously, any non-constant meromorphic solution to the Eq. (2) is neither odd nor even. Using Baker's result in [12], it follows that (2) has a meromorphic solution, for example,

$$y(z)=\frac{\sqrt[3]{2}}{2}\frac{(1+\frac{\varphi'(h(z))}{\sqrt{3}})}{\varphi(h(z))},$$

where φ is the Weierstrass φ-function that satisfies $(\varphi')^2=4\varphi^3-1$ and $h(z)$ is any odd function. If $n=-3$, then we can get similar results as above.

In Theorem 2, if $y(z)^4-2y(z)^2$ is even, then $y(z)$ may has no parity. The following Example 1 shows that the case may happen.

Example 1 Let $h(z)=\frac{1+\mathrm{i}}{\sqrt{2}}\mathrm{e}^{\mathrm{i}z}$. Thus $h(z)h(-z)=\mathrm{i}$. We have

$$y(z)=\frac{h(z)+1/h(z)}{\sqrt{2}}=\cos z-\sin z.$$

Obviously, $y(z)=\cos z-\sin z$ is a meromorphic solution to $y(z)^2+y(-z)^2=2$, and $y(z)$ is neither odd nor even.

Remark 2 By Theorem 2, if $y(z)^2-2y(z)$ is even, then $y(z)$ cannot be an odd function. So, $y(z)$ may be even, or it may not have parity. For instance, $y(z)=\cos z$ is even, $y(z)^2-2y(z)=(\cos z)^2-2\cos z$ is even; $y(z)=\sin z+1$ has no parity, $y(z)^2-2y(z)=(\sin z)^2-1$ is even.

Remark 3 Liu et al [11, Remark 1.2] obtained that $y(z)^n+y(-z)^n=0$ has no any non-zero meromorphic solution when n is an even number. Hence, $y(-z)=\pm\mathrm{i}y(z)$ does not have non-zero meromorphic solution.

Yang [13, Theorem 1] showed that: there are no non-constant entire solutions $y(z)$ and $g(z)$ that satisfy $a(z)y(z)^n+c(z)g(z)^m=1$ provided that $\frac{1}{m}+\frac{1}{n}<1$, where $a(z)$, $c(z)$ are small functions with respect to $y(z)$. The above result shows that: if $y(z)$ is a non-constant entire function and $y(z)^{2n}-2y(z)^n$ is even, then $y(z)^n$ is even when $n\geqslant 3$.

Liu et al [11, Theorem 2.1] gave a result on the parity of $y(z)$ with the differential polynomial $y(z)^n y^{(k)}(z)$.

Theorem 3 ([11]) *Let y be a non-constant meromorphic function.*

(i) *If $y(z)^n y'(z)$ is even, then $y(z)$ is odd when $n\geqslant 3$ and $y(z)=\frac{1}{2}(h(z)+\frac{1}{h(z)})$ when $n=1$, where $h(z)$ satisfies $h(-z)h(z)=\mathrm{i}$.*

(ii) *If $y(z)^n y'(z)$ is odd, then $y(z)$ must be even, or odd if n is odd.*

Remark 4 The case of higher derivatives in Theorem 3 cannot valid. For instance, $y(z)=\cos z+\sin z$ has no parity, however, $y(z)y'''(z)=\sin^2 z-\cos^2 z$ is even. If $y(z)=\mathrm{e}^z+\mathrm{e}^{-z}$ is even, it follows that $y(z)y''(z)=(\mathrm{e}^z+\mathrm{e}^{-z})^2$ is even; and $y(z)=\mathrm{e}^z-\mathrm{e}^{-z}$ is odd, we have that $y(z)^3y'''(z)=(\mathrm{e}^z-\mathrm{e}^{-z})^2(\mathrm{e}^{2z}-\mathrm{e}^{-2z})$ is odd.

In addition, Liu et al [11, Theorem 2.3] also considered the parity of q-difference polynomial $y(z)^n y(qz)$. They obtained the following Theorem 4 by using q-difference analogues of Nevanlinna theory of meromorphic functions.

Theorem 4 ([11]) *Let y be a meromorphic function with the order $\sigma(y)<1$ and $|q|\neq 0,1$.*

(i) *If $y(z)^n y(qz)$ is even, then $y(z)^{n+1}$ is even.*

(ii) *If $y(z)^n y(qz)$ is odd and n is even, then $y(z)$ is odd. If n is odd, then $y(z)^n y(qz)$ cannot be an odd function.*

In Theorem 3 and Theorem 4, n is a positive integer. It is natural to ask: if n is a negative integer, what do we get? In the following, we will answer the above question, and obtain the following results.

Theorem 5 *Let y be a non-constant meromorphic function, and let n be an integer.*

(i) *Suppose that $y'(z)/y(z)^n$ is even.*

① *If $n\geqslant 5$, then $y(z)$ is odd. If $n=3$, then $y(z)$ is neither odd nor even, and $y(z)=\frac{2h(z)}{\sqrt{A}(h(z)^2+1)}$, where $h(z)$ satisfies $h(-z)h(z)=\mathrm{i}$, A is a non-zero constant.*

② *If $n=1$, then $y(z)$ is neither odd nor even.*

③ *If $n=2$, then $y(z)$ cannot be an even function.*

(ii) *Suppose that $y'(z)/y(z)^n$ is odd.*

① *If $n\geqslant 3$, then $y(z)$ must be even, or odd if n is odd.*

② *If $n=2$, then $y(z)$ cannot be odd.*

In Theorem 5, if $y'(z)/y(z)$ is odd, then $y(z)$ can be odd or even. The following Example 2 shows that the case may happen.

Example 2 $y(z)=\sin z$ is odd, $\frac{y'(z)}{y(z)}=\frac{\cos z}{\sin z}=\cot z$ is odd. $y(z)=\cos z$ is even, $\frac{y'(z)}{y(z)}=-\frac{\sin z}{\cos z}=-\tan z$ is odd.

Remark 5 The case of higher derivatives in Theorem 5 cannot valid. For instance, $y(z)=\sin z$ is odd, we have that $y'''(z)/y(z)^5=-\frac{\cos z}{\sin^5 z}$ is odd. $y(z)=\cos z$ is even, it follows that $y''(z)/y(z)^5=-\frac{1}{\cos^4 z}$ is even.

Theorem 6 *Let y be a non-constant meromorphic function with the order $\sigma(y)<1$, q be a non-zero complex constant, and let n be a positive integer.*

(i) *Suppose that $y(qz)/y(z)^n$ is even.*

① *If $n=1$ and $|q|\neq 1$, then $y(z)^2$ is even;*

② *If $n\geqslant 2$ and $|q|\leqslant n$, then $y(z)^{n+1}$ is even.*

(ii) *If $y(qz)/y(z)^n$ is odd and n is even satisfying $|q|\leqslant n$, then $y(z)$ is odd.*

(iii) *Suppose that n is odd.*

① *If $|q|\neq 1$, then $y(qz)/y(z)$ cannot be an odd function.*

② *If $n \geqslant 3$ and $|q| \leqslant n$, then $y(qz)/y(z)^n$ cannot be an odd function.*

Based on Theorem 4, we pose the question as follows.
Question 2 Let y be a transcendental meromorphic function with the order $\sigma(y) < 1$. If $y(z)^n y(qz)^m$ has a certain parity, what do we get?

In the special case $m = 1$, Liu et al [11] have proved Theorem 4. We proceed to give our last result to consider the case $m \geqslant 2$.

Theorem 7 *Let y be a meromorphic function with the order $\sigma(y) < 1$, and let m, n be positive integers satisfying $n \geqslant m$, $|q| \neq 0, 1$.*
(i) *If $y(z)^n y(qz)^m$ is even, then $y(z)^{n+m}$ is even.*
(ii) *If $y(z)^n y(qz)^m$ is odd and $n+m$ is old, then $y(z)$ is odd. If $n+m$ is even, then $y(z)^n y(qz)^m$ cannot be an odd function.*

SOME LEMMAS

We need the following lemmas to prove our results.

Lemma 1 ([14]) *Let n be an integer satisfying $n \geqslant 4$. Then there are no non-constant meromorphic solutions $y(z)$ and $g(z)$ that satisfy*

$$y(z)^n + g(z)^n = 1.$$

Lemma 2 ([15]) *Let $y(z)$ be any meromorphic function, and q be any non-zero complex constant. Then*

$$T(r, y(qz)) = T(|q|r, y) + O(1).$$

Lemma 3 ([16]) *Let $q \neq 0, 1$. The meromorphic solutions of*

$$y(z)^2 + y(qz)^2 = 1$$

satisfy $y(z) = \frac{h(z)+1/h(z)}{2}$, where $h(z)$ is a meromorphic function satisfying one of the following cases:
(i) *$h(qz) = \mathrm{i}h(z)$;*
(ii) *$h(qz)h(z) = \mathrm{i}$.*

PROOF OF Theorem 2

(i): Suppose that $P \circ y$ is even for $P(z) = z^{2n} - 2z^n$. Then

$$y(z)^{2n} - 2y(z)^n = y(-z)^{2n} - 2y(-z)^n. \tag{3}$$

Eq. (3) implies that

$$[y(z)^n - y(-z)^n][y(z)^n + y(-z)^n] = 2[y(z)^n - y(-z)^n]. \tag{4}$$

If n is a positive integer and $y(z)^n \neq y(-z)^n$, we have by (4) that

$$y(z)^n + y(-z)^n = 2. \tag{5}$$

By Lemma 1 and $n \geqslant 4$, we have that (5) does not possess non-constant meromorphic solutions.

If n is a negative integer and $y(z)^n \neq y(-z)^n$, Eq. (4) shows

$$\left(\frac{1}{y(z)}\right)^{-n} + \left(\frac{1}{y(-z)}\right)^{-n} = 2.$$

By Lemma 1 and $-n \geqslant 4$, we have that the above equation does not possess non-constant meromorphic solutions.

Hence, if $|n| \geqslant 4$, then $y(z)^n = y(-z)^n$, that is $y(z)^n$ is even.

(ii): If $n = 2$, that is Theorem 1.

Assume $n = -2$. If $y(z)^{-2} = y(-z)^{-2}$, then $y(z)^{-2}$ is even. Hence, $y(z)$ is even or odd. If $y(z)^{-2} \neq y(-z)^{-2}$, Eq. (4) implies

$$\left(\frac{1}{y(z)}\right)^{2} + \left(\frac{1}{y(-z)}\right)^{2} = 2.$$

By Lemma 3, Remark 3 and the above equation, we have

$$\frac{1}{\sqrt{2}y(z)} = \frac{h(z) + 1/h(z)}{2},$$

where $h(z)$ satisfies $h(-z)h(z) = \mathrm{i}$. That is

$$y(z) = \frac{\sqrt{2}h(z)}{h(z)^2 + 1},$$

where $h(z)$ satisfies $h(-z)h(z) = \mathrm{i}$.

(iii): If $n = 1$, then $P(z) = z^2 - 2z$. Suppose that $P \circ y = y(z)^2 - 2y(z)$ is even. Then

$$y(z)^2 - 2y(z) = y(-z)^2 - 2y(-z).$$

That is

$$[y(z) - y(-z)][y(z) + y(-z)] = 2[y(z) - y(-z)]. \tag{6}$$

If $y(z) \neq y(-z)$, Eq. (6) shows

$$y(z) + y(-z) = 2. \tag{7}$$

By Eq. (7), we have that $y(z)$ cannot be odd.

Using the same reasoning as above, we know that $y(z)$ cannot be odd when $n = -1$.

Thus, Theorem 2 is proved.

PROOF OF Theorem 5

(i)①: Suppose that $y'(z)/y(z)^n$ is even and $n \geqslant 5$. Let $g(z) = 1/y(z)$. Then $y'(z) = -g'(z)/g(z)^2$. Thus, $y'(z)/y(z)^n = -g(z)^{n-2}g'(z)$. Obviously, $n-2 \geqslant 3$. By Theorem 3, we obtain that $y(z)$ is odd.

If $n = 3$ and $y'(z)/y(z)^3$ is even, then we have

$$\frac{y'(z)}{y(z)^3} = \frac{y'(-z)}{y(-z)^3}.$$

Integrating the above equation, we get

$$\frac{1}{y(z)^2} + \frac{1}{y(-z)^2} = A, \tag{8}$$

where A is a constant. It follows $A \neq 0$ from Remark 3 and Eq. (8). If $y(z) = \pm y(-z)$, then we have by (8) that $y(z)$ is a constant, which is impossible. Hence, $y(z)$ is neither odd nor even. Furthermore, by Lemma 3 and Eq. (8), we get $1/y(z) = \frac{\sqrt{A}}{2}(h(z) + 1/h(z))$. Thus, $y(z) = 2h(z)/\sqrt{A}(h(z)^2 + 1)$, where $h(z)$ satisfies $h(z)h(-z) = \mathrm{i}$, A is a non-zero constant.

②: If $n = 1$ and $y'(z)/y(z)$ is even, then

$$\frac{y'(z)}{y(z)} = \frac{y'(-z)}{y(-z)}.$$

Integrating the above equation, we get $\ln y(z) = -\ln y(-z) + C$. It follows that

$$y(z)y(-z) = A,$$

where A is a constant.

On the contrary, if $y(z) = y(-z)$, then we have $y(z)^2 = A$, which is impossible. If $y(z) = -y(-z)$, then we have $-y(z)^2 = A$, which implies that $y(z)$ is a constant. It's a contradiction. So, $y(z)$ is neither odd nor even.

③: If $n = 2$ and $y'(z)/y(z)^2$ is even, then

$$\frac{y'(z)}{y(z)^2} = \frac{y'(-z)}{y(-z)^2}.$$

Integrating the above equation, we have that

$$\frac{1}{y(z)} + \frac{1}{y(-z)} = C,$$

where C is a constant.

If $y(z) = y(-z)$, then $y(z)$ is a constant, which is impossible. Hence, $y(z)$ cannot be even.

(ii)①: Suppose that $y'(z)/y(z)^n$ is odd and $n \geqslant 3$. Let $g(z) = 1/y(z)$. Then $y'(z)/y(z)^n = -g(z)^{n-2}g'(z)$. Obviously, $n - 2 \geqslant 1$. By Theorem 3, we obtain the result.

②: If $n = 2$, by $y'(z)/y(z)^2$ is odd, we have

$$\frac{y'(z)}{y(z)^2} = -\frac{y'(-z)}{y(-z)^2}.$$

Integrating the above equation, we get $-1/y(z) = -1/y(-z) + C$. It follows that

$$\frac{1}{y(z)} - \frac{1}{y(-z)} = A.$$

If $y(z) = -y(-z)$, then $y(z)$ is a constant, which is impossible. Hence, $y(z)$ cannot be odd.

Thus, Theorem 5 is proved.

PROOF OF Theorem 6

(i)①: If $y(qz)/y(z)$ is an even function, then

$$\frac{y(qz)}{y(z)} = \frac{y(-qz)}{y(-z)}. \tag{9}$$

Set $H(z) = y(z)/y(-z)$. By Lemma 2, $\sigma(y(-z)) = \sigma(y(z)) < 1$. So, $\sigma(H) < 1$. And we get

$$H(z)H(-z) = 1. \tag{10}$$

From (9), we obtain

$$H(z) = H(qz). \tag{11}$$

Since $|q| \neq 0, 1$, without loss of generality, suppose that $0 < |q| < 1$.

Suppose that there exists a zero $z_1(\neq 0)$ of $H(z)$. Substitute z_1 for z in (11), we have

$$H(z_1) = H(qz_1). \tag{12}$$

By (12) and $H(z_1) = 0$, we conclude that qz_1 is a zero of $H(z)$. Replacing z by qz_1 in (11), we get

$$H(qz_1) = H(q^2 z_1).$$

By the above equation and $H(qz_1) = 0$, we conclude that $q^2 z_1$ is a zero of $H(z)$.

We proceed to follow the step as above. We will find that $q^k z_1$ is a zero of $H(z)$. Thus, there is a sequence $\{q^k z_1, k = 0, 1, 2, \dots\}$ which are the zeros of $H(z)$.

Since $0 < |q| < 1$, then the sequence $\{q^k z_1, k = 0, 1, 2, \dots\}$ of zeros of $H(z)$ has an accumulation point at the origin. It is a contradiction.

Similar analysis for the poles of $H(z)$ follows that $H(z)$ cannot have any non-zero poles either. Hence, $H(z)$ has no non-zero poles and zeros. We conclude that $H(z)$ must be a rational function by using the fact $\sigma(H) < 1$ and applying the Hadamard factorization theorem. Therefore, $H(z)$ should be a constant. Let $H(z) = H$. From (10), we have $H^2 = 1$, that is $y(z)^2 = y(-z)^2$. Thus, $y(z)^2$ is even.

②: If $y(qz)/y(z)^n$ is an even function, then

$$\frac{y(qz)}{y(z)^n} = \frac{y(-qz)}{y(-z)^n}. \tag{13}$$

Set again $H(z) = y(z)/y(-z)$. By Lemma 2, we get $\sigma(H) < 1$. From (13), we obtain

$$H(qz) = H(z)^n. \tag{14}$$

Our conclusion holds for the cases.

Case 1: $0 < |q| < 1$. Suppose that there exists a pole of $H(z)$ at $z_0(\neq 0)$ with multiplicity τ. By (14), we have

$$H(qz_0) = H(z_0)^n. \tag{15}$$

By (15) and $H(z_0) = \infty$, we conclude that qz_0 is a pole of $H(z)$ of multiplicity $t_1 = n\tau$.

Replacing z by qz_0 in (14), we get

$$H(q^2 z_0) = H(qz_0)^n. \tag{16}$$

By (16) and $H(qz_0) = \infty$, we conclude that $q^2 z_0$ is a pole of $H(z)$ of multiplicity $t_2 = n^2\tau$.

Iterating the equation (14) we have poles of $H(z)$ at $q^k z_0$ with multiplicity $t_k = n^k\tau$ for all non-negative integers k. Obviously, $|q^k z_0| \to 0$ as $k \to \infty$ since $0 < |q| < 1$. It is a contradiction.

Similar analysis for the zeros of $H(z)$ follows that $H(z)$ cannot have any non-zero zeros either. Hence, $H(z)$ has no non-zero poles and zeros. Applying the same reasoning as above, we know that $H(z)$ should be a constant. Let $H(z) = H$. By (10) and (14), we get $H^{n+1} = 1$. Thus, $y(z)^{n+1}$ is even.

Case 2: $|q| \geqslant 1$. Using the same reasoning as Case 1, we may construct poles $z_k = q^k z_0$ of $H(z)$ of multiplicity t_k for all non-negative integers k, satisfying $t_k = n^k\tau$. Obviously, $t_k = n^k\tau \to \infty$ as $k \to \infty$.

If $|q| = 1$, then $|q^k z_0| = |z_0|$. Thus, $H(z)$ is not a meromorphic function. It is a contradiction.

If $|q| > 1$, then $|q^k z_0| \to \infty$, as $k \to \infty$. It is clear that, for large enough k, say $k > k_0$,

$$n^k\tau \leqslant \tau(1 + n + \cdots + n^k) \leqslant n(|q^k z_0|, H).$$

Thus, for each sufficiently large r, there exists a k such that $r \in [|q|^k|z_0|, |q|^{k+1}|z_0|)$, that is $k > \frac{\log r - \log|qz_0|}{\log|q|}$. Hence, we have

$$n(r, H) \geqslant n(|q|^k|z_0|, H) \geqslant n^k\tau \geqslant K n^{\log r/\log|q|},$$

where $K = n^{-\log|qz_0|/\log|q|}\tau$.

Finally, since $K n^{\log r/\log|q|} \leqslant n(r, H) \leqslant \frac{1}{\log 2} T(2r, H)$ for all $r \geqslant r_0$, we immediately obtain $\mu(H) \geqslant \log n/\log|q|$.

Since $|q| \leqslant n$, we can get $\mu(H) \geqslant \log n/\log|q| \geqslant 1$. This is a contradiction.

Applying the same reasoning as above, we know that $H(z)$ should be a constant. Thus, $y(z)^{n+1}$ is even.

(ii): If $y(qz)/y(z)^n$ is odd, then

$$\frac{y(qz)}{y(z)^n} = -\frac{y(-qz)}{y(-z)^n}. \tag{17}$$

By (17), we obtain

$$-H(qz) = H(z)^n, \tag{18}$$

where $H(z) = y(z)/y(-z)$. Using a similar method as above, we see that $H(z)$ is also a constant. Combining (10) and (18), we have $H^2 = 1$ and $H^{n+1} = -1$. Thus, n cannot be odd. If n is even, then $H = -1$. So, $y(z)$ is an odd function.

(iii)①: On the contrary, if $y(qz)/y(z)$ is an odd function, then

$$\frac{y(qz)}{y(z)} = -\frac{y(-qz)}{y(-z)}. \tag{19}$$

Set again $H(z) = y(z)/y(-z)$. From (19), we obtain

$$-H(qz) = H(z). \tag{20}$$

If $|q| > 1$, (20) can be rewritten as

$$-H(z) = H\left(\tfrac{1}{q}z\right).$$

Obviously, $0 < |1/q| < 1$. So, without loss of generality, suppose that $0 < |q| < 1$.

Suppose that there exists a zero $z_1 (\neq 0)$ of $H(z)$. Substitute z_1 for z in (20), we have

$$-H(qz_1) = H(z_1). \tag{21}$$

By (21) and $H(z_1) = 0$, we conclude that qz_1 is a zero of $H(z)$.

We proceed to follow the step as above. We will find zeros of $H(z)$ at $q^k z_1$ for all $k \in \mathbb{N}$. Thus in this case the zeros of $H(z)$ have an accumulation point at the origin since $0 < |q| < 1$. It is a contradiction.

Using a similar method as above, we see that $H(z)$ is a constant. From (20), we get $-H = H$. Therefore, $H(z) = 0$, this is impossible. Hence, $y(qz)/y(z)$ cannot be an odd function.

②: On the contrary, suppose that $y(qz)/y(z)^n$ is an odd function when $n \geqslant 3$. Then

$$\frac{y(qz)}{y(z)^n} = -\frac{y(-qz)}{y(-z)^n}. \tag{22}$$

From (22), we obtain (18).

Our conclusion holds for the cases.

Case 1: $0 < |q| < 1$. Suppose that there exists a pole of $H(z)$ at $z_0 (\neq 0)$ with multiplicity τ. By (18), we have

$$-H(qz_0) = H(z_0)^n. \tag{23}$$

By (23) and $H(z_0) = \infty$, we conclude that qz_0 is a pole of $H(z)$ of multiplicity $t_1 = n\tau$.

Iterating the equation (18) we have poles of $H(z)$ at $q^k z_0$ with multiplicity $t_k = n^k\tau$ for all non-negative integers k. Obviously, $|q^k z_0| \to 0$ as $k \to \infty$ since $0 < |q| < 1$. It is a contradiction.

Applying the same reasoning as above, we know that $H(z)$ should be a constant. Let $H(z) = H$. By (10) and (18), we get $H^2 = 1$ and $H^n = -H$. Thus, $H = 0$ since n is odd, this is impossible. Hence, $y(qz)/y(z)^n$ cannot be an odd function.

Case 2: $|q| \geqslant 1$. Using the same reasoning as Case 1, we may construct poles $z_k = q^k z_0$ of $H(z)$ of multiplicity t_k for all non-negative integers k, satisfying $t_k = n^k\tau$. Obviously, $t_k = n^k\tau \to \infty$ as $k \to \infty$.

If $|q| = 1$, then $|q^k z_0| = |z_0|$. Thus, $H(z)$ is not a meromorphic function. It is a contradiction.

If $|q| > 1$, then $|q^k z_0| \to \infty$, as $k \to \infty$. Similarly as Case 2 in (i), we have

$$n(r, H) \geqslant K n^{\log r/\log|q|},$$

where K is a positive constant.

Applying the same reasoning as above, we immediately obtain $\mu(H) \geqslant \log n/\log|q| \geqslant 1$ since $|q| \leqslant n$. This is a contradiction.

Using a similar method as above, we have that $H(z) = 0$. It is a contradiction. Thus, $y(qz)/y(z)^n$ cannot be an odd function.

Thus, Theorem 6 is proved.

PROOF OF Theorem 7

(i): We need to discuss the following two cases.

Case 1: $n = m$. If $y(z)^n y(qz)^n$ is an even function, then

$$y(z)^n y(qz)^n = y(-z)^n y(-qz)^n. \quad (24)$$

Set $H(z) = y(z)/y(-z)$. From (24), we obtain

$$H(z)^n H(qz)^n = 1.$$

Similar analysis for the proof of [11, Theorem 2.3], we have that $H(z)$ should be a constant. So $y(z)^{2n}$ is even.

Case 2: $n > m$. If $y(z)^n y(qz)^m$ is an even function, then

$$y(z)^n y(qz)^m = y(-z)^n y(-qz)^m.$$

The above equation shows

$$H(z)^n H(qz)^m = 1.$$

Using a similar method as Case 1, we have that $H(z)$ is a constant. Hence, $y(z)^{n+m}$ is even.

(ii): If $y(z)^n y(qz)^m$ is odd, then

$$y(z)^n y(qz)^m = -y(-z)^n y(-qz)^m. \quad (25)$$

Eq. (25) shows

$$H(z)^n H(qz)^m = -1,$$

where $H(z) = \frac{y(z)}{y(-z)}$.

Using a similar method as above, we have that $H(z)$ is a constant. Let $H(z) = H$.

If $n+m$ is old, then $H = -1$ since $H^{n+m} = -1$ and $H^2 = 1$. So, $y(z)$ is an odd function.

If $n+m$ is even, then we have $H^2 = -1$ and $H^2 = 1$. It is a contradiction. Thus, $y(z)^n y(qz)^m$ cannot be odd.

Thus, Theorem 7 is proved.

Acknowledgements: This research is supported by the National Natural Science Foundation of China (Nos. 12261023,11861023), the Foundation of Science and Technology Project of Guizhou Province of China (No. QIANKEHEJICHU-ZK[2021]Ordinary313), and the Doctoral Research Start-up Project of Guiyang University (GYU-KY-[2024]).

REFERENCES

1. Hayman WK (1964) *Meromorphic Functions*, Clarendon Press, Oxford.
2. Yang CC, Yi HX (2003) *Uniqueness Theory of Meromorphic Functions, Mathematics and Its Applications*, Kluwer Academic Publishers Group, Dordrecht.
3. Yang L (1982) *Value Distribution Theory and Its New Research*, Science Press, Beijing.
4. Liu K, Wei YM, Yu PY (2020) Generalized Yang's conjecture on the periodicity of entire functions. *Bull Korean Math Soc* **57**, 1259–1267.
5. Liu K, Yu PY (2019) A note on the periodicity of entire functions. *Bull Aust Math Soc* **100**, 290–296.
6. Liu XL, Korhonen R (2020) On the periodicity of transcendental entire functions. *Bull Aust Math Soc* **101**, 354–465.
7. Liu Y, Jiang S (2022) The periodicity on a transcendental entire function with its differential-difference polynomials. *ScienceAsia* **48**, 759–763.
8. Wang Q, Hu PC (2018) Zeros and periodicity of entire functions. *Acta Math Sci Ser A Chin Ed* **38**, 209–214.
9. Beardon AF (2003) Even and odd entire functions. *J Aust Math Soc* **74**, 19–23.
10. Horwitz AL (1997) Even compositions of entire functions and related matters. *J Aust Math Soc Ser A* **63**, 225–237.
11. Liu XL, Liu K, Korhonen R (2022) Inverse problems on the parity of meromorphic functions. *J Math Anal Appl* **512**, 126129.
12. Baker IN (1966) On a class of meromorphic functions. *Proc Amer Math Soc* **17**, 819–822.
13. Yang CC (1970) A generalization of a theorem of P. Montel on entire functions. *Proc Amer Math Soc* **26**, 332–334.
14. Gross F (1966) On the equation $f^n + g^n = 1$. *Bull Am Math Soc* **72**, 86–88.
15. Bergweiler W, Ishizaki K, Yanagihara N (1998) Meromorphic solutions of some functional equations. *Methods Appl Anal* **5**, 248–258.
16. Liu K, Yang LZ (2016) A note on meromorphic solutions of Fermat types equations. *An Ştiinţ Univ Al I Cuza Iaşi Mat(N S)* **62**, 317–325.