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Theory of Δ conditional randomized truth degree in Gödel *n*-valued propositional logic system of adding Δ operator

Bo Wang[a,*], **Xiao-Jing Hui**[b]

[a] College of Mathematics, Si'chuan University, Chen'du 610000 China
[b] College of Mathematics and Computer Science, Yan'an University, Yan'an 716000 China

*Corresponding author, e-mail: 1536011862@qq.com

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ABSTRACT: In this paper, Δ conditional randomized truth degree of propositional formula is put forward in Gödel *n*-valued propositional logic system. It adds Δ operator compared with conditional randomized truth degree. On this basis, some inference rules such as MP, HS, intersection inference, union inference and their related properties are studied. At last, the concepts of Δ conditional randomized similarity degree, Δ conditional randomized pseudo-metric between propositional formulas are given, and their related good properties are discussed.

KEYWORDS: Δconditional randomized truth degree, Δ conditional randomized similarity degree, Δ conditional randomized logic metric space

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INTRODUCTION

As we all know, mathematical logic is a formal theory characterized by symbolization. It focuses on formal reasoning rather than numerical calculation. However, numerical calculation pays more attention to solving problems and rarely uses formal reasoning methods. There are great differences, but the quantitative logic is initiated by Wang [1–4] is a new branch of research that tries to connect the two, which is the product of the combination of mathematical logic and probability calculation.

The idea of introducing probability methods into mathematical logic has gradually emerged since the 1970s, and a monograph on "probabilistic logic" has been published [5]. Later, many experts studied on this basis and some results are obtained. It is worth noting that Hui et al [6–8] combined quantitative logic with probability logic, the concepts of randomized truth degree for binary and ternary logic systems are put forward, the randomized logic metric spaces are established.

Among the logical systems that have received widespread attention at present, related research has been hindered due to the strong negation in the Gödel system and the Goguen system. In order to solve this problem, the basic connectives ∼ and Δ are introduced in [9–12]. The quantification of Δ fuzzy logic system $SBL_\sim$ is realized by Hui in [13], the theory of t-randomized truth degree on Gödel *n*-valued propositional logic system of adding two operators is proposed by Zhu in [14]. It is a very meaningful subject to combine the conditional probability part of probabilistic logic with the truth degree through an appropriate way [5].

Following the research results of the theory of t-randomized truth degree in Gödel *n*-valued propositional logic system. In this paper, Δ conditional randomized truth degree of propositional formula is put forward in Gödel *n*-valued propositional logic system. It adds Δ operator compared with conditional randomized truth degree. On this basis, some inference rules such as MP, HS, intersection inference, union inference and their related properties are studied. At last, the concepts of Δ conditional randomized similarity degree, Δ conditional randomized pseudo-metric between propositional formulas are given, and their related good properties are discussed.

PRELIMINARIES

Definition 1 ([11]) The axiom system of BL_Δ is as follows:

(BL) the axiom system of BL
(Δ1) $\Delta A \vee \neg\Delta A$;
(Δ2) $\Delta(A\vee B) \to (\Delta A \vee \Delta B)$;
(Δ3) $\Delta A \to B$;
(Δ4) $\Delta A \to \Delta\Delta A$;
(Δ5) $\Delta(A\to B) \to (\Delta A \to \Delta B)$.

The inference rules in BL_Δ are MP rule and Δ rule, The MP rule is from $A, A\to B$, inferred B, the Δ rule is from A inferred ΔA.

Theorem 1 ([15], Δ deduction theorem) *Let L be an axiomatic extension of BL_Δ, then for any theory Γ, the formulas A and B, we have*

$\Gamma, A \vdash B$ *if and only if* $\Gamma \vdash \Delta A \to B$.

Definition 2 ([14]) Let $S=\{p_1, p_2, \ldots\}$ be a countable set, Δ is unary operation on S, ∨, ∧, → are binary operations on S, respectively, $F(S)$ is a free algebra of type (1,2,2,2) generated by S. Then the elements in $F(S)$ are called propositional formulas or formulas, and the elements in S are called atomic formulas.

Definition 3 ([14]) Let $L=\{0,\frac{1}{n-1},\dots,\frac{n-2}{n-1},1\}$. It is stipulated in L: $\forall x,\ y\in L$, $\Delta x=\{^{1,\ x=1}_{0,\ x<1}$, $x\vee y=\max\{x,y\}$, $x\wedge y=\min\{x,y\}$, $x\to y=$, then it is called type (1,2,2,2) algebra, which is called the expansion of Gödel n-valued propositional logic system. It is abbreviated as G_Δ, if there is no special description, it is expanded in G_Δ.

Definition 4 ([14]) Let $A=A(p_1,p_2,\dots,p_m)\in F(S)$, Then A corresponds to an n-valued m-element function $\overline{A}$, in G_Δ, $\{0,\frac{1}{n-1},\dots,\frac{n-2}{n-1},1\}^m\to[0,1]$. Here $\overline{A}(x_1,\dots,x_m)$ is formed by the operation symbol Δ, $\vee$, $\wedge$, $\to$ connecting $x_1,\dots,x_m$, in the same way as $A=A(p_1,p_2,\dots,p_m)\in F(S)$ is formed by connecting the atomic formula $p_1,\dots,p_m$ by the conjunction $\Delta,\vee,\wedge,\to$ then $\overline{A}$ is called the function induced by the formula A.

Definition 5 ([16]) Let $N=(1,\ 2,\ \dots)$, $D=(p_1,\ p_2,\ p_3)$, $0<p_n<1$ $(n=1,\ 2,\ \dots)$. Then D is called a randomized numbers sequence in $(0,1)$.

Definition 6 ([16]) Let $D_0=(p_{01},\ p_{02},\ \dots)$, $D_{\frac{1}{n-1}}=\{p_{\frac{1}{n-1}1},\ p_{\frac{1}{n-1}2},\ \dots\}$, $D_1=(p_{11},\ p_{12},\ \dots)$ be a series of randomized numbers in $(0,1)$, and $p_{0k}+p_{\frac{1}{n-1}k}+\dots+p_{1k}=1$ $(k=1,2,\dots)$. Then D_0, $D_{\frac{1}{n-1}}$, …, $D_{\frac{n-2}{n-1}}$, D_1 $(n\geqslant 2)$ is said to be a n-valued randomized numbers sequence in $(0,1)$.

Definition 7 ([16]) Let D_0, $D_{\frac{1}{n-1}}$, …, $D_{\frac{n-2}{n-1}}$, D_1 $(n\geqslant 2)$ be a series of n randomized numbers in $(0,1)$, $\forall a=(x_1,x_2,\dots,x_m)\in\{0,\frac{1}{n-1},\dots,\frac{n-2}{n-1},1\}^m$. Let $\varphi(\alpha)=Q_1\times\dots\times Q_m$, Here, when $x_k=0$, $x_k=d_{0k}$; when $x_k=\frac{i}{n-1}$, $Q_k=d_{\frac{i}{n-1}k}$ $(i=1,2,\dots,n-2)$; when $x_k=1$, $Q_k=d_{1k}$ $(k=1,2,\dots,m)$, then a mapping $\varphi:\{0,\frac{1}{n-1},\dots,\frac{n-2}{n-1},1\}^m\to[0,1]$. Then φ is called the D-randomization map of $\{0,\frac{1}{n-1},\dots,\frac{n-2}{n-1},1\}^m$.

Definition 8 ([16]) Let be a D-randomization map of $\{0,\frac{1}{n-1},\dots,\frac{n-2}{n-1},1\}^m$, then

$$\sum\left\{\varphi(\alpha):\alpha\in\left\{0,\frac{1}{n-1},\dots,\frac{n-2}{n-1},1\right\}^m\right\}=1.$$

Definition 9 ([14]) Let $A=A(p_1,p_2,\dots,p_m)\in F(S)$, $D_0,D_{\frac{1}{n-1}},\dots,D_{\frac{n-2}{n-1}},D_1$ $(n\geqslant 2)$ be an n-valued randomized numbers sequence in $(0,1)$, and

$$[\Delta A]_{\frac{i}{n-1}}=\overline{\Delta A}^{-1}\left(\frac{i}{n-1}\right)$$
$$\mu(\{\Delta A]_{\frac{i}{n-1}})=\sum\left\{\varphi(\alpha):\alpha\in\overline{\Delta A}^{-1}\left(\frac{i}{n-1}\right)\right\}$$
$$\mu[\Delta A]=\sum_{i=1}^{n-1}\mu([\Delta A]_{\frac{i}{n-1}}),\quad i=1,2,\dots n-1.$$

Denote $\mu[\Delta A]$ as $\tau_D(\Delta A)$, then $\tau_D(\Delta A)$ is called the randomized truth degree of Δ of the propositional formula A.

Theorem 2 ([14]) *Let $A,B\in F(S)$, D_0, $D_{\frac{1}{n-1}}$, …, $D_{\frac{n-2}{n-1}}$, D_1 $(n\geqslant 2)$ be an n-valued randomized numbers sequence in $(0,1)$. Then,*
(i) *A is tautology if and only if $\tau_D(\Delta A)=1$;*
(ii) *if $A\approx B$, then $\tau_D(\Delta A)=\tau_D(\Delta B)$;*
(iii) $\tau_D(\Delta A\vee\Delta B)=\tau_D(\Delta A)+\tau_D(\Delta B)-\tau_D(\Delta A\wedge\Delta B)$.

Δ CONDITIONAL RANDOMIZED TRUTH DEGREE

Definition 10 Let $A=A(p_1,p_2,\dots,p_m)\in F(S)$, D_0, $D_{\frac{1}{n-1}}$, …, $D_{\frac{n-2}{n-1}}$, D_1 $(n\geqslant 2)$ be an n-valued randomized numbers sequence in $(0,1)$, $\Lambda\in F(S)$, $\tau_D(\Delta\Lambda)>0$, and

$$\tau_D(\Delta A\mid\Delta\Lambda)=\frac{\tau_D(\Delta A\wedge\Delta B)}{\tau_D(\Delta\Lambda)}.$$

Then $\tau_D(\Delta A\mid\Delta\Lambda)$ is called the Δ conditional randomized truth degree of formula A under condition Λ.

Remark 1 Definition 9 gives the Δ randomized truth degree of the propositional formula, after converting Definition 9, the following Proposition 1 is obtained.

Proposition 1 *Let $A=A(p_1,p_2,\dots,p_m)\in F(S)$, D_0, $D_{\frac{1}{n-1}}$, …, $D_{\frac{n-2}{n-1}}$, D_1 $(n\geqslant 2)$ be an n-valued randomized numbers sequence in $(0,1)$, $\overline{A}(x_1,\dots,x_m)$ is the induction function of A and φ is the D-randomization map of $\{0,\frac{1}{n-1},\dots,\frac{n-2}{n-1},1\}^m$. Then,*

$$\tau_D(\Delta A)=\sum\left\{\overline{\Delta A}(\alpha)\varphi(\alpha)\mid\alpha\in\{0,\tfrac{1}{n-1},\dots,\tfrac{n-2}{n-1},1\}\right\}.$$

Proof:

$$\begin{aligned}\tau_D(\Delta A)&=\sum_{i=1}^{n-1}\tfrac{i}{n-1}\left\{\sum\{\varphi(\alpha):\alpha\in\overline{\Delta A}^{-1}(\tfrac{i}{n-1})\}\right\}\\&=\sum_{i=1}^{n-1}\tfrac{i}{n-1}\left\{\sum\{\varphi(\alpha):\overline{\Delta A}(\alpha)=(\tfrac{i}{n-1})\}\right\}\\&=\sum_{i=1}^{n-1}\left\{\sum\{\tfrac{i}{n-1}\varphi(\alpha):\overline{\Delta A}(\alpha)=(\tfrac{i}{n-1})\}\right\}\\&=\sum_{i=1}^{n-1}\left\{\sum\{\overline{\Delta A}(\alpha)\varphi(\alpha):\overline{\Delta A}(\alpha)=(\tfrac{i}{n-1})\}\right\}\\&=\sum\left\{\overline{\Delta A}(\alpha)\varphi(\alpha)\mid\alpha\in\{0,\tfrac{1}{n-1},\cdots,\tfrac{n-2}{n-1},1\}\right\}\end{aligned}$$

□

Definition 11 Let $A=A(p_1,p_2,\dots,p_m)\in F(S)$, D_0, $D_{\frac{1}{n-1}}$, …, $D_{\frac{n-2}{n-1}}$, D_1 $(n\geqslant 2)$ be an n-valued randomized numbers sequence in $(0,1)$, $\Lambda\in F(S)$, and $\tau_D(\Delta\Lambda)>0$.
(i) If there is $\overline{\Delta\Lambda}^{-1}\subseteq\overline{\Delta A}^{-1}(\frac{i}{n-1})$ when $i=1,2,\dots,n-1$, then A is called tautology under the condition Λ, denoted as $A=_\Lambda 1$;

(ii) if there is $\overline{(\Delta A\wedge\Delta\Lambda)}^{-1}(\frac{i}{n-1})=\phi$ when $i=1,2,\ldots,n-1$, then A is called contradiction under the condition Λ, denoted as $A=_\Lambda 0$;

(iii) if there is $\overline{(\Delta A\wedge\Delta\Lambda)}^{-1}(\frac{i}{n-1})=\overline{(\Delta B\wedge\Delta\Lambda)}^{-1}(\frac{i}{n-1})$ when $i=1,2,\ldots,n-1$, then A and B are said to be logically equivalent under the condition Λ, denoted as $A\approx_\Lambda B$.

Theorem 3 *Let $A,B\in F(S)$, $D_0,D_{\frac{1}{n-1}},\ldots,D_{\frac{n-2}{n-1}},D_1$ $(n\geqslant 2)$ be an n-valued randomized numbers sequence in $(0,1)$, $\Lambda\in F(S)$, and $\tau_D(\Delta\Lambda)>0$.*
(i) *If $A=_\Lambda 1$, then $\tau_D(\Delta A\mid\Delta\Lambda)=1$;*
(ii) *if $A=_\Lambda 0$, then $\tau_D(\Delta A\mid\Delta\Lambda)=0$;*
(iii) *if $A=_\Lambda B$, then $\tau_D(\Delta A\mid\Delta\Lambda)=\tau_D(\Delta B\mid\Delta\Lambda)$;*
(iv) *$\tau_D((\Delta A\vee\Delta B)\mid\Delta\Lambda)=\tau_D(\Delta A\mid\Delta\Lambda)+\tau_D(\Delta B\mid\Delta\Lambda)-\tau_D((\Delta A\wedge\Delta B)\mid\Delta\Lambda)$.*

Proof: **(i)**: Let A and Λ contain the same atomic formulas $q_1,\cdots,q_m$.

$\forall\alpha\in\overline{(\Delta A\wedge\Delta\Lambda)}^{-1}(\frac{i}{n-1})$, that is $\overline{(\Delta A\wedge\Delta\Lambda)}(\alpha)=\frac{i}{n-1}$. It follows from the homomorphism of the induced function that $\overline{\Delta A}(\alpha)\wedge\overline{\Delta\Lambda}(\alpha)=\frac{i}{n-1}$, that is, $\min\{\overline{\Delta A}(\alpha),\overline{\Delta\Lambda}(\alpha)\}=\frac{i}{n-1}$. It follows from $A=\Lambda_1$ that $\overline{\Delta\Lambda}^{-1}(\frac{i}{n-1})\subseteq\overline{\Delta A}^{-1}(\frac{i}{n-1})$. If $\alpha\in\overline{\Delta\Lambda}^{-1}(\frac{i}{n-1})$, there must be $\alpha\in\overline{\Delta A}^{-1}(\frac{i}{n-1})$. Then it follows from $\min\{\overline{\Delta A}(\alpha),\overline{\Delta\Lambda}(\alpha)\}=\frac{i}{n-1}$ that $\overline{\Delta\Lambda}(\alpha)=\frac{i}{n-1}$, that is, $\alpha\in\overline{\Delta\Lambda}^{-1}(\frac{i}{n-1})$. Conversely, $\forall\alpha\in\overline{\Delta\Lambda}^{-1}(\frac{i}{n-1})$, that is $\overline{\Delta\Lambda}(\alpha)=\frac{i}{n-1}$, it follows from $\overline{\Delta\Lambda}^{-1}(\frac{i}{n-1})\subseteq\overline{\Delta A}^{-1}(\frac{i}{n-1})$ that $\overline{\Delta A}(\alpha)=\frac{i}{n-1}$. Hence, $\overline{(\Delta A\wedge\Delta\Lambda)}(\alpha)=\frac{i}{n-1}$, that is $\forall\alpha\in\overline{(\Delta A\wedge\Delta\Lambda)}^{-1}(\frac{i}{n-1})$. Hence, $\overline{(\Delta A\wedge\Delta\Lambda)}^{-1}(\frac{i}{n-1})=\overline{\Delta\Lambda)}^{-1}(\frac{i}{n-1})$, that is $\tau_D(\Delta A\mid\Delta\Lambda)=\tau_D(\Delta\Lambda)$. Hence, $\tau_D(\Delta A\mid\Delta\Lambda)=1$.

(ii): Because $\tau_D(\Delta A\wedge\Delta\Lambda)=\sum_{i=1}^{n-1}\frac{i}{n-1}\sum\{\varphi(\alpha):\alpha\in\overline{(\Delta A\wedge\Delta\Lambda)}^{-1}(\frac{i}{n-1})\}$, it follows from $\overline{(\Delta A\wedge\Delta\Lambda)}^{-1}(\frac{i}{n-1})=\phi$ that $\tau_D(\Delta A\wedge\Delta\Lambda)=0$. Hence, $\tau_D(\Delta A\mid\Delta\Lambda)=0$.

(iii): It follows from $A\approx_\Lambda B$ that $\overline{(\Delta A\wedge\Delta\Lambda)}^{-1}(\frac{i}{n-1})=\overline{(\Delta B\wedge\Delta\Lambda)}^{-1}(\frac{i}{n-1})$. Hence,

$$\begin{aligned}&\tau_D(\Delta A\wedge\Delta\Lambda)\\&\quad=\sum_{i=1}^{n-1}\frac{i}{n-1}\sum\{\varphi(\alpha):\alpha\in\overline{(\Delta A\wedge\Delta\Lambda)}^{-1}(\tfrac{i}{n-1})\}\\&\quad=\sum_{i=1}^{n-1}\frac{i}{n-1}\sum\{\varphi(\alpha):\alpha\in\overline{(\Delta B\wedge\Delta\Lambda)}^{-1}(\tfrac{i}{n-1})\}\\&\quad=\tau_D(\Delta B\wedge\Delta\Lambda).\end{aligned}$$

Hence, $\tau_D(\Delta A\mid\Delta\Lambda)=\tau_D(\Delta B\mid\Delta\Lambda)$.

(iv): It follows from Theorem 2(iii) that $\tau_D((\Delta A\vee\Delta B)\wedge\Delta\Lambda)=\tau_D((\Delta A\wedge\Delta\Lambda)\vee(\Delta B\wedge\Delta\Lambda))=\tau_D(\Delta A\wedge\Delta\Lambda)+\tau_D(\Delta B\wedge\Delta\Lambda)-\tau_D((\Delta A\wedge\Delta B)\wedge\Delta\Lambda)$. Dividing both sides by $\tau_D(\Delta\Lambda)$ to get $\tau_D((\Delta A\vee\Delta B)\mid\Delta\Lambda)=\tau_D(\Delta A\mid\Delta\Lambda)+\tau_D(\Delta B\mid\Delta\Lambda)-\tau_D((\Delta A\wedge\Delta B)\mid\Delta\Lambda)$. □

Theorem 4 ***(Δ conditional randomized truth degree MP rule)*** *Let $A,B\in F(S)$, D_0, $D_{\frac{1}{n-1}}$, …, $D_{\frac{n-2}{n-1}}$, D_1 $(n\geqslant 2)$ be an n-valued randomized numbers sequence in $(0,1)$, $\Lambda\in F(S)$, and $\tau_D(\Delta\Lambda)>0$. If $\tau_D(\Delta A\mid\Delta\Lambda)\geqslant\alpha$, $\tau_D((\Delta A\to\Delta B)\mid\Delta\Lambda)\geqslant\beta$, then $\tau_D(\Delta B\mid\Delta\Lambda)\geqslant\alpha+\beta-1$.*

Proof: Let A, B and Λ contain the same atomic formulas $q_1,\ldots,q_m$. For all $a,b,c\in G_\Delta$ we have $\Delta b\wedge\Delta c\geqslant\Delta a\wedge\Delta c+(\Delta a\to\Delta b)\wedge\Delta c-\Delta c$, that is, $\forall\alpha\in\{0,\frac{1}{n-1},\ldots,\frac{n-2}{n-1},1\}^m$ there is $\overline{\Delta B\wedge\Delta\Lambda}(\alpha)\geqslant\overline{\Delta A\wedge\Delta\Lambda}(\alpha)+\overline{(\Delta A\to\Delta B)\wedge\Delta\Lambda}(\alpha)-\overline{\Delta\Lambda}(\alpha)$. Hence,

$$\begin{aligned}&\sum\{\overline{\Delta B\wedge\Delta\Lambda}(\alpha)\varphi(\alpha)\}\geqslant\sum\{\overline{\Delta A\wedge\Delta\Lambda}(\alpha)\varphi(\alpha)\}\\&\quad+\sum\{\overline{(\Delta A\to\Delta B)\wedge\Delta\Lambda}(\alpha)\varphi(\alpha)\}-\sum\{\overline{\Delta\Lambda}(\alpha)\varphi(\alpha)\}.\end{aligned}$$

It follows from Proposition 1 that $\tau_D(\Delta B\wedge\Delta\Lambda)\geqslant\tau_D(\Delta A\wedge\Delta\Lambda)+\tau_D((\Delta A\to\Delta B)\wedge\Delta\Lambda)-\tau_D(\Delta\Lambda)$. Dividing both sides by $\tau_D(\Delta\Lambda)$ to get $\tau_D(\Delta B\mid\Delta\Lambda)\geqslant\alpha+\beta-1$. □

Theorem 5 ***(Δ conditional randomized truth degree HS rule)*** *Let $A,B,C\in F(S)$, D_0, $D_{\frac{1}{n-1}}$, …, $D_{\frac{n-2}{n-1}}$, D_1 $(n\geqslant 2)$ be an n-valued randomized numbers sequence in $(0,1)$, $\Lambda\in F(S)$, and $\tau_D(\Delta\Lambda)>0$. If $\tau_D((\Delta A\to\Delta B)\mid\Delta\Lambda)\geqslant\alpha$, $\tau_D((\Delta B\to\Delta C)\mid\Delta\Lambda)\geqslant\beta$, then $\tau_D((\Delta A\to\Delta C)\mid\Delta\Lambda)\geqslant\alpha+\beta-1$.*

Proof: Let A, B, C and Λ contain the same atomic formulas $q_1,\ldots,q_m$. For all $a,b,c\in G_\Delta$ we have $(\Delta a\to\Delta b)\to((\Delta b\to\Delta c)\to(\Delta a\to\Delta c))=1$. Hence, $\tau_D((\Delta A\to\Delta B)\to((\Delta B\to\Delta C)\to(\Delta A\to\Delta C)))=1$. Hence, $\tau_D(((\Delta A\to\Delta B)\to((\Delta B\to\Delta C)\to(\Delta A\to\Delta C)))\mid\Delta\Lambda)=1$. It follows from Theorem 4 that $\tau_D(((\Delta B\to\Delta C)\to(\Delta A\to\Delta C))\mid\Delta\Lambda)\geqslant\alpha$, and that $\tau_D((\Delta A\to\Delta C)\mid\Delta\Lambda)\geqslant\alpha+\beta-1$. □

Theorem 6 *Let $A,B,C\in F(S)$, D_0, $D_{\frac{1}{n-1}}$, …, $D_{\frac{n-2}{n-1}}$, D_1 $(n\geqslant 2)$ be an n-valued randomized numbers sequence in $(0,1)$, $\Lambda\in F(S)$, and $\tau_D(\Delta\Lambda)>0$. Then $\tau_D((\Delta A\to(\Delta B\wedge\Delta C))\mid\Delta\Lambda)=\tau_D((\Delta A\to\Delta B)\mid\Delta\Lambda)+\tau_D((\Delta A\to\Delta C)\mid\Delta\Lambda)-\tau_D(((\Delta A\to\Delta B)\vee(\Delta A\to\Delta C))\mid\Delta\Lambda)$.*

Proof: Let A, B, C and Λ contain the same atomic formulas $q_1,\ldots,q_m$. For all $a,b,c\in G_\Delta$ we have $\Delta a\to(\Delta b\wedge\Delta c)=(\Delta a\to\Delta b)\wedge(\Delta a\to\Delta c)$. Hence, $((\Delta a\to(\Delta b\wedge\Delta c))\wedge\Delta d)=(((\Delta a\to\Delta b)\wedge(\Delta a\to\Delta c))\wedge\Delta d)$. Hence, $((\Delta A\to(\Delta B\wedge\Delta C))\wedge\Delta\Lambda)=(((\Delta A\to\Delta B)\wedge(\Delta A\to\Delta C))\wedge\Delta\Lambda)$. It follows from Theorem 2(ii) that $\tau_D((\Delta A\to(\Delta B\wedge\Delta C))\wedge\Delta\Lambda)=\tau_D(((\Delta A\to\Delta B)\wedge(\Delta A\to\Delta C))\wedge\Delta\Lambda)$. Hence,

$$\begin{aligned}&\frac{\tau_D((\Delta A\to(\Delta B\wedge\Delta C))\wedge\Delta\Lambda)}{\tau_D(\Delta\Lambda)}\\&\quad=\frac{\tau_D(((\Delta A\to\Delta B)\wedge(\Delta A\to\Delta C))\wedge\Delta\Lambda)}{\tau_D\Delta\Lambda}.\end{aligned}$$

That is, $\tau_D((\Delta A \to (\Delta B \wedge \Delta C)) \mid \Delta\Lambda) = \tau_D(((\Delta A \to \Delta B) \wedge (\Delta A \to \Delta C)) \mid \Delta\Lambda)$. It follows from Theorem 3(iv) that $\tau_D(((\Delta A \to \Delta B) \wedge (\Delta A \to \Delta C)) \mid \Delta\Lambda) = \tau_D((\Delta A \to \Delta B) \mid \Delta\Lambda) + \tau_D((\Delta A \to \Delta C) \mid \Delta\Lambda) - \tau_D(((\Delta A \to \Delta B) \vee (\Delta A \to \Delta C)) \mid \Delta\Lambda)$. Hence, $\tau_D((\Delta A \to (\Delta B \wedge \Delta C)) \mid \Delta\Lambda) = \tau_D((\Delta A \to \Delta B) \mid \Delta\Lambda) + \tau_D((\Delta A \to \Delta C) \mid \Delta\Lambda) - \tau_D(((\Delta A \to \Delta B) \vee (\Delta A \to \Delta C)) \mid \Delta\Lambda)$. □

Corollary 1 ***(Δ conditional randomized truth degree intersection inference rule)*** *Let $A, B, C \in F(S)$, D_0, $D_{\frac{1}{n-1}}$, …, $D_{\frac{n-2}{n-1}}$, D_1 $(n \geqslant 2)$ be an n-valued randomized numbers sequence in $(0,1)$, $\Lambda \in F(S)$, and $\tau_D(\Delta\Lambda) > 0$. If $\tau_D((\Delta A \to \Delta B) \mid \Delta\Lambda) \geqslant \alpha$, $\tau_D((\Delta A \to \Delta C) \mid \Delta\Lambda) \geqslant \beta$, then $\tau_D((\Delta A \to (\Delta B \wedge \Delta C)) \mid \Delta\Lambda) \geqslant \alpha + \beta - 1$.*

Theorem 7 *Let $A, B, C \in F(S)$, D_0, $D_{\frac{1}{n-1}}$, …, $D_{\frac{n-2}{n-1}}$, D_1 $(n \geqslant 2)$ be an n-valued randomized numbers sequence in $(0,1)$, $\Lambda \in F(S)$, and $\tau_D(\Delta\Lambda) > 0$, then $\tau_D(((\Delta A \vee \Delta B) \to \Delta C) \mid \Delta\Lambda) = \tau_D((\Delta A \to \Delta C) \mid \Delta\Lambda) + \tau_D((\Delta B \to \Delta C) \mid \Delta\Lambda) - \tau_D(((\Delta A \to \Delta C) \vee (\Delta B \to \Delta C)) \mid \Delta\Lambda)$.*

Proof: Let A, B, C and Λ contain the same atomic formulas $q_1, \dots, q_m$. For all $a, b, c \in G_\Delta$ we have $((\Delta a \vee \Delta b) \to \Delta c) = (\Delta a \to \Delta c) \wedge (\Delta b \to \Delta c)$. Hence, $(((\Delta a \vee \Delta b) \to \Delta c) \wedge \Delta d) = (((\Delta a \to \Delta c) \wedge (\Delta b \to \Delta c)) \wedge \Delta d)$. Hence, $(((\Delta A \vee \Delta B) \to \Delta C) \wedge \Delta\Lambda) = (((\Delta A \to \Delta C) \wedge (\Delta B \to \Delta C)) \wedge \Delta\Lambda)$. It follows from Theorem 2(ii) that $\tau_D(((\Delta A \vee \Delta B) \to \Delta C) \wedge \Delta\Lambda) = \tau_D(((\Delta A \to \Delta C) \wedge (\Delta B \to \Delta C)) \wedge \Delta\Lambda)$. Hence,

$$\frac{\tau_D(((\Delta A \vee \Delta B) \to \Delta C) \wedge \Delta\Lambda)}{\tau_D(\Delta\Lambda)} = \frac{\tau_D(((\Delta A \to \Delta C) \wedge (\Delta B \to \Delta C)) \wedge \Delta\Lambda)}{\tau_D(\Delta\Lambda)}.$$

That is, $\tau_D(((\Delta A \vee \Delta B) \to \Delta C) \mid \Delta\Lambda) = \tau_D(((\Delta A \to \Delta C) \wedge (\Delta B \to \Delta C)) \mid \Delta\Lambda)$. It follows from Theorem 3(iv) that $\tau_D(((\Delta A \to \Delta C) \wedge (\Delta B \to \Delta C)) \mid \Delta\Lambda) = \tau_D((\Delta A \to \Delta C) \mid \Delta\Lambda) + \tau_D((\Delta B \to \Delta C) \mid \Delta\Lambda) - \tau_D(((\Delta A \to \Delta C) \vee (\Delta B \to \Delta C)) \mid \Delta\Lambda)$. Hence, $\tau_D(((\Delta A \vee \Delta B) \to \Delta C) \mid \Delta\Lambda) = \tau_D((\Delta A \to \Delta C) \mid \Delta\Lambda) + \tau_D((\Delta B \to \Delta C) \mid \Delta\Lambda) - \tau_D(((\Delta A \to \Delta C) \vee (\Delta B \to \Delta C)) \mid \Delta\Lambda)$. □

Corollary 2 ***(Δ conditional randomized truth degree union inference rule)*** *Let $A, B, C \in F(S)$, D_0, $D_{\frac{1}{n-1}}$, …, $D_{\frac{n-2}{n-1}}$, D_1 $(n \geqslant 2)$ be an n-valued randomized numbers sequence in $(0,1)$, $\Lambda \in F(S)$, and $\tau_D(\Delta\Lambda) > 0$. If $\tau_D((\Delta A \to \Delta C) \mid \Delta\Lambda) \geqslant \alpha$, $\tau_D((\Delta B \to \Delta C) \mid \Delta\Lambda) \geqslant \beta$, then $\tau_D(((\Delta A \vee \Delta B) \to \Delta C) \mid \Delta\Lambda) \geqslant \alpha + \beta - 1$.*

Theorem 8 *Let $A, B \in F(S)$, D_0, $D_{\frac{1}{n-1}}$, …, $D_{\frac{n-2}{n-1}}$, D_1 $(n \geqslant 2)$ be an n-valued randomized numbers sequence in $(0,1)$, $\Lambda \in F(S)$, and $\tau_D(\Delta\Lambda) > 0$. Then $\tau_D((\Delta A \to \Delta B) \mid \Delta\Lambda) = \tau_D((\Delta A \wedge \Delta B) \mid \Delta\Lambda) - \tau_D(\Delta A \mid \Delta\Lambda) + 1$.*

Proof: Let A, B and Λ contain the same atomic formulas $q_1, \dots, q_m$. For all $a, b, c \in G_\Delta$ we have $(\Delta a \to \Delta b) \wedge \Delta c = (\Delta a \wedge \Delta b) \wedge \Delta c - \Delta a \wedge \Delta c + \Delta c$. That is, $\forall \alpha \in \{0, \frac{1}{n-1}, \dots, \frac{n-2}{n-1}, 1\}^m$ there is $\overline{(\Delta A \to \Delta B) \wedge \Delta\Lambda}(\alpha) = \overline{(\Delta A \wedge \Delta B) \wedge \Delta\Lambda}(\alpha) - \overline{\Delta A \wedge \Delta\Lambda}(\alpha) + \overline{\Delta\Lambda}(\alpha)$. Hence,

$$\sum\{\overline{(\Delta A \to \Delta B) \wedge \Delta\Lambda}\varphi(\alpha)\} = \sum\{\overline{(\Delta A \wedge \Delta B) \wedge \Delta\Lambda}\varphi(\alpha)\} - \sum\{\overline{\Delta A \wedge \Delta\Lambda}\varphi(\alpha)\} + \sum\{\overline{\Delta\Lambda}\varphi(\alpha)\}.$$

It follows from Proposition 1 that $\tau_D((\Delta A \to \Delta B) \wedge \Delta\Lambda) = \tau_D((\Delta A \wedge \Delta B) \wedge \Delta\Lambda) - \tau_D(\Delta A \wedge \Delta\Lambda) + \tau_D(\Delta\Lambda)$. Dividing both sides by $\tau_D(\Delta\Lambda)$ to get $\tau_D((\Delta A \to \Delta B) \mid \Delta\Lambda) = \tau_D((\Delta A \wedge \Delta B) \mid \Delta\Lambda) - \tau_D(\Delta A \mid \Delta\Lambda) + 1$. □

Δ CONDITIONAL RANDOMIZED SIMILARITY DEGREE AND Δ CONDITIONAL RANDOMIZED PSEUDO-DISTANCE

Definition 12 Let $A, B \in F(S)$, D_0, $D_{\frac{1}{n-1}}$, …, $D_{\frac{n-2}{n-1}}$, D_1 $(n \geqslant 2)$ be an n-valued randomized numbers sequence in $(0,1)$, $\Lambda \in F(S)$, and $\tau_D(\Delta\Lambda) > 0$. Let

$$\xi_D((\Delta A, \Delta B) \mid \Delta\Lambda) = \tau_D(((\Delta A \to \Delta B) \wedge (\Delta B \to \Delta A)) \mid \Delta\Lambda),$$

then, $\xi_D((\Delta A, \Delta B) \mid \Delta\Lambda)$ is called the Δ conditional randomized similarity degree between Proposition 1 formulas A and B.

Theorem 9 *Let $A, B \in F(S)$, D_0, $D_{\frac{1}{n-1}}$, …, $D_{\frac{n-2}{n-1}}$, D_1 $(n \geqslant 2)$ be an n-valued randomized numbers sequence in $(0,1)$, $\Lambda \in F(S)$, and $\tau_D(\Delta\Lambda) > 0$.*
(i) If $A \approx_\Lambda B$, then $\xi_D((\Delta A, \Delta B) \mid \Delta\Lambda) = 1$;
(ii) $\xi_D((\Delta A, \Delta B) \mid \Delta\Lambda) = \xi_D((\Delta B, \Delta A) \mid \Delta\Lambda)$;
(iii) $\xi_D((\Delta A \vee \Delta B, \Delta B) \mid \Delta\Lambda) = \tau_D((\Delta A \to \Delta B) \mid \Delta\Lambda)$;
(iv) $\xi_D((\Delta A \wedge \Delta B, \Delta B) \mid \Delta\Lambda) = \tau_D((\Delta B \to \Delta A) \mid \Delta\Lambda)$.

Proof: **(i)**: It follows from $A \approx_\Lambda B$ that both $A \to B$ and $B \to A$ are tautologies based on Λ, then $(A \to B) \wedge (B \to A)$ is also a tautology based on Λ, that is $((A \to B) \wedge (B \to A)) =_\Lambda 1$. It follows from Theorem 3(i) that $\tau_D(((\Delta A \to \Delta B) \wedge (\Delta B \to \Delta A)) \mid \Delta\Lambda) = 1$, that is $\xi_D((\Delta A, \Delta B) \mid \Delta\Lambda) = 1$.

(ii): For all $a, b, c \in G_\Delta$, we have $(\Delta a \to \Delta b) \wedge (\Delta b \to \Delta a) = (\Delta b \to \Delta a) \wedge (\Delta a \to \Delta b)$. Hence, $((\Delta a \to \Delta b) \wedge (\Delta b \to \Delta a) \wedge \Delta c) = ((\Delta b \to \Delta a) \wedge (\Delta a \to \Delta b) \wedge \Delta c)$. That is, $((\Delta A \to \Delta B) \wedge (\Delta B \to \Delta A) \wedge \Delta\Lambda) = ((\Delta B \to \Delta A) \wedge (\Delta A \to \Delta B) \wedge \Delta\Lambda)$. It follows from Theorem 2(ii) that $\tau_D((\Delta A \to \Delta B) \wedge (\Delta B \to \Delta A) \wedge \Delta\Lambda) = \tau_D((\Delta B \to \Delta A) \wedge (\Delta A \to \Delta B) \wedge \Delta\Lambda)$. Dividing both sides by $\tau_D(\Delta\Lambda)$ to get $\xi_D((\Delta A, \Delta B) \mid \Delta\Lambda) = \xi_D((\Delta B, \Delta A) \mid \Delta\Lambda)$.

(iii): For all $a, b, c \in G_\Delta$, we have $(\Delta a \vee \Delta b) \to \Delta b = (\Delta a \to \Delta b) \wedge (\Delta b \to \Delta b) = \Delta a \to \Delta b$, then $\Delta b \to (\Delta a \vee \Delta b) = (\Delta b \to \Delta a) \vee (\Delta b \to \Delta b) = \Delta b \to$

Δb. Hence,

$$\begin{aligned}
&\xi_D((\Delta A \vee \Delta B, \Delta B) \mid \Delta\Lambda) \\
&= \tau_D((((\Delta A \vee \Delta B) \to \Delta B) \wedge (\Delta B \to (\Delta A \vee \Delta B))) \mid \Delta\Lambda) \\
&= \frac{\tau_D((((\Delta A \vee \Delta B) \to \Delta B) \wedge (\Delta B \to (\Delta A \vee \Delta B))) \mid \Delta\Lambda)}{\tau_D \Delta\Lambda} \\
&= \frac{\tau_D(((\Delta A \to \Delta B) \wedge (\Delta B \to \Delta B)) \wedge \Delta\Lambda)}{\tau_D(\Delta\Lambda)} \\
&= \frac{\tau_D((\Delta A \to \Delta B) \wedge \Delta\Lambda)}{\tau_D(\Delta\Lambda)} \\
&= \tau_D((\Delta A \to \Delta B) \mid \Delta\Lambda).
\end{aligned}$$

(iv): For all $a, b, c \in G_\Delta$, we have $(\Delta a \wedge \Delta b) \to \Delta b = (\Delta a \to \Delta b) \vee (\Delta b \to \Delta b) = \Delta b \to \Delta b$, then $\Delta b \to (\Delta a \wedge \Delta b) = (\Delta b \to \Delta a) \wedge (\Delta b \to \Delta b) = \Delta b \to \Delta a$. Hence,

$$\begin{aligned}
&\xi_D((\Delta A \wedge \Delta B, \Delta B) \mid \Delta\Lambda) \\
&= \tau_D((((\Delta A \wedge \Delta B) \to \Delta B) \wedge (\Delta B \to (\Delta A \wedge \Delta B))) \mid \Delta\Lambda) \\
&= \frac{\tau_D((((\Delta A \wedge \Delta B) \to \Delta B) \wedge (\Delta B \to (\Delta A \wedge \Delta B))) \mid \Delta\Lambda)}{\tau_D \Delta\Lambda} \\
&= \frac{\tau_D(((\Delta B \to \Delta B) \wedge (\Delta B \to \Delta A)) \wedge \Delta\Lambda)}{\tau_D(\Delta\Lambda)} \\
&= \frac{\tau_D((\Delta B \to \Delta A) \wedge \Delta\Lambda)}{\tau_D(\Delta\Lambda)} \\
&= \tau_D((\Delta B \to \Delta A) \mid \Delta\Lambda).
\end{aligned}$$

□

Theorem 10 *Let $A, B \in F(S)$, D_0, $D_{\frac{1}{n-1}}$, $\dots$, $D_{\frac{n-2}{n-1}}$, D_1 $(n \geqslant 2)$ be an n-valued randomized numbers sequence in $(0,1)$, $\Lambda \in F(S)$, and $\tau_D(\Delta\Lambda) > 0$. Then, $\xi_D((\Delta A, \Delta B) \mid \Delta\Lambda) = \tau_D((\Delta A \to \Delta B) \mid \Delta\Lambda) + \tau_D((\Delta B \to \Delta A) \mid \Delta\Lambda) - 1$.*

Proof: Let A, B and Λ contain the same atomic formulas $q_1, \dots, q_m$. It follows from Definition 12 and Theorem 3(iv) that $\xi_D((\Delta A, \Delta B) \mid \Delta\Lambda) = \tau_D(((\Delta A \to \Delta B) \wedge (\Delta B \to \Delta A)) \mid \Delta\Lambda) = \tau_D((\Delta A \to \Delta B) \mid \Delta\Lambda) + \tau_D((\Delta B \to \Delta A) \mid \Delta\Lambda) - \tau_D(((\Delta A \to \Delta B) \vee (\Delta B \to \Delta A)) \mid \Delta\Lambda)$. Since $\tau_D((\Delta A \to \Delta B) \vee (\Delta B \to \Delta A)) = 1$, hence, $\tau_D(((\Delta A \to \Delta B) \vee (\Delta B \to \Delta A)) \mid \Delta\Lambda) = 1$, that is, $\xi_D((\Delta A, \Delta B) \mid \Delta\Lambda) = \tau_D((\Delta A \to \Delta B) \mid \Delta\Lambda) + \tau_D((\Delta B \to \Delta A) \mid \Delta\Lambda) - 1$. □

Theorem 11 *Let $A, B, C \in F(S)$, D_0, $D_{\frac{1}{n-1}}$, $\dots$, $D_{\frac{n-2}{n-1}}$, D_1 $(n \geqslant 2)$ be an n-valued randomized numbers sequence in$(0,1)$, $\Lambda \in F(S)$, and $\tau_D(\Delta\Lambda) > 0$. Then, $\xi_D((\Delta A, \Delta C) \mid \Delta\Lambda) \geqslant \xi_D((\Delta A, \Delta B) \mid \Delta\Lambda) + \xi_D((\Delta B, \Delta C) \mid \Delta\Lambda) - 1$.*

Proof: Let A, B, C and Λ contain the same atomic formulas $q_1, \dots, q_m$. It follows from Theorem 5 and Theorem 10 that $\xi_D((\Delta A, \Delta B) \mid \Delta\Lambda) + \xi_D((\Delta B, \Delta C) \mid \Delta\Lambda) - 1 = [\tau_D((\Delta A \to \Delta B) \mid \Delta\Lambda) + \tau_D((\Delta B \to \Delta A) \mid \Delta\Lambda) - 1] + [\tau_D((\Delta B \to \Delta C) \mid \Delta\Lambda) + \tau_D((\Delta C \to \Delta B) \mid \Delta\Lambda) - 1] = [\tau_D((\Delta A \to \Delta B) \mid \Delta\Lambda) + \tau_D((\Delta B \to \Delta C) \mid \Delta\Lambda) - 1] + [\tau_D((\Delta C \to \Delta B) \mid \Delta\Lambda) + \tau_D((\Delta B \to \Delta A) \mid \Delta\Lambda) - 1] \leqslant \tau_D((\Delta A \to \Delta C) \mid \Delta\Lambda) + \tau_D((\Delta A \to \Delta C) \mid \Delta\Lambda) - 1 = \xi_D((\Delta A, \Delta C) \mid \Delta\Lambda)$. □

Definition 13 Let $A, B \in F(S)$, D_0, $D_{\frac{1}{n-1}}$, $\dots$, $D_{\frac{n-2}{n-1}}$, D_1 $(n \geqslant 2)$ be an n-valued randomized numbers sequence in $(0,1)$, $\Lambda \in F(S)$, and $\tau_D(\Delta\Lambda) > 0$, regulation $\rho_0 : F(S) \times F(S) \to [0,1]$. Let

$$\rho_D((\Delta A, \Delta B) \mid \Delta\Lambda) = 1 - \xi_D((\Delta A, \Delta B) \mid \Delta\Lambda).$$

The $\rho_D((\Delta A, \Delta B) \mid \Delta\Lambda)$ is called the Δ conditional randomized pseudo-distance on $F(S)$, and $(F(S), \rho_D)$ is called the Δ conditional randomized logical metric space.

Theorem 12 *Let $A, B \in F(S)$, D_0, $D_{\frac{1}{n-1}}$, $\dots$, $D_{\frac{n-2}{n-1}}$, D_1 $(n \geqslant 2)$ be an n-valued randomized numbers sequence in $(0,1)$, $\Lambda \in F(S)$, and $\tau_D(\Delta\Lambda) > 0$.*
(i) *If $A \approx_\Lambda B$, then $\rho_D((\Delta A, \Delta B) \mid \Delta\Lambda) = 0$;*
(ii) *$\rho_D((\Delta A, \Delta B) \mid \Delta\Lambda) = \rho_D((\Delta B, \Delta A) \mid \Delta\Lambda)$;*
(iii) *$\rho_D((\Delta A \vee \Delta B, \Delta B) \mid \Delta\Lambda) = 1 - \tau_D((\Delta A \to \Delta B) \mid \Delta\Lambda)$;*
(iv) *$\rho_D((\Delta A \wedge \Delta B, \Delta B) \mid \Delta\Lambda) = 1 - \tau_D((\Delta B \to \Delta A) \mid \Delta\Lambda)$;*

Proof: It follows from Theorem 9. □

Theorem 13 *Let $A, B \in F(S)$, D_0, $D_{\frac{1}{n-1}}$, $\dots$, $D_{\frac{n-2}{n-1}}$, D_1 $(n \geqslant 2)$ be an n-valued randomized numbers sequence in $(0,1)$, $\Lambda \in F(S)$, and $\tau_D(\Delta\Lambda) > 0$. Then, $\rho_D((\Delta A, \Delta B) \mid \Delta\Lambda) = 2 - \tau_D((\Delta A \to \Delta B) \mid \Delta\Lambda) - \tau_D((\Delta B \to \Delta A) \mid \Delta\Lambda)$.*

Proof: It follows from Theorem 10. □

Theorem 14 *Let $A, B, C \in F(S)$, D_0, $D_{\frac{1}{n-1}}$, $\dots$, $D_{\frac{n-2}{n-1}}$, D_1 $(n \geqslant 2)$ be an n-valued randomized numbers sequence in $(0,1)$, $\Lambda \in F(S)$, and $\tau_D(\Delta\Lambda) > 0$. Then $\rho_D((\Delta A, \Delta C) \mid \Delta\Lambda) \leqslant \rho_D((\Delta A, \Delta B) \mid \Delta\Lambda) + \rho_D((\Delta B, \Delta C) \mid \Delta\Lambda)$.*

Proof: It follows from Theorem 11. □

CONCLUSION

In this paper, Δ conditional randomized truth degree of propositional formula is put forward in Gödel n-valued propositional logic system. It adds Δ operator compared with conditional randomized truth degree. On this basis, some inference rules such as MP, HS, intersection inference, union inference and their related properties are studied. At last, the concepts of Δ conditional randomized similarity degree, Δ conditional randomized pseudo-metric between propositional formulas are given, and their related good properties are discussed. Thus, how to further develop the approximate reasoning and topological properties in the Δ conditional randomized metric space of the Gödel n-valued propositional logic system with the addition of Δ operator will be discussed in another paper.

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