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Local derivations and 2-local Lie derivations of triangular algebras

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ABSTRACT: Let $\mathscr{T}=\mathrm{Tri}(A,M,B)$ be a triangular algebra. In this paper, we prove that under certain conditions, every local derivation from $\mathscr{T}$ into itself is a derivation; every additive 2-local Lie derivation from $\mathscr{T}$ into itself is a Lie derivation.

KEYWORDS: derivation, local derivation, 2-local Lie derivation, triangular algebra

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INTRODUCTION

The local derivations problem, initiated by Kadison [1] and Larson and Sourour [2], is to find conditions implying that a local derivation is a derivation. Let R be a commutative ring with identity. Suppose that A is a unital algebra over R and M be an A-bimodule. We say that a linear map $\varphi: A\to M$ is a derivation if $\varphi(ab)=\varphi(a)b+a\varphi(b)$ for all $a,b\in A$; and an inner derivation if there exists $x\in A$ such that $\varphi(a)=xa-ax$ for all $a\in A$.

In [3], Christensen has proved that each derivation d of nest algebras on a Hilbert space H is an inner derivation. In [4], Hou and Han have proved that every derivation of CSL algebras on Banach spaces is continuous and obtained that additive derivations of nest algebras on Banach spaces are inner derivations.

A linear map $\varphi: A\to M$ is called a local derivation, if for every $a\in A$, there exists a derivation φ_a of A, depending on a, such that $\varphi(a)=\varphi_a(a)$. The relationship between local derivations and derivations on self-adjoint algebras or non-self-adjoint algebras has been discussed by many authors, see [5–12]. In [1], Kadison has proved that every norm-continuous local derivation from a von Neumann algebra into its dual normal bimodule is a derivation. A similar result for local derivatins on $B(X)$ was obtained in [2], where $B(X)$ is an algebra of all bounded linear operators on a Banach space X. In [9], Hadwin and Li investigated bounded local derivations of certain CSL algebras. In [13], Alizadeh and Bitarafan have proved that if $\varphi: M_n(R)\to M_n(M)$ is a local derivation, then φ is a derivation for $n\geqslant 3$.

A linear map φ from an algebra A into an A-bimodule M is called a Lie derivation if $\varphi([a,b])=[\varphi(a),b]+[a,\varphi(b)]$ for all $a,b\in A$, where $[a,b]=ab-ba$ is the usual Lie product, also called a commutator. A Lie derivation φ is standard if it can be decomposed as $\varphi=d+\tau$, where d is a derivation from A into M and τ is a linear map from A into the relative center of M vanishing on each commutator. The classical problem, which has been studied for many years, is to find conditions on A under which each Lie derivation is standard or standard-like. This problem has been investigated for general operator algebras [14].

In [15], Semrl introduced the concepts of 2-local maps. A map φ of an algebra A is called a 2-local Lie derivation if for each $a,b\in A$, there exists a Lie derivation $\varphi_{a,b}$ such that $\varphi(a)=\varphi_{a,b}(a)$ and $\varphi(b)=\varphi_{a,b}(b)$. In [16], Chen, Lu and Wang have proved that each 2-local Lie derivation of $B(X)$, where X is a Banach space of dimension $\geqslant 2$, is a Lie derivation. Later, in [17], Liu has proved that each additive 2-local Lie derivation of nest subalgebras of factors is a Lie derivation which is standard, and provided an example to show the additivity of 2-local Lie derivations is necessary. In [10], Yang investigated 2-local Lie derivation on a von Neumann algebra without central summands of type I_1 and the results showed that every 2-local Lie derivation $\varphi: A\to A$ can be decomposed as $\varphi=d+\tau$, where d is an inner derivation of a finite von Neumann algebra without central summands of type I_1 and τ is a homogeneous map of A vanishing on each commutator. The purpose of the present paper is to study local derivations and 2-local Lie derivations of triangular algebras.

Suppose that A and B are unital algebras over R, with unit 1_A and 1_B, respectively; and M is a unital (A,B)-bimodule. We assume that M is faithful as a left A-module and also as a right B-module. Under the usual matrix operations,

$$\mathrm{Tri}(A,M,B)=\left\{\begin{pmatrix} a & m\\ 0 & b\end{pmatrix}: a\in A, m\in M, b\in B\right\}$$

is called a triangular algebra. The main examples of triangular algebras are nest algebras and (block) upper triangular matrix algebras. For more details, see [14, 18]. The local derivation problems on triangular algebras have been studied extensively (see [19]).

Let $Z(\mathscr{T})$ be the center of $\mathscr{T}$. It follows from [20, Proposition 3] that

$$Z(\mathscr{T})=\left\{\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}: am=mb \quad \text{for all} \quad m\in M\right\}.$$

Let $\pi_A:\mathscr{T}\to A$ and $\pi_B:\mathscr{T}\to B$ be two maps defined by

$$\pi_A:\begin{pmatrix} a & m\\ 0 & b\end{pmatrix}\mapsto a \quad \text{and} \quad \pi_B:\begin{pmatrix} a & m\\ 0 & b\end{pmatrix}\mapsto b.$$

Furthermore, $\pi_A(Z(\mathscr{T}))\subseteq Z(A)$ and $\pi_B(Z(\mathscr{T}))\subseteq Z(B)$, and there exists a unique algebra isomorphism η from $\pi_A(Z(\mathscr{T}))$ to $\pi_B(Z(\mathscr{T}))$ such that $am=m\eta(a)$ for all $m\in M$ (see [20]).

Consider a triangular algebra $\mathscr{T}=\mathrm{Tri}(A,M,B)$. Let 1 be the identity of $\mathscr{T}$. Set

$$p_1=\begin{pmatrix} 1_A & 0\\ 0 & 0\end{pmatrix},\quad p_2=1-p_1=\begin{pmatrix} 0 & 0\\ 0 & 1_B\end{pmatrix}$$

and

$$\mathscr{T}_{ij}=p_i\mathscr{T}p_j \quad \text{for all } 1\leqslant i\leqslant j\leqslant 2.$$

It is clear that $\mathscr{T}$ can be represented as

$$\mathscr{T}=\mathscr{T}_{11}\oplus\mathscr{T}_{12}\oplus\mathscr{T}_{22}.$$

In this paper, the subalgebra of A generated by all idempotents in A will be denoted by $\mathscr{J}(A)$.

LOCAL DERIVATIONS

Our main result is the following theorem.

Theorem 1 *Let $\mathscr{T}=\mathrm{Tri}(A,M,B)$ be a triangular algebra. If $A=\mathscr{J}(A)$ and $B=\mathscr{J}(B)$, then every local derivation $\varphi:\mathscr{T}\to\mathscr{T}$ is a derivation.*

By the condition of $A=\mathscr{J}(A)$ and $B=\mathscr{J}(B)$, we can obtain that every $a_{kk}\in\mathscr{T}_{kk}$ can be written as a linear combination of some elements $a_{kk}^{(i_1)}a_{kk}^{(i_2)}\cdots a_{kk}^{(i_{n_i})}$ ($i=1,2,\dots,m$), where $a_{kk}^{(i_1)},a_{kk}^{(i_2)},\dots,a_{kk}^{(i_{n_i})}$ are idempotents in $\mathscr{T}_{kk}$ ($k=1,2$).

In the following, φ is a local derivation and, for any $x\in\mathscr{T}$, the symbol φ_x stands for a derivation from $\mathscr{T}$ into itself such that $\varphi(x)=\varphi_x(x)$.

To prove our main theorem, we need the following lemmas.

Lemma 1 *For every idempotents $p,q\in\mathscr{T}$ and $x\in\mathscr{T}$, we have $\varphi(pxq)=\varphi(px)q+p\varphi(xq)-p\varphi(x)q$.*

The proof of the Lemma 1 is similar to [21, Lemma 3.2].

Lemma 2 *For any $a_{ij}\in\mathscr{T}_{ij}(1\leqslant i\leqslant j\leqslant 2)$, we have*
(i) $\varphi(p_1),\varphi(a_{12})\in\mathscr{T}_{12}$;
(ii) $\varphi(a_{11})=p_1\varphi(a_{11})p_1+a_{11}\varphi(p_1)p_2$, $\varphi(a_{22})=-p_1\varphi(p_1)a_{22}+p_2\varphi(a_{22})p_2$.

Proof: **(i)**: It follows from $\varphi(p_1)=\varphi_{p_1}(p_1)=\varphi(p_1)p_1+p_1\varphi(p_1)$ that $p_1\varphi(p_1)p_1=p_2\varphi(p_1)p_2=0$. So, $\varphi(p_1)=p_1\varphi(p_1)p_2\in\mathscr{T}_{12}$. For any $a_{12}\in\mathscr{T}_{12}$, we have

$$0=\varphi_{a_{12}}(a_{12}p_1)=\varphi(a_{12})p_1+a_{12}\varphi_{a_{12}}(p_1)$$

and

$$\varphi(a_{12})=\varphi_{a_{12}}(p_1a_{12})=\varphi_{a_{12}}(p_1)a_{12}+p_1\varphi(a_{12}).$$

This implies that $p_1\varphi(a_{12})p_1=0$, $p_2\varphi(a_{12})p_2=0$. Hence $\varphi(a_{12})=p_1\varphi(a_{12})p_2\in\mathscr{T}_{12}$.

(ii): Let $b_{11}\in\mathscr{T}_{11}$ and $a_{12}\in\mathscr{T}_{12}$. Taking $p=a_{11}^{(1)}$, $x=b_{11}$ and $q=a_{12}+p_1$ in Lemma 1, we know that

$$\begin{aligned}\varphi(a_{11}^{(1)}&b_{11}(p_1+a_{12}))\\ &=\varphi(a_{11}^{(1)}b_{11})(p_1+a_{12})+a_{11}^{(1)}\varphi(b_{11}+b_{11}a_{12})\\ &\quad -a_{11}^{(1)}\varphi(b_{11})(p_1+a_{12})\\ &=\varphi(a_{11}^{(1)}b_{11})p_1+\varphi(a_{11}^{(1)}b_{11})a_{12}+a_{11}^{(1)}\varphi(b_{11})p_2\\ &\quad +a_{11}^{(1)}\varphi(b_{11}a_{12})-a_{11}^{(1)}\varphi(b_{11})a_{12}. \end{aligned} \tag{1}$$

Taking $a_{12}=0$ in Eq. (1), we have

$$\varphi(a_{11}^{(1)}b_{11})p_2=a_{11}^{(1)}\varphi(b_{11})p_2. \tag{2}$$

In particular,

$$\varphi(a_{11}^{(1)})p_2=a_{11}^{(1)}\varphi(p_1)p_2. \tag{3}$$

By Eqs. (2)–(3), then

$$\begin{aligned}\varphi(a_{11}^{(1)}a_{11}^{(2)}\cdots a_{11}^{(n)})p_2&=a_{11}^{(1)}\varphi(a_{11}^{(2)}\cdots a_{11}^{(n)})p_2\\ &=a_{11}^{(1)}a_{11}^{(2)}\cdots a_{11}^{(n-1)}\varphi(a_{11}^{(n)})p_2\\ &=a_{11}^{(1)}a_{11}^{(2)}\cdots a_{11}^{(n)}\varphi(p_1)p_2\end{aligned}$$

for any idempotents $a_{11}^{(1)},a_{11}^{(2)},\dots,a_{11}^{(n)}\in\mathscr{T}_{11}$. By $A=\mathscr{J}(A)$, we know that $\varphi(a_{11})p_2=a_{11}\varphi(p_1)p_2$ for all $a_{11}\in\mathscr{T}_{11}$. Thus $\varphi(a_{11})=p_1\varphi(a_{11})p_1+a_{11}\varphi(p_1)p_2$ for all $a_{11}\in\mathscr{T}_{11}$. Similarly, we can obtain from Lemma 1 and the fact $\varphi(1)=0$ that

$$p_1\varphi(a_{22})=p_1\varphi(p_2)a_{22}=-p_1\varphi(p_1)a_{22}$$

for all $a_{22}\in\mathscr{T}_{22}$. Hence $\varphi(a_{22})=-p_1\varphi(p_1)a_{22}+p_2\varphi(a_{22})p_2$ for all $a_{22}\in\mathscr{T}_{22}$. □

Next, we define $\delta:\mathscr{T}\to\mathscr{T}$ by $\delta(x)=\varphi(x)-[x,\varphi(p_1)]$. Then δ is also a local derivation and by Lemma 2.2, $\delta(p_1)=0$ and $\delta(\mathscr{T}_{ij})\subseteq\mathscr{T}_{ij}$ for $1\leqslant i\leqslant j\leqslant 2$.

Lemma 3 (i) $\delta(a_{11}a_{12}) = \delta(a_{11})a_{12} + a_{11}\delta(a_{12})$ *for all* $a_{11} \in \mathcal{T}_{11}$ *and* $a_{12} \in \mathcal{T}_{12}$*;*
(ii) $\delta(a_{12}a_{22}) = \delta(a_{12})a_{22} + a_{12}\delta(a_{22})$ *for all* $a_{12} \in \mathcal{T}_{12}$ *and* $a_{22} \in \mathcal{T}_{22}$*.*

Proof: **(i)**: To prove this statement, it is sufficient to prove that

$$\delta(a_{11}^{(1)}a_{11}^{(2)}\cdots a_{11}^{(n)}a_{12}) = \delta(a_{11}^{(1)}a_{11}^{(2)}\cdots a_{11}^{(n)})a_{12} + a_{11}^{(1)}a_{11}^{(2)}\cdots a_{11}^{(n)}\delta(a_{12}) \quad (4)$$

for any idempotent $a_{11}^{(1)}, a_{11}^{(2)}, \dots, a_{11}^{(n)} \in \mathcal{T}_{11}$ and $a_{12} \in \mathcal{T}_{12}$. Eqs. (1) and (2) imply that

$$\delta(a_{11}^{(1)}b_{11}a_{12}) = \delta(a_{11}^{(1)}b_{11})a_{12} + a_{11}^{(1)}\delta(b_{11}a_{12}) - a_{11}^{(1)}\delta(b_{11})a_{12}. \quad (5)$$

Taking $b_{11} = p_1$ in Eq. (5), we have from Lemma 2(i) that

$$\delta(a_{11}^{(1)}a_{12}) = \delta(a_{11}^{(1)})a_{12} + a_{11}^{(1)}\delta(a_{12}).$$

This implies that Eq. (4) is true for $n = 1$. Suppose that Eq. (4) is true for $n = k-1$. Then for $n = k$, we have from Eq. (5) that

$$\begin{aligned}
&\delta(a_{11}^{(1)}a_{11}^{(2)}\cdots a_{11}^{(k)}a_{12}) \\
&= \delta(a_{11}^{(1)}a_{11}^{(2)}\cdots a_{11}^{(k)})a_{12} + a_{11}^{(1)}\delta(a_{11}^{(2)}\cdots a_{11}^{(k)}a_{12}) \\
&\quad - a_{11}^{(1)}\delta(a_{11}^{(2)}\cdots a_{11}^{(k)})a_{12} \\
&= \delta(a_{11}^{(1)}a_{11}^{(2)}\cdots a_{11}^{(k)})a_{12} + a_{11}^{(1)}\delta(a_{11}^{(2)}\cdots a_{11}^{(k)})a_{12} \\
&\quad + a_{11}^{(1)}a_{11}^{(2)}\cdots a_{11}^{(k)}\delta(a_{12}) - a_{11}^{(1)}\delta(a_{11}^{(2)}\cdots a_{11}^{(k)})a_{12} \\
&= \delta(a_{11}^{(1)}a_{11}^{(2)}\cdots a_{11}^{(k)})a_{12} + a_{11}^{(1)}a_{11}^{(2)}\cdots a_{11}^{(k)}\delta(a_{12}).
\end{aligned}$$

Thus Eq. (4) is true for all n.

The statement (ii) can be proven with a similar calculation. □

Lemma 4 (i) $\delta(a_{11}b_{11}) = \delta(a_{11})b_{11} + a_{11}\delta(b_{11})$ *for all* $a_{11}, b_{11} \in \mathcal{T}_{11}$*;*
(ii) $\delta(a_{22}b_{22}) = \delta(a_{22})b_{22} + a_{22}\delta(b_{22})$ *for all* $a_{22}, b_{22} \in \mathcal{T}_{22}$*.*

Proof: Let $a_{11}, b_{11} \in \mathcal{T}_{11}$. For any $c_{12} \in \mathcal{T}_{12}$, by Lemma 2.3, on one hand, we have

$$\begin{aligned}
\delta(a_{11}b_{11}c_{12}) &= \delta(a_{11})b_{11}c_{12} + a_{11}\delta(b_{11}c_{12}) \\
&= \delta(a_{11})b_{11}c_{12} + a_{11}\delta(b_{11})c_{12} + a_{11}b_{11}\delta(c_{12}).
\end{aligned}$$

On the other hand,

$$\delta(a_{11}b_{11}c_{12}) = \delta(a_{11}b_{11})c_{12} + a_{11}b_{11}\delta(c_{12}).$$

Comparing these two equalities, we get

$$\{\delta(a_{11}b_{11}) - \delta(a_{11})b_{11} - a_{11}\delta(b_{11})\}c_{12} = 0$$

for any $c_{12} \in \mathcal{T}_{12}$. Since $\mathcal{T}_{12}$ is a faithful left $\mathcal{T}_{11}$ module, we get that $\delta(a_{11}b_{11}) = \delta(a_{11})b_{11} + a_{11}\delta(b_{11})$.

Similarly, we can obtain $\delta(a_{22}b_{22}) = \delta(a_{22})b_{22} + a_{22}\delta(b_{22})$. □

Proof of Theorem 1

Proof: For any $x, y \in \mathcal{T}$, then

$$x = a_{11} + a_{12} + a_{22}, \quad y = b_{11} + b_{12} + b_{22},$$

where $a_{ij}, b_{ij} \in \mathcal{T}_{ij}$. Considering Lemmas 3 and 4 we have that

$$\begin{aligned}
\delta(xy) &= \delta((a_{11} + a_{12} + a_{22})(b_{11} + b_{12} + b_{22})) \\
&= \delta(a_{11}b_{11}) + \delta(a_{11}b_{12}) + \delta(a_{12}b_{22}) + \delta(a_{22}b_{22}) \\
&= \delta(a_{11})b_{11} + a_{11}\delta(b_{11}) + \delta(a_{11})b_{12} + a_{11}\delta(b_{12}) \\
&\quad + \delta(a_{12})b_{22} + a_{12}\delta(b_{22}) + \delta(a_{22})b_{22} + a_{22}\delta(b_{22}).
\end{aligned}$$

On the other hand, we have from $\delta(\mathcal{T}_{ij}) \subseteq \mathcal{T}_{ij}$ that

$$\begin{aligned}
\delta(x)y + x\delta(y) &= \delta(a_{11} + a_{12} + a_{22})(b_{11} + b_{12} + b_{22}) \\
&\quad + (a_{11} + a_{12} + a_{22})\delta(b_{11} + b_{12} + b_{22}) \\
&= \delta(a_{11})b_{11} + a_{11}\delta(b_{11}) + \delta(a_{11})b_{12} + a_{11}\delta(b_{12}) \\
&\quad + \delta(a_{12})b_{22} + a_{12}\delta(b_{22}) + \delta(a_{22})b_{22} + a_{22}\delta(b_{22}).
\end{aligned}$$

Hence $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{T}$, i.e., δ is a derivation. The proof is complete. □

As a consequence of Theorem 1, we have the followings.

Let $M_{n\times k}(R)$ be the set of all $n \times k$ matrices over R, where R is a commutative ring with unit 1. For $n \geqslant 2$ and $m \leqslant n$, the block upper triangular matrix algebra $T_n^{\bar{k}}(R)$ is a subalgebra of $M_n(R)$ of the form

$$\begin{pmatrix}
M_{k_1}(R) & M_{k_1\times k_2}(R) & \cdots & M_{k_1\times k_m}(R) \\
0 & M_{k_2}(R) & \cdots & M_{k_2\times k_m}(R) \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & M_{k_m}(R)
\end{pmatrix}$$

where $\bar{k} = (k_1, k_2, \dots, k_m) \in \mathbb{N}^m$ is an ordered m-vector of positive integers such that $k_1 + k_2 + \cdots + k_m = n$.

Corollary 1 *Let* $T_n^{\bar{k}}(R)$ *be a block upper triangular matrix algebra. If* $\Delta : T_n^{\bar{k}}(R) \to T_n^{\bar{k}}(R)$ *is a local derivation, then there exists a derivation* d *such that* $\Delta(x) = d(x) + [x, \Delta(p_1)]$*, thus* Δ *is a derivation.*

Let A be a unital algebra over a field. By [6], we know that the matrix algebra $M_n(A)$ is generated by its idempotents for $n \geqslant 2$.

Corollary 2 *Let* A *and* B *be unital algebras over the field* $\mathbb{F}$ *and let* W *be a unital faithful* (A, B)*-bimodule. If a linear map* $\Delta : \mathrm{T}ri(M_n(A), M_{n\times k}(W), M_k(B)) \to \mathrm{T}ri(M_n(A), M_{n\times k}(W), M_k(B))$ *is a local derivation, then it is a derivation for* $n, k \geqslant 2$*.*

Let A and B be norm closed unital subalgebras of $B(H)$ and $B(K)$ respectively. In [3, 22], Gilfeather and Smith defined an operator algebra analog $A\sharp B$, which is called the join of A and B, as a subalgebra of $B(H \oplus K)$ of the form

$$\begin{pmatrix} A & 0 \\ B(H, K) & B \end{pmatrix}.$$

Corollary 3 *If $A=\mathscr{J}(A)$ and $B=\mathscr{J}(B)$, then every local derivation of $A\sharp B$ is a derivation.*

Next, we will give an example to show that the condition the algebras A and B are generated by all its idempotents is indispensable in Theorem 1.

Example 1 We denote by $\{e_{ij}\}$ the standard matrix units of $M_3(\mathbb{C})$. Let $A=\text{span}\{e_{11}+e_{22},e_{12}\}$, $B=\text{span}\{e_{33}\}$, $M=\text{span}\{e_{13},e_{23}\}$. Set $\mathscr{T}=\text{Tri}(A,M,B)$, then $\mathscr{T}$ is a triangular algebra. One can verify that $A\neq\mathscr{J}(A)$.

Let $\varphi:\mathscr{T}\to\mathscr{T}$ be a linear map. We can show that φ is a derivation if there exist scalars $\lambda_i\in\mathbb{C}(i=1,2,3,4)$ such that $\varphi(I)=0,\varphi(e_{12})=\lambda_1 e_{12},\varphi(e_{23})=\lambda_2 e_{13}+\lambda_3 e_{23}$ and $\varphi(e_{13})=(\lambda_1+\lambda_3)e_{13}$. For each $x=(a_{ij})$ in $\mathscr{T}$, we define $\Delta(x)=(2a_{13}-a_{23})e_{13}+a_{12}e_{12}$. One can verify that Δ is a linear map of $\mathscr{T}$ onto itself, and

$$\Delta(I)=0,\ \Delta(e_{12})=e_{12},\ \Delta(e_{23})=-e_{13},\ \Delta(e_{13})=2e_{13}.$$

If $a_{23}\neq 0$, let φ_1 be a linear map with $\varphi_1(I)=0,\varphi_1(e_{12})=e_{12}$, $\varphi_1(e_{23})=(a_{23}^{-1}a_{13}-1)e_{13}$, and $\varphi_1(e_{13})=e_{13}$. Then, φ_1 is a derivation of $\mathscr{T}$. By the definition of Δ, we can obtain that $\Delta(x)=\varphi_1(x)$. If $a_{23}=0$, let φ_2 be a linear map with $\varphi_2(I)=0,\varphi_2(e_{12})=e_{12}$, $\varphi_2(e_{23})=e_{23}$, and $\varphi_2(e_{13})=2e_{13}$. Then, φ_2 is a derivation of $\mathscr{T}$. The definition of Δ implies that $\Delta(x)=\varphi_2(x)$. Therefore, Δ is a local derivation of $\mathscr{T}$. Let $x=e_{12}$ and $y=e_{12}+e_{23}$, we have

$$\Delta(xy)\neq\Delta(x)y+x\Delta(y).$$

We conclude that Δ is a local derivation, which is not a derivation of $\mathscr{T}$.

2-LOCAL LIE DERIVATIONS

Proposition 1 ([7]) *Let $\mathscr{T}=\text{Tri}(A,M,B)$ be a triangular algebra. If $Z(A)=\pi_A(Z(\mathscr{T}))$ and $Z(B)=\pi_B(Z(\mathscr{T}))$, then every Lie derivation $\varphi:\mathscr{T}\to\mathscr{T}$ is standard, that is, φ is the sum of a derivation d and a linear central-valued map τ vanishing on each commutator.*

In this section, our main result is the following theorem.

Theorem 2 *Let $\mathscr{T}=\text{Tri}(A,M,B)$ be a triangular algebra. Suppose that $Z(A)=\pi_A(Z(\mathscr{T}))$ and $Z(B)=\pi_B(Z(\mathscr{T}))$. Then every additive 2-local Lie derivation φ from $\mathscr{T}$ into itself is a Lie derivation.*

To prove Theorem 2, we need some lemmas. In the following, for any $x,y\in\mathscr{T}$, the symbol $\varphi_{x,y}$ stands for a Lie derivation from $\mathscr{T}$ into itself such that $\varphi(x)=\varphi_{x,y}(x),\varphi(y)=\varphi_{x,y}(y)$.

Lemma 5 *φ is homogeneous and $\varphi(0)=0$.*

Proof: For $x\in\mathscr{T}$, $\lambda\in\mathscr{F}$, there exists a Lie derivation $\varphi_{x,\lambda x}$ such that

$$\varphi(\lambda x)=\varphi_{x,\lambda x}(\lambda x)=\lambda\varphi_{x,\lambda x}(x)=\lambda\varphi(x).$$

So φ is homogeneous and then $\varphi(0)=0\varphi(0)=0$. □

Lemma 6 *For any $a_{ij}\in\mathscr{T}_{ij}(1\leqslant i\leqslant j\leqslant 2)$, we have $p_1\varphi(p_1)p_1+p_2\varphi(p_1)p_2\in Z(\mathscr{T})$ and $\varphi(a_{12})\in\mathscr{T}_{12}$.*

Proof: For any $a_{12}\in\mathscr{T}_{12}$, we have

$$\begin{aligned}\varphi(a_{12})&=\varphi_{p_1,a_{12}}([p_1,a_{12}])\\&=[\varphi_{p_1,a_{12}}(p_1),a_{12}]+[p_1,\varphi_{p_1,a_{12}}(a_{12})]\\&=\varphi(p_1)a_{12}-a_{12}\varphi(p_1)+p_1\varphi(a_{12})p_2.\end{aligned}$$

This implies that $p_1\varphi(p_1)a_{12}=a_{12}\varphi(p_1)p_2$, and so

$$p_1\varphi(p_1)p_1+p_2\varphi(p_1)p_2\in Z(\mathscr{T}).$$

Comparing the above two equations, we get $\varphi(a_{12})=p_1\varphi(a_{12})p_2\in\mathscr{T}_{12}$. □

In the sequel, we define $\phi(x)=\varphi(x)-[x,p_1\varphi(p_1)p_2]$. One can verify that ϕ is also a 2-local Lie derivation. Moreover, by Lemma 6, we have $\phi(p_1)=p_1\varphi(p_1)p_1+p_2\varphi(p_1)p_2\in Z(\mathscr{T})$ and $\phi(a_{12})=\varphi(a_{12})\in\mathscr{T}_{12}$.

Lemma 7 *There exists a linear map $\tau_i:\mathscr{T}_{ii}\to Z(\mathscr{T})$ such that $\phi(a_{ii})-\tau_i(a_{ii})\in\mathscr{T}_{ii}$.*

Proof: For any $a_{ij}\in\mathscr{T}_{ij}$, $1\leqslant i,j\leqslant 2$, we have

$$\begin{aligned}0&=\phi_{p_1,a_{11}}([p_1,a_{11}])\\&=[\phi_{p_1,a_{11}}(p_1),a_{11}]+[p_1,\phi_{p_1,a_{11}}(a_{11})]\\&=[\phi(p_1),a_{11}]+[p_1,\phi(a_{11})]\\&=p_1\phi(a_{11})-\phi(a_{11})p_1.\end{aligned}$$

This implies that $\phi(a_{11})=p_1\phi(a_{11})p_1+p_2\phi(a_{11})p_2$.

Similarly, we can obtain $\phi(a_{22})=p_1\phi(a_{22})p_1+p_2\phi(a_{22})p_2$.

Noting that

$$\begin{aligned}0&=\phi_{a_{11},a_{22}}([a_{11},a_{22}])\\&=[\phi_{a_{11},a_{22}}(a_{11}),a_{22}]+[a_{11},\phi_{a_{11},a_{22}}(a_{22})]\\&=[\phi(a_{11}),a_{22}]+[a_{11},\phi(a_{22})]\\&=\phi(a_{11})a_{22}-a_{22}\phi(a_{11})+a_{11}\phi(a_{22})-\phi(a_{22})a_{11}.\end{aligned}$$

Comparing the above three equations, we get $p_2\phi(a_{11})a_{22}=a_{22}\phi(a_{11})p_2$ and $p_1\phi(a_{22})a_{11}=a_{11}\phi(a_{22})p_1$. Thus, $p_2\phi(a_{11})p_2\in Z(\mathscr{T}_{22})$ and $p_1\phi(a_{22})p_1\in Z(\mathscr{T}_{11})$.

By the hypothesis of Theorem 2, there exists a unique algebra isomorphism $\eta:Z(\mathscr{T}_{22})\to Z(\mathscr{T}_{11})$ such that $\eta(a_{22})\oplus a_{22}\in Z(\mathscr{T})$ for any $a_{22}\in Z(\mathscr{T}_{22})$. For each

$a_{11} \in \mathscr{T}_{11}$, we define $\tau_1 : \mathscr{T}_{11} \to Z(\mathscr{T})$ by $\tau_1(a_{11}) = \eta(p_2\phi(a_{11})p_2) \oplus p_2\phi(a_{11})p_2$. Thus we get

$$\begin{aligned}\phi(a_{11}) - \tau_1(a_{11}) &= p_1\phi(a_{11})p_1 + p_2\phi(a_{11})p_2 \\ &\quad - \eta(p_2\phi(a_{11})p_2) - p_2\phi(a_{11})p_2 \\ &= p_1\phi(a_{11})p_1 - \eta(p_2\phi(a_{11})p_2) \in \mathscr{T}_{11}.\end{aligned}$$

Similarly, we can define a linear map $\tau_2 : \mathscr{T}_{22} \to Z(\mathscr{T})$ by $\tau_2(a_{22}) = p_1\phi(a_{22})p_1 \oplus \eta^{-1}(p_1\phi(a_{22})p_1)$. Then

$$\begin{aligned}\phi(a_{22}) - \tau_2(a_{22}) &= p_1\phi(a_{22})p_1 + p_2\phi(a_{22})p_2 \\ &\quad - p_1\phi(a_{22})p_1 - \eta^{-1}(p_1\phi(a_{22})p_1) \\ &= p_2\phi(a_{22})p_2 - \eta^{-1}(p_1\phi(a_{22})p_1) \in \mathscr{T}_{22}.\end{aligned}$$

Now for any $x \in \mathscr{T}$, we define $\tau : \mathscr{T} \to Z(\mathscr{T})$ and $\delta : \mathscr{T} \to \mathscr{T}$ by

$$\tau(x) = \tau_1(p_1xp_1) + \tau_2(p_2xp_2) \text{ and } \delta(x) = \phi(x) - \tau(x).$$

It is easy to verify that $\delta(\mathscr{T}_{ij}) \subseteq \mathscr{T}_{ij}$ for $1 \leqslant i, j \leqslant 2$ and $\delta(a_{12}) = \phi(a_{12})$ for all $a_{12} \in \mathscr{T}_{12}$. □

Lemma 8 (i) $\delta(a_{11}a_{12}) = \delta(a_{11})a_{12} + a_{11}\delta(a_{12})$ *for all* $a_{11} \in \mathscr{T}_{11}$ *and* $a_{12} \in \mathscr{T}_{12}$;
(ii) $\delta(a_{12}a_{22}) = \delta(a_{12})a_{22} + a_{12}\delta(a_{22})$ *for all* $a_{12} \in \mathscr{T}_{12}$ *and* $a_{22} \in \mathscr{T}_{22}$.

Proof: **(i)**: Let $a_{11} \in \mathscr{T}_{11}, a_{12} \in \mathscr{T}_{12}$. There exists a Lie derivation $\delta_{p_1+a_{12}, a_{11}+a_{11}a_{12}}$ such that

$$\begin{aligned}\delta(p_1 + a_{12}) &= \delta_{p_1+a_{12},a_{11}+a_{11}a_{12}}(p_1 + a_{12}), \\ \delta(a_{11} + a_{11}a_{12}) &= \delta_{p_1+a_{12},a_{11}+a_{11}a_{12}}(a_{11} + a_{11}a_{12}).\end{aligned}$$

So it follows from $[p_1 + a_{12}, a_{11} + a_{11}a_{12}] = 0$ that

$$\begin{aligned}0 &= \delta_{p_1+a_{12},a_{11}+a_{11}a_{12}}([p_1 + a_{12}, a_{11} + a_{11}a_{12}]) \\ &= [\delta_{p_1+a_{12},a_{11}+a_{11}a_{12}}(p_1 + a_{12}), a_{11} + a_{11}a_{12}] \\ &\quad + [p_1 + a_{12}, \delta_{p_1+a_{12},a_{11}+a_{11}a_{12}}(a_{11} + a_{11}a_{12})].\end{aligned}$$

We have from Lemmas 7 and 8 that

$$\begin{aligned}0 &= [\delta(p_1+a_{12}), a_{11}+a_{11}a_{12}] + [p_1+a_{12}, \delta(a_{11}+a_{11}a_{12})] \\ &= [\delta(a_{12}), a_{11} + a_{11}a_{12}] + [p_1 + a_{12}, \delta(a_{11} + a_{11}a_{12})] \\ &= -a_{11}\delta(a_{12}) + \delta(a_{11}a_{12}) - \delta(a_{11})a_{12}.\end{aligned}$$

Thus, $\delta(a_{11}a_{12}) = \delta(a_{11})a_{12} + a_{11}\delta(a_{12})$.

Similarly, we can get $\delta(a_{12}a_{22}) = \delta(a_{12})a_{22} + a_{12}\delta(a_{22})$. □

Lemma 9 (i) $\delta(a_{11}b_{11}) = \delta(a_{11})b_{11} + a_{11}\delta(b_{11})$ *for all* $a_{11}, b_{11} \in \mathscr{T}_{11}$;
(ii) $\delta(a_{22}b_{22}) = \delta(a_{22})b_{22} + a_{22}\delta(b_{22})$ *for all* $a_{22}, b_{22} \in \mathscr{T}_{22}$.

Proof: **(i)**: Let $a_{11}, b_{11} \in \mathscr{T}_{11}$. For any $c_{12} \in \mathscr{T}_{12}$, by Lemma 8, on one hand, we have

$$\begin{aligned}\delta(a_{11}b_{11}c_{12}) &= \delta(a_{11})b_{11}c_{12} + a_{11}\delta(b_{11}c_{12}) \\ &= \delta(a_{11})b_{11}c_{12} + a_{11}\delta(b_{11})c_{12} + a_{11}b_{11}\delta(c_{12}).\end{aligned}$$

On the other hand,

$$\delta(a_{11}b_{11}c_{12}) = \delta(a_{11}b_{11})c_{12} + a_{11}b_{11}\delta(c_{12}).$$

Comparing these two equalities, we have

$$\{\delta(a_{11}b_{11}) - \delta(a_{11})b_{11} - a_{11}\delta(b_{11})\}c_{12} = 0$$

for any $c_{12} \in \mathscr{T}_{12}$. Since M is a faithful left A module, we get $\delta(a_{11}b_{11}) = \delta(a_{11})b_{11} + a_{11}\delta(b_{11})$.

Similarly, we can show that statement (ii) is valid. □

Proof of Theorem 2

Proof: For any $x, y \in \mathscr{T}$, we have

$$x = a_{11} + a_{12} + a_{22}, \quad y = b_{11} + b_{12} + b_{22},$$

where $a_{ij}, b_{ij} \in \mathscr{T}_{ij}$. It is easily checked that $\delta(xy) = \delta(x)y + x\delta(y)$, i.e., δ is a derivation. We omit the proof here.

By the definition of τ, we have $\tau([a_{11}, b_{12}]) = \tau([a_{22}, b_{12}]) = 0$. We have $\phi([a_{11}, b_{11}]) = p_1\phi([a_{11}, b_{11}])p_1 + p_2\phi([a_{11}, b_{11}])p_2 \in \mathscr{T}_{11} \oplus \mathscr{T}_{22}$. On the other hand, Proposition 1 implies that for any $a_{11}, b_{11} \in \mathscr{T}_{11}$, there exist a derivation $d : \mathscr{T} \to \mathscr{T}$ and a linear map $h : \mathscr{T} \to Z(\mathscr{T})$ vanishing on each commutator such that $\phi([a_{11}, b_{11}]) = d([a_{11}, b_{11}]) + h([a_{11}, b_{11}]) = d([a_{11}, b_{11}]) \in \mathscr{T}_{11} \oplus \mathscr{T}_{12}$. Thus, $p_2\phi([a_{11}, b_{11}])p_2 = 0$. This implies that $\tau([a_{11}, b_{11}]) = \tau_1([a_{11}, b_{11}]) = \eta(p_1\phi([a_{11}, b_{11}])p_1) + p_2\phi([a_{11}, b_{11}])p_2 = 0$.

Similarly, we can get $\tau([a_{11}, b_{22}]) = 0$. Thus, we show that τ vanishes on all commutators $[x, y]$ for all $x, y \in \mathscr{T}$. Thus, $\varphi(x) = \phi(x) + [x, p_1\varphi(p_1)p_2] = \delta(x) + [x, p_1\varphi(p_1)p_2] + \tau(x)$, where δ is a derivation and τ is a central-valued map which vanishing on each commutator. The proof is completed. □

CONCLUSION

In this paper, we prove that under certain conditions, every local derivation from $\mathscr{T}$ into itself is a derivation; every additive 2-local Lie derivation from $\mathscr{T}$ into itself is a Lie derivation. The main result is then applied to block upper triangular matrix algebras. We also give an example to show that the condition the algebras A and B are generated by all its idempotents is indispensable in Theorem 1.

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