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On the principal eigenvectors of general hypergraphs

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ABSTRACT: Let G be a connected general hypergraph of order n with rank r. The unique positive eigenvector x with $\sum_{i=1}^{n} x_i^r = 1$ corresponding to the spectral radius $\rho(G)$ is called the principal eigenvector of G. In this paper, the relation between each entry of the principal eigenvector of G and the vertex degree associated with this entry is presented. And some bounds for the extreme entries of the principal eigenvector are obtained. As applications, we give some bounds of the spectral radius of G.

KEYWORDS: hypergraph, tensor, spectral radius, principal eigenvector

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INTRODUCTION

The spectral graph theory concerns the relations of structure, parameters of a graph and the eigenvalues and eigenvectors of matrices associated with that graph. It has a wide range of applications in physics, chemistry, computer science and other fields. The principal eigenvector is an important topic in the research of spectral graph theory. In [1], Bonacich studied the centrality of networks by the principal eigenvector for graphs, the value of each entry of principal eigenvector may be seen as a spectral measure of the centrality of the vertex associated with this entry. The study of the principal eigenvector for graphs is important, and it has attracted extensive attention [2–4].

The spectral hypergraph theory is a natural generalization of spectral graph theory. The important tool that has been used in spectral hypergraph theory is tensor. In 2005, Qi [5] and Lim [6] independently proposed the concept of tensor eigenvalues. In 2012, Cooper and Dutle [7] defined the adjacency tensor of uniform hypergraphs. In [8], the definition of adjacency tensor for general hypergraphs is presented by Banerjee et al. In [9], Sun et al proposed another definition of the adjacency tensor for general hypergraphs. In this paper, we use the definition of adjacency tensor presented in [8].

The principal eigenvector is an important topic in the research of spectral hypergraph theory. In [10–13], some bounds on entries for the principal eigenvector of uniform hypergraphs are obtained. In [14], Cardoso et al presented some bounds for the extreme entries of the principal eigenvector of general hypergraphs, and studied inequalities involving the ratio and difference between the two extreme entries of this vector. In [15], Wang et al posed some bounds on entries of the positive unit eigenvector corresponding to the α-spectral radius of general hypergraphs. In [16], Kang et al gave some bounds on entries of the nonnegative unit eigenvector corresponding to the p-spectral radius of general hypergraphs. In [17], Benson studied the centrality of hypergraphs by the principal eigenvector.

The bound of the spectral radius can be used to estimate the convergence rate of algorithms, and it can be regarded as a measure of the irregularity for hypergraphs. It can be obtained by the bound on entries of the principal eigenvector. Recently, many researchers paid attention to the bound on entries of the principal eigenvector. However, some existing bounds can be further improved. In this paper, we study the bound on entries of the principal eigenvector of general hypergraphs, and improve some existing results. As applications, we obtain some bounds of the spectral radius of general hypergraphs.

PRELIMINARIES

For a positive integer n, let $[n]=\{1,2,\ldots,n\}$. An order m dimension n tensor $\mathscr{A}=\left(a_{i_1 i_2\cdots i_m}\right)$ is a multidimensional array with n^m entries, where $i_j\in[n]$, $j\in[m]$, see [18]. When $m=2$, $\mathscr{A}$ is an $n\times n$ matrix. Let $\mathbb{C}^{[m,n]}$ be the set of order m dimension n tensors over the complex field $\mathbb{C}$, and $\mathbb{C}^n$ be the set of n-vectors over the complex field $\mathbb{C}$. For $\mathscr{A}=\left(a_{i_1 i_2\cdots i_m}\right)\in\mathbb{C}^{[m,n]}$, if all the entries $a_{i_1 i_2\cdots i_m}\geqslant 0$, then $\mathscr{A}$ is called nonnegative.

For $\mathscr{A}=\left(a_{i_1 i_2\cdots i_m}\right)\in\mathbb{C}^{[m,n]}$ and $x=(x_1,\,x_2,\,\ldots,\,x_n)^\top\in\mathbb{C}^n$, $\mathscr{A}x^{m-1}$ is an n-vector whose i-th component is

$$\left(\mathscr{A}x^{m-1}\right)_i=\sum_{i_2,\ldots,i_m=1}^{n} a_{i i_2\cdots i_m}x_{i_2}\cdots x_{i_m}.$$

In 2005, Qi [5] and Lim [6] defined the eigenvalues of tensors, respectively.

Definition 1 ([5,6]) For $\mathscr{A}=\left(a_{i_1 i_2\cdots i_m}\right)\in\mathbb{C}^{[m,n]}$, if there exists a number $\lambda\in\mathbb{C}$ and a nonzero vector $x=(x_1,\dots,x_n)^\top\in\mathbb{C}^n$ such that

$$\mathscr{A}x^{m-1}=\lambda x^{[m-1]},$$

then λ is called an eigenvalue of $\mathscr{A}$, x is called an eigenvector of $\mathscr{A}$ corresponding to λ, where $x^{[m-1]}=\left(x_1^{m-1},x_2^{m-1},\dots,x_n^{m-1}\right)^\top$. The spectral radius of $\mathscr{A}$ is the largest modulus of its eigenvalues, denoted by $\rho(\mathscr{A})$.

A (general) hypergraph G is a pair $(V(G),E(G))$, where $E(G)\subseteq P(V(G))\backslash\{\varnothing\}$ and $P(V(G))$ stands for the power set of $V(G)$. The elements of $V(G)$ and $E(G)$ are called vertices and edges, respectively [19]. The number of vertices in G is called the order of G. The rank (resp., co-rank) of G is $r(G)=\max\{|e|:e\in E(G)\}$ (resp., $cr(G)=\min\{|e|:e\in E(G)\}$). In this paper, all hypergraphs have co-rank at least two. If $r(G)=cr(G)=r$, then G is called r-uniform. For all $i\in V(G)$, $E_i(G)$ denotes the set of edges containing i, and $d_i=|E_i(G)|$ denotes the degree of i, $\Delta=\max_i\{d_i\}$ and $\delta=\min_i\{d_i\}$. If $\Delta=\delta$, then G is called regular. Otherwise, G is called irregular. A path P of a hypergraph G is an alternating sequence of vertices and edges $v_0e_1v_1e_2\cdots v_{l-1}e_lv_l$, where $v_0,\dots,v_l$ are distinct vertices of G, $e_1,\dots,e_l$ are distinct edges of G and $v_{i-1},v_i\in e_i$, for $i=1,\dots,l$. If there exists a path between any two vertices of G, then G is called connected.

For a general hypergraph, the adjacency tensor is defined as follows.

Definition 2 ([8]) Let $G=(V(G),E(G))$ be a general hypergraph of order n with rank r. The adjacency tensor of G is

$$\mathscr{A}(G)=(a_{i_1i_2\dots i_r}),\quad 1\leqslant i_1,i_2,\dots,i_r\leqslant n.$$

For all edges $e=\{j_1,j_2,\dots,j_s\}\in E(G)$ of cardinality $s\leqslant r$,

$$a_{i_1i_2\cdots i_r}=\frac{s}{\alpha(s)},$$

where $\alpha(s)=\sum\limits_{\substack{k_1,k_2,\dots,k_s\geqslant 1,\\ k_1+k_2+\cdots+k_s=r}}\frac{r!}{k_1!k_2!\cdots k_s!}$, and $i_1,i_2,\dots,i_r$ are chosen in all possible way from $J=\{j_1,j_2,\dots,j_s\}$ with at least once for each element of the set J. Other entries of the tensor are zero.

Let $G=(V(G),E(G))$ be a general hypergraph of order n with rank r, $\mathscr{A}(G)=(a_{i_1i_2\dots i_r})$ be the adjacency tensor of G. Then

$$d_i=\sum_{i_2,\dots,i_r=1}^{n}a_{ii_2\dots i_r},\ i\in V(G).\qquad\text{(see [8])}$$

The spectral radius of $\mathscr{A}(G)$ is called the spectral radius of G, denoted by $\rho(G)$. A general hypergraph is connected if and only if $\mathscr{A}(G)$ is nonnegative weakly irreducible [20,21]. For a connected general hypergraph G of order n with rank r. According to the Perron-Frobenius theorem of nonnegative weakly irreducible tensors [20], $\rho(G)$ is a positive eigenvalue of $\mathscr{A}(G)$ and there exists the unique positive eigenvector $x=(x_1,\dots,x_n)^\top$ with $\sum\limits_{i=1}^{n}x_i^r=1$ corresponding to $\rho(G)$, which is called the principal eigenvector of G. The maximum and minimum entries of x are denoted by $x_{\max}$ and $x_{\min}$, respectively. Hypergraph G is regular if and only if its principal eigenvector $x=\left(\frac{1}{\sqrt[r]{n}},\frac{1}{\sqrt[r]{n}},\cdots,\frac{1}{\sqrt[r]{n}}\right)^\top$ (see [14]).

Some helpful lemmas are introduced below, which will be used in the sequel.

Lemma 1 ([14]) *Let G be a connected general hypergraph with rank r. Then*

$$\frac{x_{\max}}{x_{\min}}\geqslant\left(\frac{\Delta}{\delta}\right)^{\frac{1}{2(r-1)}}.$$

Lemma 2 ([21]) *Let G be a general hypergraph of order n. Then*

$$\overline{d}\leqslant\rho(G)\leqslant\Delta,$$

where $\overline{d}=\frac{1}{n}\sum\limits_{i=1}^{n}d_i$.

Lemma 3 ([22]) *Let $a_1,a_2,\dots,a_n$ be nonnegative numbers ($n\geqslant 2$). Then*

$$\frac{a_1+a_2+\cdots+a_n}{n}-(a_1a_2\cdots a_n)^{\frac{1}{n}}\geqslant\frac{1}{n(n-1)}\sum_{1\leqslant i<j\leqslant n}(\sqrt{a_i}-\sqrt{a_j})^2,$$

the equality holds if and only if $a_1=a_2=\cdots=a_n$.

Lemma 4 ([23]) *Let a,b,k_1,k_2 be positive numbers. Then*

$$a(k_1-k_2)^2+bk_2^2\geqslant\frac{ab}{a+b}k_1^2,$$

the equality holds if and only if $k_2=\frac{ak_1}{a+b}$.

MAIN RESULTS

In this section, some bounds for the entries of principal eigenvector of a general hypergraph are given.

Theorem 1 *For a connected general hypergraph $G=(V(G),E(G))$ with rank r. Let $\rho(G)$ and $x=\left(x_1,\dots,x_{|V(G)|}\right)^\top$ be the spectral radius and the principal eigenvector of G, respectively. Then*

$$x_i\leqslant\left(\frac{\sum\limits_{e\in E_i(G)}|e|^{-|e|/r}}{\rho(G)}\right)^{1/r},\qquad i\in V(G).$$

Proof: Let $\mathscr{A}(G)=(a_{ii_2\cdots i_r})$ be the adjacency tensor of G, and $|V(G)|=n$. Then

$$\mathscr{A}(G)x^{r-1}=\rho(G)x^{[r-1]}.$$

For all $i\in V(G)$. We get

$$\rho(G)x_i^{r-1}=\left(\mathscr{A}(G)x^{r-1}\right)_i=\sum_{i_2,\dots,i_r=1}^{n}a_{ii_2\cdots i_r}x_{i_2}\cdots x_{i_r}$$
$$=\sum_{\{i,j_2,\dots,j_s\}\in E_i(G)}\left(\frac{s}{\alpha(s)}\sum_{\substack{k_1\geqslant 0,k_2,\dots,k_s\geqslant 1,\\ k_1+k_2+\cdots+k_s=r-1}}\frac{(r-1)!}{k_1!k_2!\cdots k_s!}x_i^{k_1}x_{j_2}^{k_2}\cdots x_{j_s}^{k_s}\right)$$
$$\leqslant\sum_{\{i,j_2,\dots,j_s\}\in E_i(G)}\left(x_{j_2}\cdots x_{j_s}\frac{s}{\alpha(s)}\sum_{\substack{k_1\geqslant 0,k_2,\dots,k_s\geqslant 1,\\ k_1+k_2+\cdots+k_s=r-1}}\frac{(r-1)!}{k_1!k_2!\cdots k_s!}\right),$$

where $\alpha(s)=\sum_{\substack{k_1,k_2,\dots,k_s\geqslant 1,\\ k_1+k_2+\cdots+k_s=r}}\frac{r!}{k_1!k_2!\cdots k_s!}$. Since

$$s\sum_{\substack{k_1\geqslant 0,k_2,\dots,k_s\geqslant 1,\\ k_1+k_2+\cdots+k_s=r-1}}\frac{(r-1)!}{k_1!k_2!\cdots k_s!}$$
$$=s\sum_{\substack{k_1,k_2,\dots,k_s\geqslant 1,\\ k_1+k_2+\cdots+k_s=r}}\frac{(r-1)!}{(k_1-1)!k_2!\cdots k_s!}$$
$$=\sum_{i=1}^{s}\sum_{\substack{k_1,k_2,\dots,k_s\geqslant 1,\\ k_1+k_2+\cdots+k_s=r}}\frac{(r-1)!}{k_1!\cdots k_{i-1}!(k_i-1)!k_{i+1}!\cdots k_s!}$$
$$=\sum_{\substack{k_1,k_2,\dots,k_s\geqslant 1,\\ k_1+k_2+\cdots+k_s=r}}\sum_{i=1}^{s}\frac{(r-1)!}{k_1!\cdots k_{i-1}!(k_i-1)!k_{i+1}!\cdots k_s!}$$
$$=\sum_{\substack{k_1,k_2,\dots,k_s\geqslant 1,\\ k_1+k_2+\cdots+k_s=r}}\frac{(r-1)!}{k_1!k_2!\cdots k_s!}(k_1+k_2+\cdots+k_s)$$
$$=\sum_{\substack{k_1,k_2,\dots,k_s\geqslant 1,\\ k_1+k_2+\cdots+k_s=r}}\frac{r!}{k_1!k_2!\cdots k_s!}=\alpha(s).$$

We obtain

$$\rho(G)x_i^{r-1}\leqslant\sum_{\{i,j_2,\dots,j_s\}\in E_i(G)}x_{j_2}\cdots x_{j_s}. \tag{1}$$

It follows from AM-GM inequality (i.e., the arithmetic mean-geometric mean inequality) [24, 25], the power mean (PM) inequality [25] and (1) that

$$\rho(G)x_i^{r}\leqslant\sum_{\{i,j_2,\dots,j_s\}\in E_i(G)}x_ix_{j_2}\cdots x_{j_s}$$
$$\leqslant\sum_{\{i,j_2,\dots,j_s\}\in E_i(G)}\frac{x_i^s+x_{j_2}^s+\cdots+x_{j_s}^s}{s}$$
$$\leqslant\sum_{\{i,j_2,\dots,j_s\}\in E_i(G)}\left(\frac{x_i^r+x_{j_2}^r+\cdots+x_{j_s}^r}{s}\right)^{s/r}.$$

Notice that $\sum_{j=1}^{n}x_j^r=1$. We get

$$\rho(G)x_i^{r}\leqslant\sum_{\{i,j_2,\dots,j_s\}\in E_i(G)}\left(\frac{x_i^r+x_{j_2}^r+\cdots x_{j_s}^r}{s}\right)^{s/r}$$
$$\leqslant\sum_{\{i,j_2,\dots,j_s\}\in E_i(G)}s^{-s/r}=\sum_{e\in E_i(G)}|e|^{-|e|/r},$$

which implies that

$$x_i\leqslant\left(\frac{\sum_{e\in E_i(G)}|e|^{-|e|/r}}{\rho(G)}\right)^{1/r}.$$

□

Corollary 1 *For a connected general hypergraph $G=(V(G),E(G))$ with rank r, and co-rank c, we have*

$$x_i\leqslant\left(\frac{d_i}{c^{c/r}\rho(G)}\right)^{1/r},\qquad i\in V(G).$$

When G is a connected uniform hypergraph, the following result is obtained by Theorem 1.

Corollary 2 *For a connected k-uniform hypergraph $G=(V(G),E(G))$, we have*

$$x_i\leqslant\left(\frac{d_i}{k\rho(G)}\right)^{1/k},\qquad i\in V(G).$$

Remark 1 For a connected k-uniform hypergraph G, Nikiforov [10] proved that

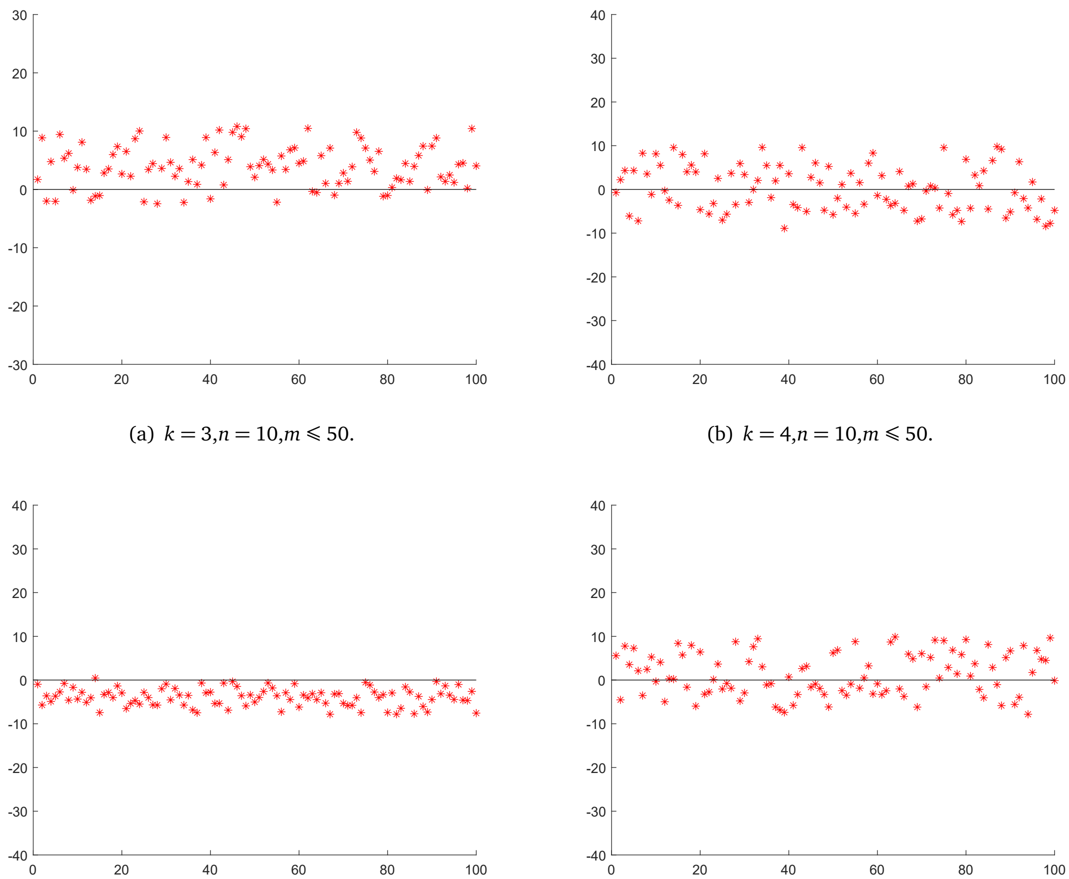
$$x_i\leqslant\left(\frac{d_i}{[(\rho(G))^k(k-1)!]^{1/(k-1)}}\right)^{1/k},\qquad i\in V(G). \tag{2}$$

When $\rho(G)<\frac{k^{k-1}}{(k-1)!}$, the bound in Corollary 2 is better than the bound in (2). When $\rho(G)>\frac{k^{k-1}}{(k-1)!}$, the bound in (2) is better than the bound in Corollary 2.

Next we randomly construct some k-uniform hypergraphs. By calculating $\rho(G)-\frac{k^{k-1}}{(k-1)!}$, the bound in (2) and the bound in Corollary 2 are demonstrated, as shown in the Fig. 1.

In every subfigure of Fig. 1, $\rho(G)-k^{k-1}/(k-1)!$ is denoted by star symbol. We show the results of 100 generated k-uniform connected hypergraphs with n vertices and m edges. The x-axis refers to these 100 random generated cases. For the star symbol, there are 17%, 51%, 99% and 45% of cases below the x-axis in subfigures (a), (b), (c), and (d), respectively. The star symbol below the x-axis implies $\rho(G)-k^{k-1}/(k-1)!<0$, i.e., the bound in Corollary 2 is better than the bound in (2).

By Corollary 2, the following result is presented.



(a) $k=3, n=10, m \leqslant 50$. (b) $k=4, n=10, m \leqslant 50$.

(c) $k=4, n=20, m \leqslant 50$. (d) $k=4, n=20, m \leqslant 100$.

Fig. 1: The randomly generated results.

Corollary 3 *Let G be a connected k-uniform hypergraph. Then*

$$x_{\min} \leqslant \left(\frac{\delta}{k\rho(G)}\right)^{1/k}.$$

The following theorem gives an upper bound on $x_{\min}$ for a connected hypergraph.

Theorem 2 *Let $G=(V(G),E(G))$ be a connected general hypergraph of order n with rank r, and co-rank c. Then*

$$x_{\min} \leqslant \frac{\delta^{1/r}}{\left(c(\rho(G))^{r/c}+(n-c)\delta^{r/c}\right)^{c/r^2}}.$$

Proof: From (1), we have

$$\rho(G)x_i^{r-1} \leqslant \sum_{\{i,i_2,\ldots,i_s\}\in E_i(G)} x_{i_2}\cdots x_{i_s}, \quad i\in V(G).$$

Let $d_u=\delta$, $u\in V(G)$. It follows from AM-GM inequality [24, 25] and PM inequality [25] that

$$\begin{aligned}
\rho(G)x_{\min}^r &\leqslant \rho(G)x_u^r \\
&= \sum_{\{u,i_2,\ldots,i_s\}\in E_u(G)} x_u x_{i_2}\cdots x_{i_s} \\
&\leqslant \sum_{\{u,i_2,\ldots,i_s\}\in E_u(G)} \frac{1}{s}\left(x_u^s+x_{i_2}^s+\cdots+x_{i_s}^s\right) \\
&\leqslant \sum_{\{u,i_2,\ldots,i_s\}\in E_u(G)} \left(\frac{x_u^r+x_{i_2}^r+\cdots+x_{i_s}^r}{s}\right)^{s/r} \\
&\leqslant \sum_{\{u,i_2,\ldots,i_s\}\in E_u(G)} \left(\frac{\sum_{j=1}^{n} x_j^r-(n-s)x_{\min}^r}{s}\right)^{s/r}.
\end{aligned}$$

Since G is connected. We get x is a positive vector.

Notice that $\sum_{j=1}^{n} x_j^r = 1$ and $s \geqslant c$. We obtain

$$\begin{aligned}\rho(G)x_{\min}^r &\leqslant \sum_{\{u,i_2,\ldots,i_s\}\in E_u(G)} \left(\frac{\sum_{j=1}^{n} x_j^r - (n-s)x_{\min}^r}{s}\right)^{s/r} \\ &\leqslant \sum_{\{u,i_2,\ldots,i_s\}\in E_u(G)} \left(\frac{1-(n-c)x_{\min}^r}{c}\right)^{c/r} \\ &= \frac{\delta}{c^{c/r}}\left(1-(n-c)x_{\min}^r\right)^{c/r}.\end{aligned}$$

Therefore,

$$\left(\frac{\rho(G)}{\delta}\right)^{r/c} c x_{\min}^{r^2/c} \leqslant 1-(n-c)x_{\min}^r \leqslant 1-(n-c)x_{\min}^{r^2/c},$$

which implies that

$$x_{\min} \leqslant \frac{\delta^{1/r}}{(c(\rho(G))^{r/c} + (n-c)\delta^{r/c})^{c/r^2}}.$$

□

The following result is obtained by Theorem 2.

Corollary 4 *Let G be a connected k-uniform hypergraph. Then*

$$x_{\min} \leqslant \left(\frac{\delta}{k\rho(G)+(n-k)\delta}\right)^{1/k}.$$

Remark 2 Let G be a connected general hypergraph of order n with rank r, and co-rank c. Cardoso and Trevisan [14] proved that

$$x_{\min} \leqslant \left(\frac{\delta}{\rho(G)+(n-1)\delta}\right)^{1/r}, \tag{3}$$

and

$$x_{\min} \leqslant \left(\frac{1}{\left(\frac{\Delta}{\delta}\right)^{r/2(r-1)} + (n-1)}\right)^{1/r}. \tag{4}$$

When $\rho(G) > (\Delta^r \delta^{r-1})^{1/2(r-1)}$, the bound in (3) is better than the bound in (4). When G is a connected irregular uniform hypergraph, the bound in Theorem 2 is better than the bound in (3). Thus, for a connected irregular uniform hypergraph G with $\rho(G) > (\Delta^r \delta^{r-1})^{1/2(r-1)}$, the bound in Theorem 2 is better than the bounds in (3) and (4).

The following results give several lower bounds on $x_{\max}$ for hypergraphs.

Theorem 3 *Let G be a connected irregular general hypergraph of order n with rank r. Then*

$$x_{\max} \geqslant \left(\frac{\rho(G)-\delta}{n(\overline{d}-\delta)}\right)^{1/r}.$$

where $\overline{d} = \frac{1}{n}\sum_{i=1}^{n} d_i$.

Proof: Let $\mathscr{A}(G) = (a_{ii_2\cdots i_r})$ and $x = (x_1, \ldots, x_n)^\top$ be the adjacency tensor and the principal eigenvector of $G = (V(G), E(G))$, respectively. Then

$$\rho(G)x_i^{r-1} = \left(\mathscr{A}(G)x^{r-1}\right)_i = \sum_{i_2,\ldots,i_r=1}^{n} a_{ii_2\cdots i_r} x_{i_2}\cdots x_{i_r},$$

$$\rho(G) = \rho(G)\sum_{i=1}^{n} x_i^r = \sum_{i,i_2,\ldots,i_r=1}^{n} a_{ii_2\cdots i_r} x_i x_{i_2}\cdots x_{i_r}.$$

So,

$$\begin{aligned}&\rho(G)-\delta \\ &= \sum_{i_1,i_2,\ldots,i_r=1}^{n} a_{i_1 i_2\cdots i_r} x_{i_1}x_{i_2}\cdots x_{i_r} - \sum_{i=1}^{n} d_i x_i^r + \sum_{i=1}^{n} d_i x_i^r - \delta \\ &= \sum_{\{i_1,i_2,\ldots,i_s\}\in E(G)} \Bigg(\frac{s}{\alpha(s)} \sum_{\substack{k_1,k_2,\ldots,k_s\geqslant 1,\\ k_1+k_2+\cdots+k_s=r}} \frac{r!}{k_1!k_2!\cdots k_s!} x_{i_1}^{k_1}x_{i_2}^{k_2}\cdots x_{i_s}^{k_s} \\ &\qquad - \sum_{j=1}^{s} x_{i_j}^r\Bigg) + \sum_{i=1}^{n}(d_i-\delta)x_i^r \\ &= \sum_{i=1}^{n}(d_i-\delta)x_i^r - \sum_{\{i_1,i_2,\ldots,i_s\}\in E(G)} \frac{s}{\alpha(s)}\Bigg(\frac{\alpha(s)}{s}\sum_{j=1}^{s} x_{i_j}^r \\ &\qquad - \sum_{\substack{k_1,k_2,\ldots,k_s\geqslant 1,\\ k_1+k_2+\cdots+k_s=r}} \frac{r!}{k_1!k_2!\cdots k_s!} x_{i_1}^{k_1}x_{i_2}^{k_2}\cdots x_{i_s}^{k_s}\Bigg),\end{aligned} \tag{5}$$

where $\alpha(s) = \sum_{\substack{k_1,k_2,\ldots,k_s\geqslant 1,\\ k_1+k_2+\cdots+k_s=r}} \frac{r!}{k_1!k_2!\cdots k_s!}$.

Since

$$\begin{aligned}&\sum_{\substack{k_1,k_2,\ldots,k_s\geqslant 1,\\ k_1+k_2+\cdots+k_s=r}} \frac{(r-1)!}{k_1!k_2!\cdots k_s!} k_t x_{i_t}^r \\ &= \frac{x_{i_t}^r}{s}\sum_{t=1}^{s} \sum_{\substack{k_1,k_2,\ldots,k_s\geqslant 1,\\ k_1+k_2+\cdots+k_s=r}} \frac{(r-1)!k_t}{k_1!k_2!\cdots k_s!} \\ &= \frac{x_{i_t}^r}{s} \sum_{\substack{k_1,k_2,\ldots,k_s\geqslant 1,\\ k_1+k_2+\cdots+k_s=r}} \sum_{t=1}^{s} \frac{(r-1)!k_t}{k_1!k_2!\cdots k_s!} \\ &= \frac{\alpha(s)x_{i_t}^r}{s},\end{aligned}$$

for all $t \in [s]$. We get

$$\begin{aligned}&\frac{\alpha(s)}{s}\sum_{j=1}^{s} x_{i_j}^r - \sum_{\substack{k_1,k_2,\ldots,k_s\geqslant 1,\\ k_1+k_2+\cdots+k_s=r}} \frac{r!}{k_1!k_2!\cdots k_s!} x_{i_1}^{k_1}x_{i_2}^{k_2}\cdots x_{i_s}^{k_s} \\ &= \sum_{\substack{k_1,k_2,\ldots,k_s\geqslant 1,\\ k_1+k_2+\cdots+k_s=r}} \frac{r!}{k_1!k_2!\cdots k_s!} \\ &\times \left(\frac{k_1x_{i_1}^r + k_2x_{i_2}^r + \cdots + k_sx_{i_s}^r}{r} - x_{i_1}^{k_1}x_{i_2}^{k_2}\cdots x_{i_s}^{k_s}\right).\end{aligned} \tag{6}$$

Let $k_1, k_2, \dots, k_s \geqslant 1$, and $k_1 + k_2 + \cdots + k_s = r$. From Lemma 3, we get

$$\frac{k_1 x_{i_1}^r + k_2 x_{i_2}^r + \cdots + k_s x_{i_s}^r}{r} - x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_s}^{k_s} \geqslant \frac{1}{r(r-1)} \sum_{1\leqslant a<b\leqslant s} (x_{i_a}^{r/2} - x_{i_b}^{r/2})^2 \geqslant 0.$$

It follows from (5) and (6) that

$$\rho(G) - \delta \leqslant \sum_{i=1}^{n} (d_i - \delta)\, x_i^r \leqslant n(\overline{d} - \delta) x_{\max}^r.$$

Since G is irregular. We get $\overline{d} > \delta$. Thus,

$$x_{\max} \geqslant \left(\frac{\rho(G) - \delta}{n(\overline{d} - \delta)} \right)^{1/r}.$$

□

The following result is obtained by Theorem 3.

Corollary 5 *Let G be a connected irregular k-uniform hypergraph. Then*

$$x_{\max} \geqslant \left(\frac{\rho(G) - \delta}{n(\overline{d} - \delta)} \right)^{1/k},$$

where $\overline{d} = \frac{1}{n} \sum_{i=1}^{n} d_i$.

Remark 3 Let G be a connected irregular k-uniform hypergraph G with n vertices, m edges. Then (see [13, 14])

$$x_{\max} \geqslant \left(\frac{\rho\,(G)}{km} \right)^{1/k}. \tag{7}$$

Assume that $\frac{\rho(G)-\delta}{n(\overline{d}-\delta)} < \frac{\rho(G)}{km}$. Then we get

$$\rho(G) n(\overline{d} - \delta) > km(\rho(G) - \delta) = n\overline{d}(\rho(G) - \delta),$$

which implies that $\overline{d} > \rho(G)$. This is a contradiction to Lemma 2. Thus,

$$x_{\max} \geqslant \left(\frac{\rho(G) - \delta}{n(\overline{d} - \delta)} \right)^{1/k} \geqslant \left(\frac{\rho\,(G)}{km} \right)^{1/k}.$$

The bound in Corollary 5 is better than the bound in (7).

Lemma 5 *Let $G = (V(G), E(G))$ be a connected general hypergraph with rank r. Then*

$$x_{\max} \geqslant \left(\frac{\Delta}{\rho(G) + \Delta(n-1)} \right)^{1/r},$$

equality holds if and only if there is a vertex $u \in V(G)$ such that $x_u = x_{\min} = \left(\frac{\rho(G)}{\Delta}\right)^{1/r} x_{\max}$, and for all $v \in V(G)\backslash\{u\}$, $x_v = x_{\max}$.

Proof: Let $\mathscr{A}(G) = (a_{ii_2\cdots i_r})$ be the adjacency tensor of G, and $|V(G)| = n$. For all $i \in V(G)$,

$$\rho(G) x_i^{r-1} = \left(\mathscr{A}(G) x^{r-1} \right)_i = \sum_{i_2,\dots,i_r=1}^{n} a_{ii_2\cdots i_r} x_{i_2} \cdots x_{i_r} \geqslant d_i x_{\min}^{r-1}.$$

Let $d_p = \Delta$, $p \in V(G)$. Then

$$\rho(G) x_{\max}^r \geqslant \rho(G) x_p^r \geqslant d_p x_{\min}^r = \Delta x_{\min}^r. \tag{8}$$

Notice that $\sum_{i=1}^{n} x_i^r = 1$. We get

$$\Delta(n-1) x_{\max}^r \geqslant \Delta(\sum_{i=1}^{n} x_i^r - x_{\min}^r). \tag{9}$$

Thus,

$$(\rho(G) + \Delta(n-1)) x_{\max}^r \geqslant \Delta x_{\min}^r + \Delta(\sum_{i=1}^{n} x_i^r - x_{\min}^r) = \Delta,$$

which implies that

$$x_{\max} \geqslant \left(\frac{\Delta}{\rho(G) + \Delta(n-1)} \right)^{1/r}.$$

By (8) and (9), we know that $x_{\max} = \left(\frac{\Delta}{\rho(G)+\Delta(n-1)}\right)^{1/r}$ if and only if there is a vertex $u \in V(G)$ such that $x_u = x_{\min} = \left(\frac{\rho(G)}{\Delta}\right)^{1/r} x_{\max}$, and for all $v \in V(G)\backslash\{u\}$, $x_v = x_{\max}$. □

The following theorem is obtained by Lemma 1 and Lemma 5.

Theorem 4 *Let G be a connected irregular general hypergraph of order n with rank r. Then*

$$x_{\max} - x_{\min} \geqslant \frac{\Delta^{\frac{(r-2)}{2r(r-1)}} \left(\Delta^{\frac{1}{2(r-1)}} - \delta^{\frac{1}{2(r-1)}} \right)}{(\rho(G) + \Delta(n-1))^{\frac{1}{r}}}.$$

Proof: It is easy to see that

$$x_{\max} - x_{\min} = x_{\max} \left(1 - \frac{x_{\min}}{x_{\max}} \right).$$

It follows from Lemma 1 and Lemma 5 that

$$x_{\max} - x_{\min} \geqslant \left(\frac{\Delta}{\rho(G) + \Delta(n-1)} \right)^{\frac{1}{r}} \left(\frac{\Delta^{\frac{1}{2(r-1)}} - \delta^{\frac{1}{2(r-1)}}}{\Delta^{\frac{1}{2(r-1)}}} \right) = \frac{\Delta^{\frac{r-2}{2r(r-1)}} \left(\Delta^{\frac{1}{2(r-1)}} - \delta^{\frac{1}{2(r-1)}} \right)}{(\rho(G) + \Delta(n-1))^{\frac{1}{r}}}.$$

□

Remark 4 Let G be a connected irregular general hypergraph of order n with rank r. Then (see [14])

$$x_{\max}-x_{\min} \geqslant \frac{\Delta^{\frac{1}{2(r-1)}}-\delta^{\frac{1}{2(r-1)}}}{n^{\frac{1}{r}}\Delta^{\frac{1}{2(r-1)}}}. \tag{10}$$

It follows from Theorem 4 and Lemma 2 that

$$\begin{aligned}
x_{\max}-x_{\min} &\geqslant \frac{\Delta^{\frac{r-2}{2r(r-1)}}\left(\Delta^{\frac{1}{2(r-1)}}-\delta^{\frac{1}{2(r-1)}}\right)}{(\rho(G)+\Delta(n-1))^{\frac{1}{r}}}\\
&=\left(\frac{\Delta}{\rho(G)+\Delta(n-1)}\right)^{\frac{1}{r}}\left(\frac{\Delta^{\frac{1}{2(r-1)}}-\delta^{\frac{1}{2(r-1)}}}{\Delta^{\frac{1}{2(r-1)}}}\right)\\
&\geqslant \frac{\Delta^{\frac{1}{2(r-1)}}-\delta^{\frac{1}{2(r-1)}}}{n^{\frac{1}{r}}\Delta^{\frac{1}{2(r-1)}}}.
\end{aligned}$$

Thus, the bound in Theorem 4 is better than the bound in (10).

APPLICATIONS

For a connected general hypergraph G, it is regular if and only if $\rho(G)=\Delta$. So $\Delta-\rho(G)$ can be regarded as a measure of the irregularity for G. In this section, some bounds of $\Delta-\rho(G)$ are obtained.

Let G be a connected hypergraph. The distance $d(u,v)$ between two distinct vertices u and v of G is the number of edges of the shortest path connecting them. The diameter D of G is the maximum distance among all pairs of vertices of G.

Theorem 5 *Let G be a connected irregular general hypergraph of order n with rank r, co-rank c. Then*

$$\Delta-\rho(G) > \frac{c^2(\Delta-\overline{d})(\Delta-\delta)}{c^2(\Delta-\delta)+2r(r-1)nD(\Delta-\overline{d})(\overline{d}-\delta)},$$

where D and $\overline{d}$ are the diameter and the average degree of G, respectively.

Proof: Let $x=(x_1,\dots,x_n)^{\top}$ be the principal eigenvector of $G=(V(G),E(G))$. Then

$$\begin{aligned}
&\Delta-\rho(G)\\
&=\Delta-\sum_{i=1}^{n}d_ix_i^r+\sum_{i=1}^{n}d_ix_i^r-\sum_{i_1,i_2,\dots,i_r=1}^{n}a_{i_1i_2\cdots i_r}x_{i_1}x_{i_2}\cdots x_{i_r}\\
&=\sum_{i=1}^{n}(\Delta-d_i)x_i^r+\sum_{\{i_1,i_2,\dots,i_s\}\in E(G)}\left(\sum_{j=1}^{s}x_{i_j}^r\right.\\
&\qquad\left.-\frac{s}{\alpha(s)}\sum_{\substack{k_1,k_2,\dots,k_s\geqslant 1,\\k_1+k_2+\cdots+k_s=r}}\frac{r!}{k_1!k_2!\cdots k_s!}x_{i_1}^{k_1}x_{i_2}^{k_2}\cdots x_{i_s}^{k_s}\right)\\
&=\sum_{i=1}^{n}(\Delta-d_i)x_i^r+\sum_{\{i_1,i_2,\dots,i_s\}\in E(G)}\frac{s}{\alpha(s)}\left(\frac{\alpha(s)}{s}\sum_{j=1}^{s}x_{i_j}^r\right.\\
&\qquad\left.-\sum_{\substack{k_1,k_2,\dots,k_s\geqslant 1,\\k_1+k_2+\cdots+k_s=r}}\frac{r!}{k_1!k_2!\cdots k_s!}x_{i_1}^{k_1}x_{i_2}^{k_2}\cdots x_{i_s}^{k_s}\right),
\end{aligned}$$

where $\alpha(s)=\sum_{\substack{k_1,k_2,\dots,k_s\geqslant 1,\\k_1+k_2+\cdots+k_s=r}}\frac{r!}{k_1!k_2!\cdots k_s!}$. Similar to the proof of Theorem 3, we have

$$\begin{aligned}
&\frac{\alpha(s)}{s}\sum_{j=1}^{s}x_{i_j}^r-\sum_{\substack{k_1,k_2,\dots,k_s\geqslant 1,\\k_1+k_2+\cdots+k_s=r}}\frac{r!}{k_1!k_2!\cdots k_s!}x_{i_1}^{k_1}x_{i_2}^{k_2}\cdots x_{i_s}^{k_s}\\
&\quad=\sum_{\substack{k_1,k_2,\dots,k_s\geqslant 1,\\k_1+k_2+\cdots+k_s=r}}\frac{r!}{k_1!k_2!\cdots k_s!}\\
&\qquad\times\left(\frac{k_1x_{i_1}^r+k_2x_{i_2}^r+\cdots+k_sx_{i_s}^r}{r}-x_{i_1}^{k_1}x_{i_2}^{k_2}\cdots x_{i_s}^{k_s}\right)\\
&\quad\geqslant\frac{\alpha(s)}{r(r-1)}\sum_{1\leqslant a<b\leqslant s}(x_{i_a}^{\frac{r}{2}}-x_{i_b}^{\frac{r}{2}})^2.
\end{aligned}$$

Since G is irregular, we have

$$\sum_{i=1}^{n}(\Delta-d_i)\,x_i^r>\sum_{i=1}^{n}(\Delta-d_i)\,x_{\min}^r=n(\Delta-\overline{d})x_{\min}^r.$$

Thus,

$$\begin{aligned}
&\Delta-\rho(G)\\
&\ >n(\Delta-\overline{d})x_{\min}^r+\sum_{\{i_1,i_2,\dots,i_s\}\in E(G)}\frac{c}{r(r-1)}\sum_{1\leqslant a<b\leqslant s}\left(x_{i_a}^{\frac{r}{2}}-x_{i_b}^{\frac{r}{2}}\right)^2\\
&\ =n(\Delta-\overline{d})x_{\min}^r+\frac{c}{r(r-1)}\sum_{\{p,q\}\subseteq e\in E(G)}\left(x_p^{\frac{r}{2}}-x_q^{\frac{r}{2}}\right)^2.
\end{aligned} \tag{11}$$

Suppose that $u,v\in V(G)$ such that $u\neq v$, $x_u=x_{\max}$ and $x_v=x_{\min}$. Let $P=v_0e_1v_1e_2\cdots v_{l-1}e_lv_l$ be the shortest path from vertex u to vertex v, where $u=v_0$, $v=v_l$. Similar to the proof of [13, Theorem 4.3], we have

$$\begin{aligned}
&\sum_{\{p,q\}\subseteq e\in E(G)}\left(x_p^{\frac{r}{2}}-x_q^{\frac{r}{2}}\right)^2\\
&\geqslant\sum_{\{p,q\}\subseteq e\in E(P)}\left(x_p^{\frac{r}{2}}-x_q^{\frac{r}{2}}\right)^2\\
&\geqslant\sum_{i=1}^{l}\left(\left(x_{v_{i-1}}^{\frac{r}{2}}-x_{v_i}^{\frac{r}{2}}\right)^2+\sum_{u_j\in e_i\setminus\{v_{i-1},v_i\}}\left(\left(x_{v_{i-1}}^{\frac{r}{2}}-x_{u_j}^{\frac{r}{2}}\right)^2+\left(x_{u_j}^{\frac{r}{2}}-x_{v_i}^{\frac{r}{2}}\right)^2\right)\right)\\
&\geqslant\sum_{i=1}^{l}\left(\left(x_{v_{i-1}}^{\frac{r}{2}}-x_{v_i}^{\frac{r}{2}}\right)^2+\frac{1}{2}\sum_{u_j\in e_i\setminus\{v_{i-1},v_i\}}\left(x_{v_{i-1}}^{\frac{r}{2}}-x_{u_j}^{\frac{r}{2}}+x_{u_j}^{\frac{r}{2}}-x_{v_i}^{\frac{r}{2}}\right)^2\right)\\
&=\sum_{i=1}^{l}\left(\left(x_{v_{i-1}}^{\frac{r}{2}}-x_{v_i}^{\frac{r}{2}}\right)^2+\frac{|e_i|-2}{2}\left(x_{v_{i-1}}^{\frac{r}{2}}-x_{v_i}^{\frac{r}{2}}\right)^2\right)\\
&\geqslant\frac{c}{2}\sum_{i=1}^{l}\left(x_{v_{i-1}}^{\frac{r}{2}}-x_{v_i}^{\frac{r}{2}}\right)^2.
\end{aligned}$$

With the help of Cauchy-Schwarz inequality [25], we obtain

$$\sum_{\{p,q\}\subseteq e\in E(G)} \left(x_p^{\frac{r}{2}} - x_q^{\frac{r}{2}}\right)^2 \geqslant \frac{c}{2l}\left(\sum_{i=1}^{l}\left(x_{v_{i-1}}^{\frac{r}{2}} - x_{v_i}^{\frac{r}{2}}\right)\right)^2$$
$$= \frac{c}{2l}\left(x_u^{\frac{r}{2}} - x_v^{\frac{r}{2}}\right)^2 \geqslant \frac{c}{2D}\left(x_{\max}^{\frac{r}{2}} - x_{\min}^{\frac{r}{2}}\right)^2. \quad (12)$$

From (11) and (12), we get

$$\Delta - \rho(G) > n(\Delta - \overline{d})x_{\min}^r + \frac{c^2}{2r(r-1)D}\left(x_{\max}^{\frac{r}{2}} - x_{\min}^{\frac{r}{2}}\right)^2.$$

It follows from Lemma 4 that

$$\Delta - \rho(G) > \frac{c^2 n(\Delta - \overline{d})}{c^2 + 2r(r-1)nD(\Delta - \overline{d})} x_{\max}^r. \quad (13)$$

By Theorem 3 and (13), we have

$$\Delta - \rho(G) > \frac{c^2(\Delta - \overline{d})(\rho(G) - \delta)}{\left(c^2 + 2r(r-1)nD(\Delta - \overline{d})\right)(\overline{d} - \delta)}.$$

Thus,

$$\Delta - \rho(G) > \frac{c^2(\Delta - \overline{d})(\Delta - \delta)}{c^2(\Delta - \delta) + 2r(r-1)nD(\Delta - \overline{d})(\overline{d} - \delta)}.$$

□

Corollary 6 *Let G be a connected irregular k-uniform hypergraph of order n. Then*

$$\Delta - \rho(G) > \frac{k(\Delta - \overline{d})(\Delta - \delta)}{k(\Delta - \delta) + 2(k-1)nD(\Delta - \overline{d})(\overline{d} - \delta)},$$

where D and $\overline{d}$ are the diameter and the average degree of G, respectively.

Remark 5 Let G be a connected irregular k-uniform hypergraph G with n vertices, m edges. It follows from Corollary 6 that

$$\begin{aligned}\Delta - \rho(G) &> \frac{k(\Delta - \overline{d})(\Delta - \delta)}{k(\Delta - \delta) + 2(k-1)nD(\Delta - \overline{d})(\overline{d} - \delta)} \\ &= \frac{k\Delta(\Delta - \overline{d})}{k\Delta + 2(k-1)nD(\Delta - \overline{d})(\overline{d} - \delta)\frac{\Delta}{\Delta - \delta}} \\ &> \frac{k\Delta(\Delta - \overline{d})}{k\Delta + 2(k-1)nD\overline{d}(\Delta - \overline{d})} \\ &= \frac{\Delta(n\Delta - km)}{n\Delta + 2(k-1)Dm(n\Delta - km)},\end{aligned}$$

which improves the result of [13, Theorem 4.1].

CONCLUSION

In this paper, several bounds on entries of the principal eigenvector for general hypergraphs are given. In particular, we present a bound on the maximum entry of the principal eigenvector, which improves the result in [13, 14]. Further, we obtain some estimations on the spectral radius of hypergraphs, which improves the existing results. Moreover, a bound on the difference between the two extreme entries of the principal eigenvector is showed, which is better than the one in [14].

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