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Two generalized constrained inverses based on the core part of the Core-EP decomposition of a complex matrix

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ABSTRACT: Let A be a complex matrix. The question when a generalized column constrained inverse coincides with a generalized row constrained inverse was answered, which lead to the generalized constrained inverse of A was introduced and this inverse coincides with the weak group inverse. Moreover, the "distance" between the generalized constrained inverse and the inverse along two matrices was given, that is, the generalized constrained inverse of A coincides with the $(A^k, (A^k)^*A)$-inverse of A, where k is the index of A.

KEYWORDS: generalized constrained inverse, weak group inverse, the inverse along two matrices

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INTRODUCTION

Let $\mathbb{C}$ be the complex filed. The set $\mathbb{C}^{m\times n}$ denotes the set of all $m\times n$ complex matrices over the complex filed $\mathbb{C}$. Let $A\in\mathbb{C}^{m\times n}$. The symbol A^* denotes the conjugate transpose of A. Notations $\mathscr{R}(A)=\{y\in\mathbb{C}^m : y=Ax, x\in\mathbb{C}^n\}$, $\mathscr{N}(A)=\{x\in\mathbb{C}^n : Ax=0, x\in\mathbb{C}^n\}$ and $\mathscr{R}\mathscr{S}(A)=\{y\in\mathbb{C}^n : y^\top=x^\top A, x\in\mathbb{C}^m\}$ will be used in the sequel. The smallest positive integer k such that $\mathrm{rank}(A^k)=\mathrm{rank}(A^{k+1})$ is called the index of $A\in\mathbb{C}^{n\times n}$ and denoted by $\mathrm{Ind}(A)$. Let $A\in\mathbb{C}^{m\times n}$. If a matrix $X\in\mathbb{C}^{n\times m}$ satisfies

$$AXA=A,\ XAX=X,\ (AX)^*=AX \text{ and } (XA)^*=XA,$$

then X is called the Moore-Penrose inverse of A [1, 2] and denoted by $X=A^\dagger$. Let $A\in\mathbb{C}^{m\times n}$ and $X\in\mathbb{C}^{n\times m}$. If $AXA=A$ holds, then X (and denoted by A^-) is called an inner inverse of A, the set $A\{1\}$ represents the class all inner inverse of A, i.e. $A\{1\}=\{X : AXA=A\}$. Let $A,X\in\mathbb{C}^{n\times n}$. If

$$AXA=A,\ XAX=X \text{ and } AX=XA,$$

then X is called a group inverse of A. If such X exists, then it is unique and denoted by $A^\#$ [3]. A necessary and sufficient condition for a given complex square matrix to have group inverse is $\mathrm{Ind}(A)\leqslant 1$. Manjunatha Prasad and Mohana [4] introduced the core-EP inverse of matrix [4, Definition 3.1]. Let $A\in\mathbb{C}^{n\times n}$. If there exists $X\in\mathbb{C}^{n\times n}$ such that $XAX=X, \mathscr{R}(X)=\mathscr{R}(X^*)=\mathscr{R}(A^k)$, then X is called the core-EP inverse of A. If such inverse exists, then it is unique and denoted by $A^{\textcircled{\dagger}}$. For a square matrix $A\in\mathbb{C}^{n\times n}$, a inner inverse of A with columns belonging to the linear manifold generated by the columns of A will be denoted by A^-_C. For a square matrix $A\in\mathbb{C}^{n\times n}$, a inner inverse of A with rows belonging to the linear manifold generated by the rows of A will be denoted by A^-_R.

By [5, Theorem 2.1], Wang introduced a new matrix decomposition, namely the Core-EP decomposition of $A\in\mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A)=k$. Given a matrix $A\in\mathbb{C}^{n\times n}$, then A can be written as the sum of matrices $A_1\in\mathbb{C}^{n\times n}$ and $A_2\in\mathbb{C}^{n\times n}$, that is $A=A_1+A_2$, where $A_1\in\mathbb{C}^{CM}_n$, $A_2^k=0$ and $A_1^*A_2=A_2A_1=0$, $\mathbb{C}^{CM}_n=\{A\in\mathbb{C}^{n\times n}\mid \mathrm{rank}\{(A)\}=\mathrm{rank}\{(A^2)\}$. By Theorems 2.3 and 2.4 in [5], Wang proved this matrix decomposition is unique and there exists a unitary matrix $U\in\mathbb{C}^{n\times n}$ such that

$$A_1=U\begin{bmatrix}T & S\\ 0 & 0\end{bmatrix}U^* \text{ and } A_2=U\begin{bmatrix}0 & 0\\ 0 & N\end{bmatrix}U^*, \qquad (1)$$

where $T\in\mathbb{C}^{r\times r}$ is nonsingular and $N\in\mathbb{C}^{(n-r)\times(n-r)}$ is nilpotent. Some new generalized inverses was investigated by using the Core-EP decomposition of $A\in\mathbb{C}^{n\times n}$, for example, the generalized WG inverse [6]. New type generalized inverse can be investigated in rings, for example, the (p,q,m)-core inverse and the $\langle p,q,n\rangle$-core inverse [7]. The EP-nilpotent decomposition of A was introduced by Wang and Liu [8].

By [9, Definition 1.2] and [10, Definition 2.1], the authors introduced the one-sided (b,c)-inverse in rings. By [11, Definition 2.7], the authors introduced the one-sided (B,C)-inverse for complex matrices. Let $A,B,C\in\mathbb{C}^{n\times n}$. We call that $X\in\mathbb{C}^{n\times n}$ is a left (B,C)-inverse of A if we have $\mathscr{N}(C)\subseteq\mathscr{N}(X)$ and $XAB=B$. We call that $Y\in\mathbb{C}^{n\times n}$ is a right (B,C)-inverse of A if we have $\mathscr{R}(Y)\subseteq\mathscr{R}(B)$ and $CAY=C$. In fact, there is an important generalized inverse was introduced in [12] by Rao and Mitra. Let $A\in\mathbb{C}^{n\times n}$. In [13], Rakić showed that Rao and Mitra's constrained inverse of A coincides with the (B,C)-inverse of A, where $B,C\in\mathbb{C}^{n\times n}$. Existence criteria and expressions of the (b,c)-inverse in rings can be found in [14, 15]. In the next section, two generalized constrained inverses were introduced by using the core part of the Core-EP decomposition of a complex matrix, the generalized column constrained inverse of A and the generalized row constrained inverse of A. The expression of a generalized column constrained inverse of A is $A^-_{gC} = A_1(A_1^2)^-$ and the expression of a generalized row constrained inverse of A is $A^-_{gR} = (A_1^2)^- A_1$. After that, we answer the question when a generalized column constrained inverse coincides with a generalized row constrained inverse, that is, if A^-_{gC} is a generalized column constrained inverse of A and A^-_{gR} is a generalized column constrained inverse of A, then $A^{ⓦ} = A^-_{gC} A_1 A^-_{gR}$. Finally, we obtained the "distance" between the generalized constrained inverse and the inverse along two matrices, that is, the generalized constrained inverse of A coincides with the $(A^k, (A^k)^* A)$-inverse of A for $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$.

TWO GENERALIZED CONSTRAINED INVERSES WERE INTRODUCED BY USING THE CORE PART OF THE CORE-EP DECOMPOSITION OF A COMPLEX MATRIX

For a square matrix $A \in \mathbb{C}^{n\times n}$, an inner inverse of A with columns belonging to the linear manifold generated by the columns of A will be denoted by A^-_C. That is, if $X \in \mathbb{C}^{n\times n}$ satisfy $AXA = A$ and $\mathscr{R}(X) \subseteq \mathscr{R}(A)$, then $X = A^-_C$ [16]. Motivated by the definition of A^-_C, we introduced the generalized column constrained inverse of A by using the core part of the Core-EP decomposition of A. As the core part of the Core-EP decomposition of A is useful in the study of some kinds of generalized inverses. We use an inner inverse of A_1 with columns belonging to the linear manifold generated by the columns of A_1 to define the generalized column constrained inverses of A.

Definition 1 Let $A \in \mathbb{C}^{n\times n}$ with $A = A_1 + A_2$ be the Core-EP decomposition of A as in (1). For the square matrix A_1, an inner inverse of A_1 with columns belonging to the linear manifold generated by the columns of A_1 will be called a generalized column constrained inverse of A and denoted by A^-_{gC}. That is, if $X \in \mathbb{C}^{n\times n}$ satisfy $A_1 X A_1 = A_1$ and $\mathscr{R}(X) \subseteq \mathscr{R}(A_1)$, then $X = A^-_{gC}$.

Lemma 1 (Corollary 2.1 in [16]) *Let $A \in \mathbb{C}^{m\times n}$ and $P \in \mathbb{C}^{n\times p}$. Then $X = P(AP)^- \in A\{1\}$ if and only if* $\mathrm{rank}(AP) = \mathrm{rank}(A)$ *for any* $(AP)^- \in AP\{1\}$.

As $A_1 \in \mathbb{C}_n^{CM}$, we have generalized column constrained inverse of A always exists by Lemma 1 and let $P = A$. In the following theorem, we will give the expression of a generalized column constrained inverse of A by using the core part of the Core-EP decomposition of A.

Theorem 1 *Let $A \in \mathbb{C}^{n\times n}$ with $A = A_1 + A_2$ be the Core-EP decomposition of A as in* (1). *Then the expression of a generalized column constrained inverse of A is $A^-_{gC} = A_1(A_1^2)^-$.*

Proof: Since A_1 is the core part of A, so $\mathrm{rank}(A_1^2) = \mathrm{rank}(A_1)$. By Lemma 1 and $\mathrm{rank}(A_1^2) = \mathrm{rank}(A_1)$, we can get $A^-_{gC} = A_1(A_1^2)^-$. □

Theorem 2 *Let $A \in \mathbb{C}^{n\times n}$ with $A = A_1 + A_2$ be the Core-EP decomposition of A as in* (1). *Then the expression of a generalized column constrained inverse of A is*

$$A^-_{gC} = U \begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix} U^*, \tag{2}$$

where $Z = TX_2 + SX_4$ for some X_2, X_4 with suitable size.

Proof: From $A_1 = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^*$, we have

$$A_1^2 = U \begin{bmatrix} T^2 & TS \\ 0 & 0 \end{bmatrix} U^*.$$

Let $X = U \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} U^*$ be a inner inverse of A_1^2, then we have

$$\begin{aligned} A_1^2 = U \begin{bmatrix} T^2 & TS \\ 0 & 0 \end{bmatrix} U^* &= A_1^2 X A_1^2 \\ &= U \begin{bmatrix} T^2 & TS \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} T^2 & TS \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (T^2X_1 + TSX_3)T^2 & (T^2X_1 + TSX_3)TS \\ 0 & 0 \end{bmatrix} U^*, \end{aligned}$$

which implies

$$\begin{cases} T^2 = (T^2X_1 + TSX_3)T^2 \\ TS = (T^2X_1 + TSX_3)TS \end{cases} \tag{3}$$

The condition $T^2 = (T^2X_1 + TSX_3)T^2$ in (3) is equivalent to $E = T^2X_1 + TSX_3$ by T is nonsingular. Then $E = T^2X_1 + TSX_3$ gives the condition $TS = (T^2X_1 + TSX_3)TS$ in (3) is always hold. Also, $E = T^2X_1 + TSX_3$ gives

$$T^{-1} = TX_1 + SX_3. \tag{4}$$

By the definition of a generalized column constrained inverse of A and Theorem 1, we have

$$\begin{aligned} A^-_{gC} &= A_1(A_1^2)^- \\ &= U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} U^* \\ &= U \begin{bmatrix} TX_1 + SX_3 & TX_2 + SX_4 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} T^{-1} & TX_2 + SX_4 \\ 0 & 0 \end{bmatrix} U^* \end{aligned} \tag{5}$$

by (4). □

Theorem 3 *Let $A \in \mathbb{C}^{n\times n}$ with $A = A_1 + A_2$ be the Core-EP decomposition of A as in (1). Then the core-EP inverse of A is a special generalized column constrained inverse of A. Moreover, if $Z = 0$, then the core-EP inverse of A is consistent with a special generalized column constrained inverse of A, where $A^-_{gC} = U\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}U^*$ with $Z = TX_2 + SX_4$ for some X_2, X_4 with suitable size.*

Proof: By [5, Theorem 3.2], we have $A^{\textcircled{\dagger}} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*$. Thus

$$\begin{aligned} A^{\textcircled{\dagger}} - A^-_{gC} &= U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* - U\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}U^* \\ &= A^k U\begin{bmatrix} 0 & Z \\ 0 & 0 \end{bmatrix}U^* \end{aligned}$$

Thus, the condition $Z = 0$ implies the core-EP inverse is consistent with a special generalized column constrained inverse of A. □

By [5, Theorem 2.3], we have $A_1 = A^k(A^k)^\dagger A = P_{A^k}A$ and $A_1^2 = A^k(A^k)^\dagger AA^k(A^k)^\dagger A = A^{k+1}(A^k)^\dagger A = AP_{A^k}A$. Thus $A^-_{gC} = A_1(A_1^2)^- = P_{A^k}A(AP_{A^k}A)^-$. By [5, Corollary 3.3], we have $A^{\textcircled{\dagger}} = A^k(A^{k+1})^{\#}$. Thus, the core-EP inverse of A is consistent with a special generalized column constrained inverse of A, that is

$$A^k(A^{k+1})^{\#} = P_{A^k}A(AP_{A^k}A)^-. \tag{6}$$

for any $(AP_{A^k}A)^- \in AP_{A^k}A\{1\}$.

In the following theorem, we will show that a special generalized column constrained inverse of A is an outer inverse of A, which is useful in the Theorem 10 in the following section.

Theorem 4 *Let $A \in \mathbb{C}^{n\times n}$. If $X \in \mathbb{C}^{n\times n}$ is a special generalized column constrained inverse of A, then $XAX = X$ and $AX^2 = X$.*

Proof: Let $A = A_1 + A_2$ be the Core-EP decomposition of $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$A_1 = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^* \text{ and } A_2 = U\begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}U^*, \tag{7}$$

where $T \in \mathbb{C}^{r\times r}$ is nonsingular and $N \in \mathbb{C}^{(n-r)\times(n-r)}$ is nilpotent.

By Theorem 2, we have $A^-_{gC} = U\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}U^*$, where $Z = TX_2 + SX_4$ for some X_2, X_4 with suitable size. Then

$$\begin{aligned} A^-_{gC}AA^-_{gC} &= U\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} E & T^{-1}S + ZN \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}U^* \\ &= A^-_{gC}, \end{aligned}$$

and

$$\begin{aligned} A(A^-_{gC})^2 &= U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\left(\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}\right)^2 U^* \\ &= U\begin{bmatrix} E & TZ \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}U^* \\ &= A^-_{gC}. \end{aligned}$$

□

Remark 1 Let $A \in \mathbb{C}^{n\times n}$. If $X \in \mathbb{C}^{n\times n}$ is a special generalized column constrained inverse of A, then $AXA \neq A$ and $XA^2 \neq A$.

Proof: Let $A = A_1 + A_2$ be the Core-EP decomposition of $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$A_1 = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^* \text{ and } A_2 = U\begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}U^*, \tag{8}$$

where $T \in \mathbb{C}^{r\times r}$ is nonsingular and $N \in \mathbb{C}^{(n-r)\times(n-r)}$ is nilpotent.

By Theorem 2, we have $A^-_{gC} = U\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}U^*$, where $Z = TX_2 + SX_4$ for some X_2, X_4 with suitable size. Then

$$\begin{aligned} AA^-_{gC}A &= U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^* \\ &= U\begin{bmatrix} E & TZ \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^* \\ &= U\begin{bmatrix} T & S + TZN \\ 0 & 0 \end{bmatrix}U^* \\ &\neq A, \end{aligned}$$

and

$$\begin{aligned} A^-_{gC}(A)^2 &= U\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}\left(\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\right)^2 U^* \\ &= U\begin{bmatrix} E & TZ \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} T & S + T^{-1}SN + ZN^2 \\ 0 & 0 \end{bmatrix}U^* \\ &\neq A. \end{aligned}$$

□

Theorem 5 *Let $A, X \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$ and $A = A_1 + A_2$ be the Core-EP decomposition of A as in (1). Then the following statements are equivalent:*

(1) *X is a generalized column constrained inverse of A;*

(2) *$A_1XA_1 = A_1$ and $X = A_1X^2$;*

(3) *$XA^k \in (A^k)^{\dagger}A\{1\}$ and $X = P_{A^k}X = AX^2$;*

(4) *$(A^k)^{\dagger}X^* \in A^*A^k\{1\}$ and $X = P_{A^k}X = AX^2$.*

Proof: (1) ⇒ (2). Let X be a generalized column constrained inverse of A. Then $A_1XA_1 = A_1$ and $\mathscr{R}(X) \subseteq \mathscr{R}(A_1)$. Thus, $X = A_1U$ for some $U \in \mathbb{C}^{n\times n}$ by $\mathscr{R}(X) \subseteq \mathscr{R}(A_1)$, which gives $X = A_1U = A_1XA_1U = A_1X^2$ by $A_1XA_1 = A_1$.

(2) ⇒ (1) is trivial.

(1) ⇒ (3). We have $X = A_1X^2 = P_{A^k}AX^2 = P_{A^k}X$ by Theorem 4 and [5, Theorem 2.3]. The condition $A_1XA_1 = A_1$ can be written as $A^k(A^k)^{\dagger}AXA^k(A^k)^{\dagger}A = A^k(A^k)^{\dagger}A$. Pre-multiplying by $(A^k)^{\dagger}$ on $A^k(A^k)^{\dagger}AXA^k(A^k)^{\dagger}A = A^k(A^k)^{\dagger}A$ gives

$$(A^k)^{\dagger}A^k(A^k)^{\dagger}AXA^k(A^k)^{\dagger}A = (A^k)^{\dagger}A^k(A^k)^{\dagger}A.$$

That is,

$$(A^k)^{\dagger}AXA^k(A^k)^{\dagger}A = (A^k)^{\dagger}A,$$

by $(A^k)^{\dagger}A^k(A^k)^{\dagger} = (A^k)^{\dagger}$, which gives $XA^k \in (A^k)^{\dagger}A\{1\}$.

(3) ⇒ (4). The condition $XA^k \in (A^k)^{\dagger}A\{1\}$ is $(A^k)^{\dagger}AXA^k(A^k)^{\dagger}A = (A^k)^{\dagger}A$, is equivalent to $(A^k)^*AXA^k(A^k)^{\dagger}A = (A^k)^*A$. Taking $*$ on $(A^k)^*AXA^k(A^k)^{\dagger}A = (A^k)^*A$ implies

$$A^*A^k(A^k)^{\dagger}X^*A^*A^k = A^*A^k$$

by $(A^k(A^k)^{\dagger})^* = A^k(A^k)^{\dagger}$. The equality $A^*A^k(A^k)^{\dagger}X^*A^*A^k = A^*A^k$ gives $(A^k)^{\dagger}X^* \in A^*A^k\{1\}$.

(4) ⇒ (1). The condition $X = P_{A^k}X = AX^2$ implies $X = P_{A^k}X = P_{A^k}AX^2 = A^k(A^k)^{\dagger}AX^2 = A_1X^2$, then we have $\mathscr{R}(X) \subseteq \mathscr{R}(A_1)$. The condition $(A^k)^{\dagger}X^* \in A^*A^k\{1\}$ is $A^*A^k(A^k)^{\dagger}X^*A^*A^k = A^*A^k$. Taking $*$ on $A^*A^k(A^k)^{\dagger}X^*A^*A^k = A^*A^k$ gives

$$(A^k)^*AXA^k(A^k)^{\dagger}A = (A^k)^*A,$$

which is equivalent to $(A^k)^{\dagger}AXA^k(A^k)^{\dagger}A = (A^k)^{\dagger}A$. Pre-multiplying by A^k on

$$(A^k)^{\dagger}AXA^k(A^k)^{\dagger}A = (A^k)^{\dagger}A$$

gives $A^k(A^k)^{\dagger}AXA^k(A^k)^{\dagger}A = A^k(A^k)^{\dagger}A$, that is $A_1XA_1 = A_1$. Thus, X is a generalized column constrained inverse of A by Definition 1. □

Definition 2 Let $A \in \mathbb{C}^{n\times n}$ with $A = A_1 + A_2$ be the Core-EP decomposition of A as in (1). For the square matrix A_1, a inner inverse of A_1 with rows belonging to the linear manifold generated by the rows of A_1 will be called a generalized row constrained inverse of A and denoted by A^-_{gR}. That is, if $Y \in \mathbb{C}^{n\times n}$ satisfy $A_1YA_1 = A_1$ and $\mathscr{RS}(Y) \subseteq \mathscr{RS}(A_1)$, then $Y = A^-_{gR}$.

By [16, Theorem 2.1], we have the following lemma.

Lemma 2 *Let $A, Q \in \mathbb{C}^{n\times n}$. Then $X = (QA)^-Q \in A\{1\}$ if and only if $\mathrm{rank}(QA) = \mathrm{rank}(A)$ for any $(QA)^- \in QA\{1\}$.*

As $A_1 \in \mathbb{C}^{CM}_n$, we have generalized row constrained inverse of A always exists by Lemma 2 and let $Q = A$. In the following theorem, we will give the expression of a generalized row constrained inverse of A by using the core part of the Core-EP decomposition of A.

Theorem 6 *Let $A \in \mathbb{C}^{n\times n}$ with $A = A_1 + A_2$ be the Core-EP decomposition of A as in (1). Then the expression of a generalized row constrained inverse of A is $A^-_{gR} = (A_1^2)^-A_1$.*

Proof: Since A_1 is the core part of A, so $\mathrm{rank}(A_1^2) = \mathrm{rank}(A_1)$. By Lemma 1 and $\mathrm{rank}(A_1^2) = \mathrm{rank}(A_1)$, we can get $A^-_{gR} = (A_1^2)^-A_1$. □

Lemma 3 (Corollary 1a.1 in [17]) *Let $A \in \mathbb{C}^{m\times n}$ and $X \in \mathbb{C}^{n\times m}$ such that*

$$\begin{cases} AX \text{ is idempotent}, \\ \mathrm{rank}(AX) = \mathrm{rank}(A), \end{cases} \tag{9}$$

then $X \in A\{1\}$.

Lemma 4 *Let $A \in \mathbb{C}^{n\times n}$ with $A = A_1 + A_2$ be the Core-EP decomposition of A as in (1). If Y is a generalized row constrained inverse of A, then $\mathrm{rank}(YA_1) = \mathrm{rank}(Y)$.*

Proof: Let Y be a generalized row constrained inverse of A, so $A_1YA_1 = A_1$ and $\mathscr{RS}(Y) \subseteq \mathscr{RS}(A_1)$ by Definition 2. Then

$$\mathrm{rank}(Y) \leqslant \mathrm{rank}(A_1)$$

by $\mathrm{rank}(Y) = \dim(\mathscr{RS}(Y)) \leqslant \dim(\mathscr{RS}(A_1)) = \mathrm{rank}(A_1)$; $\mathrm{rank}(A_1) \leqslant \mathrm{rank}(Y)$ by $A_1YA_1 = A_1$. Then we have $\mathrm{rank}(Y) = \mathrm{rank}(A_1)$. The condition $A_1YA_1 = A_1$ implies $\mathrm{rank}(YA_1) = \mathrm{rank}(A_1)$, so we can get

$$\mathrm{rank}(YA_1) = \mathrm{rank}(Y)$$

by $\mathrm{rank}(Y) = \mathrm{rank}(A_1)$ and $\mathrm{rank}(YA_1) = \mathrm{rank}(A_1)$. □

Proposition 1 *Let $A \in \mathbb{C}^{n\times n}$ with $A = A_1 + A_2$ be the Core-EP decomposition of A as in (1). Let Y be a generalized row constrained inverse of A, then $A_1 = A_1YA_1$, $Y = YA_1Y$, $Y = Y^2A_1$ and $A_1 = A_1^2Y$.*

Proof: Let Y be a generalized row constrained inverse of A, then $A_1YA_1 = A_1$ and $\mathscr{RS}(Y) \subseteq \mathscr{RS}(A_1)$ by Definition 2. Then $Y = UA_1$ for some $U \in \mathbb{C}^{n\times n}$, which gives

$$Y = UA_1 = UA_1YA_1 = Y^2A_1$$

by $A_1YA_1 = A_1$. The equality $A_1YA_1 = A_1$ implies YA_1 is idempotent, using $\mathrm{rank}(YA_1) = \mathrm{rank}(Y)$ and Lemma 3, we can get $YA_1Y = Y$.

By the proof of Lemma 4, we have $\operatorname{rank}(Y) = \operatorname{rank}(A_1)$, then $\mathscr{RS}(Y) = \mathscr{RS}(A_1)$ holds by $\mathscr{RS}(Y) \subseteq \mathscr{RS}(A_1)$ and $\operatorname{rank}(Y) = \operatorname{rank}(A_1)$. The condition $\mathscr{RS}(Y) = \mathscr{RS}(A_1)$ gives $A_1 = VY$ for some $V \in \mathbb{C}^{n\times n}$, then

$$A_1 = VY = VYA_1Y = A_1^2Y.$$

□

Proposition 2 *Let $A \in \mathbb{C}^{n\times n}$ with $A = A_1 + A_2$ be the Core-EP decomposition of A as in (1). Let $Y = U\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}U^*$ be a generalized row constrained inverse of A, then we have*

(1) $TY_1 + SY_3 = E$;

(2) $TY_2 + SY_4 = T^{-1}S$;

(3) $Y_1T^{-1}S = Y_2$;

(4) $Y_3T^{-1}S = Y_4$.

Proof: Let $Y = U\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}U^*$ be a generalized row constrained inverse of A, then $A_1 = A_1YA_1$, $Y = YA_1Y$ and $A_1 = A_1^2Y$ by Proposition 1. Then

$$\begin{aligned} A_1Y &= U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}U^* \\ &= U\begin{bmatrix} TY_1+SY_3 & TY_2+SY_4 \\ 0 & 0 \end{bmatrix}U^*. \end{aligned} \tag{10}$$

By $A_1 = A_1YA_1$ and (10), we have

$$\begin{aligned} A_1YA_1 &= U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} TY_1+SY_3 & TY_2+SY_4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} (TY_1+SY_3)T & (TY_1+SY_3)S = S \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*, \end{aligned} \tag{11}$$

which implies

$$\begin{cases} T = (TY_1+SY_3)T \\ S = (TY_1+SY_3)S \end{cases} \tag{12}$$

The condition $T = (TY_1+SY_3)T$ in (12) is equivalent to $E = TY_1+SY_3$ by T is nonsingular. Then $E = TY_1+SY_3$ gives the condition $S = (TY_1+SY_3)S$ in (12) is always hold.

By $Y = YA_1Y$, $E = TY_1+SY_3$ and (10), we have

$$\begin{aligned} YA_1Y &= U\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}U^* \\ &= U\begin{bmatrix} Y_1 & Y_1(TY_2+SY_4) \\ Y_3 & Y_3(TY_2+SY_4) \end{bmatrix}U^* = U\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}U^* \end{aligned} \tag{13}$$

which implies

$$\begin{cases} Y_2 = Y_1(TY_2+SY_4) \\ Y_4 = Y_3(TY_2+SY_4) \end{cases} \tag{14}$$

By $A_1 = A_1^2Y$, $E = TY_1+SY_3$ and (10), we have

$$\begin{aligned} A_1^2Y &= U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}U^* \\ &= U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} TY_1+SY_3 & TY_2+SY_4 \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} E & TY_2+SY_4 \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} T & T(TY_2+SY_4) \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*, \end{aligned} \tag{15}$$

which implies

$$T(TY_2+SY_4) = S.$$

That is

$$TY_2+SY_4 = T^{-1}S. \tag{16}$$

Thus, the equalities in (14) can be rewritten as

$$\begin{cases} Y_2 = Y_1T^{-1}S \\ Y_4 = Y_3T^{-1}S \end{cases} \tag{17}$$

by (16). □

WHEN A GENERALIZED COLUMN CONSTRAINED INVERSE COINCIDES WITH A GENERALIZED ROW CONSTRAINED INVERSE

Definition 3 Let $A \in \mathbb{C}^{n\times n}$ with $A = A_1 + A_2$ be the Core-EP decomposition of A as in (1). For the square matrix A_1, a inner inverse of A_1 with columns belonging to the linear manifold generated by the columns of A_1 and rows belonging to the linear manifold generated by the rows of A_1 will be called a generalized constrained inverse of A and denoted by A^-_{gRC}. That is, if $X \in \mathbb{C}^{n\times n}$ satisfy $A_1XA_1 = A_1$, $\mathscr{R}(X) \subseteq \mathscr{R}(A_1)$ and $\mathscr{RS}(X) \subseteq \mathscr{RS}(A_1)$, then $X = A^-_{gRC}$.

Lemma 5 (Proposition 8.22 in [18]) *Let $A \in \mathbb{C}^{n\times n}$. If $A = A^2X = YA^2$ for some $X, Y \in \mathbb{C}^{n\times n}$, then $A^\# = YAX$.*

Proposition 3 *Let $A \in \mathbb{C}^{n\times n}$ with $A = A_1 + A_2$ be the Core-EP decomposition of A as in (1). If X is the generalized constrained inverse of A, then we have $\mathscr{R}(X) = \mathscr{R}(A^k(A^k)^\dagger A)$ and $\mathscr{N}(X) = \mathscr{R}(A^k(A^k)^\dagger A)$.*

Proof: Let X be the generalized constrained inverse of A, then we have $A_1XA_1 = A_1$, $\mathscr{R}(X) \subseteq \mathscr{R}(A_1)$ and $\mathscr{RS}(X) \subseteq \mathscr{RS}(A_1)$ by Definition 3. Thus, the condition $\mathscr{R}(X) \subseteq \mathscr{R}(A_1)$ is equivalent to $X = A_1X^2$ by

$A_1XA_1 = A_1$ and the condition $\mathscr{RS}(X) \subseteq \mathscr{RS}(A_1)$ is equivalent to $X = X^2A_1$ by $A_1XA_1 = A_1$. Thus, $X^{\#} = A_1XA_1 = A_1$, then $A^{\#} = X$, so $A_1X = XA_1$. The proof is finished by $A_1XA_1 = A_1$, $\mathscr{R}(X) \subseteq \mathscr{R}(A_1)$, $\mathscr{RS}(X) \subseteq \mathscr{RS}(A_1)$ and $A_1 = A^k(A^k)^{\dagger}A$. □

Theorem 7 *Let $A \in \mathbb{C}^{n\times n}$. If $X \in \mathbb{C}^{n\times n}$ is a generalized constrained inverse of A, then this generalized constrained inverse of A is unique. Moreover, Then the generalized constrained inverse of A coincides with the weak group inverse, that is $A^{-}_{gRC} = A^{ⓦ}$.*

Proof: Let X be the generalized constrained inverse of A, that is $A_1XA_1 = A_1$, $\mathscr{R}(X) \subseteq \mathscr{R}(A_1)$ and $\mathscr{RS}(X) \subseteq \mathscr{RS}(A_1)$. The condition $\mathscr{R}(X) \subseteq \mathscr{R}(A_1)$ implies $X = A_1U$ for some $U \in \mathbb{C}^{n\times n}$. Then

$$X = A_1U = A_1XA_1U = A_1X^2 \tag{18}$$

by $A_1XA_1 = A_1$. The condition $\mathscr{RS}(X) \subseteq \mathscr{RS}(A_1)$ implies $X = VA_1$ for some $v \in \mathbb{C}^{n\times n}$. Then

$$X = VA_1 = VA_1XA_1U = X^2A_1 \tag{19}$$

by $A_1XA_1 = A_1$. By Lemma 5 and $A_1XA_1 = A_1$, we have $X^{\#} = A_1XA_1 = A_1$. By the definition of the group inverse, we have $A_1^{\#} = X$, that is $X = A^{ⓦ}$ by [19, Theorem 3.7]. As $A^{ⓦ}$ is unique by [19, Theorem 3.1], so the generalized constrained inverse of A is unique by $A^{-}_{gRC} = A^{ⓦ}$. □

Remark 2 By the proof of Theorem 7, one can see that the generalized constrained inverse of a complex matrix is unique.

In the following theorem, we will use the generalized column constrained inverse of A and the generalized row constrained inverse of A to give the expression of the weak group inverse of A.

Theorem 8 *Let $A \in \mathbb{C}^{n\times n}$. If A^{-}_{gC} is a generalized column constrained inverse of A and A^{-}_{gR} is a generalized column constrained inverse of A, then $A^{ⓦ} = A^{-}_{gC}A_1A^{-}_{gR}$.*

Proof: Let A^{-}_{gC} is a generalized column constrained inverse of A and A^{-}_{gR} is a generalized column constrained inverse of A. Let $A \in \mathbb{C}^{n\times n}$ with $A = A_1 + A_2$ be the Core-EP decomposition of A as in (1). Then by Theorem 2, the expression of a generalized column constrained inverse of A is

$$A^{-}_{gC} = U\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}U^*, \tag{20}$$

where $Z = TX_2 + SX_4$ for some X_2, X_4 with suitable size.

Let $Y = U\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}U^*$ be a generalized row constrained inverse of A, then by Proposition 2, we have $TY_1 + SY_3 = E$ and $TY_2 + SY_4 = T^{-1}S$. Thus

$$\begin{aligned} A_1Y &= U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}U^* \\ &= U\begin{bmatrix} TY_1 + SY_3 & TY_2 + SY_4 \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} E & T^{-1}S \\ 0 & 0 \end{bmatrix}U^*. \end{aligned} \tag{21}$$

Hence, by (20) and (21) we have

$$\begin{aligned} A^{-}_{gC}A_1A^{-}_{gR} &= U\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}\begin{bmatrix} E & T^{-1}S \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^*. \end{aligned} \tag{22}$$

By [19, Theorem 3.7], we have $A^{ⓦ} = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^*$, which gives $A^{ⓦ} = A^{-}_{gC}A_1A^{-}_{gR}$ by (22). □

THE “DISTANCE” BETWEEN THE GENERALIZED CONSTRAINED INVERSE AND THE INVERSE ALONG TWO MATRICES

Proposition 4 *Let $A, X \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$ and $A = A_1 + A_2$ be the Core-EP decomposition of A as in (1). If X is a generalized column constrained inverse of A, then $X = XA_1X$.*

Proof: Let $A = A_1 + A_2$ be the Core-EP decomposition of $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $A_1 = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*$, where $T \in \mathbb{C}^{r\times r}$ is nonsingular. By Theorem 2, we have

$$X = U\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}U^*,$$

where $Z = TX_2 + SX_4$ for some X_2, X_4 with suitable size. Then

$$\begin{aligned} XA_1X &= U\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} E & T^{-1}S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}U^* \\ &= X, \end{aligned}$$

that is $X = XA_1X$. □

Theorem 9 *Let $A \in \mathbb{C}^{n\times n}$ and $X \in \mathbb{C}^{n\times n}$ be a generalized column constrained inverse of A. Then X is the $(AA^{ⓦ}, AA^{-}_{gC})$-inverse of A_1, where A_1 is the core part of the Core-EP decomposition of A, $A^{ⓦ}$ is the weak group inverse of A.*

Proof: By the proof of [5, Corollary 3.12], we have

$$AA^{\circledS} = U\begin{bmatrix} E & T^{-1}S \\ 0 & 0 \end{bmatrix}U^*. \tag{23}$$

By Proposition 4, we have $X = XA_1X$. Next, we will prove that X coincides with the (XA_1, A_1X)-inverse of A. Then $X = XA_1X$ implies $\mathcal{R}(X) \subseteq \mathcal{R}(XA_1)$, $\mathcal{R}(XA_1) \subseteq \mathcal{R}(X)$ is obvious, hence $\mathcal{R}(X) = \mathcal{R}(XA_1)$. For any $u \in \mathcal{N}(A_1X)$, that is $A_1Xu = 0$. Then $Xu = XA_1Xu = 0$, which implies $\mathcal{N}(A_1X) \subseteq \mathcal{N}(X)$, hence $\mathcal{N}(X) = \mathcal{N}(A_1X)$ by $\mathcal{N}(X) \subseteq \mathcal{N}(A_1X)$ is trivial. Thus, A_1 is (XA_1, A_1X)-invertible by [20, Theorem 2.1(ii) and Proposition 6.1]. Therefore,

$$\begin{aligned} XA_1 &= U\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} E & T^{-1}S \\ 0 & 0 \end{bmatrix}U^* \\ &= AA^{\circledS} \end{aligned}$$

by the equality (23). Since

$$\begin{aligned} A_1X &= \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} E & TZ \\ 0 & 0 \end{bmatrix}U^* \end{aligned}$$

and

$$\begin{aligned} AX &= \begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U\begin{bmatrix} T^{-1} & Z \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} E & TZ \\ 0 & 0 \end{bmatrix}U^*, \end{aligned}$$

so we have $A_1X = AX$. □

Lemma 6 (Theorem 2.1(ii) and Proposition 6.1 in [20]) *Let $A, B, C \in \mathbb{C}^{n\times n}$. Then $Y \in \mathbb{C}^{n\times n}$ is the (B, C)-inverse of A if and only if $YAY = Y$, $\mathcal{R}(Y) = \mathcal{R}(B)$ and $\mathcal{N}(Y) = \mathcal{N}(C)$.*

Theorem 10 *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$. The generalized constrained inverse of A coincides with the $(A^k, (A^k)^*A)$-inverse of A.*

Proof: By Definition 3, a generalized constrained invertible matrix is a generalized column constrained invertible, thus we have $XAX = X$ by Theorem 4. Since

$$A^k = A^k(A^k)^\dagger A^k = A^k(A^k)^\dagger AA^{k-1},$$

so $\mathcal{R}(A^k(A^k)^\dagger A) \subseteq \mathcal{R}(A^k)$. And $\mathcal{R}(A^k) \subseteq \mathcal{R}(A^k(A^k)^\dagger A)$ is trivial, hence,

$$\mathcal{R}(A^k) = \mathcal{R}(A^k(A^k)^\dagger A). \tag{24}$$

Let $x \in \mathcal{N}((A^k)^\dagger A)$, that is $(A^k)^\dagger Ax = 0$, then $A^k(A^k)^\dagger Ax = 0$, so $\mathcal{N}((A^k)^\dagger A) \subseteq \mathcal{N}(A^k(A^k)^\dagger A)$. Let $y \in \mathcal{N}(A^k(A^k)^\dagger A)$, that is $A^k(A^k)^\dagger Ay = 0$, then

$$(A^k)^\dagger Ay = (A^k)^\dagger A^k(A^k)^\dagger Ay = (A^k)^\dagger A^k[(A^k)^\dagger Ay] = 0,$$

so $\mathcal{N}(A^k(A^k)^\dagger A) \subseteq \mathcal{N}((A^k)^\dagger A)$. Hence

$$\mathcal{N}(A^k(A^k)^\dagger A) = \mathcal{R}((A^k)^\dagger A). \tag{25}$$

Let $u \in \mathcal{N}((A^k)^\dagger A)$, that is $(A^k)^\dagger Ax = 0$, then

$$(A^k)^*Au = [A^k(A^k)^\dagger A^k]^*Au = (A^k)^*A^k(A^k)^\dagger Au = 0,$$

so $\mathcal{N}((A^k)^\dagger A) \subseteq \mathcal{N}((A^k)^*A)$. Let $v \in \mathcal{N}((A^k)^*A)$, that is $(A^k)^*Av = 0$, then

$$(A^k)^\dagger Av = (A^k)^\dagger A^k(A^k)^\dagger Av(A^k)^\dagger[(A^k)^\dagger]^*(A^k)^*Av = 0,$$

so $\mathcal{N}((A^k)^*A) \subseteq \mathcal{N}((A^k)^\dagger A)$. Hence

$$\mathcal{N}((A^k)^*A) = \mathcal{R}((A^k)^\dagger A). \tag{26}$$

By (25) and (26), we have

$$\mathcal{N}((A^k)^*A) = \mathcal{R}(A^k(A^k)^\dagger A). \tag{27}$$

Thus

$$\begin{cases} XAX = X, \quad \mathcal{R}(A^k) = \mathcal{R}(A^k(A^k)^\dagger A), \\ \mathcal{N}((A^k)^*A) = \mathcal{R}(A^k(A^k)^\dagger A). \end{cases} \tag{28}$$

by (24). By Proposition 3, we have

$$\mathcal{R}(X) = \mathcal{R}(A^k(A^k)^\dagger A),\ \mathcal{N}(X) = \mathcal{R}(A^k(A^k)^\dagger A). \tag{29}$$

Thus, by (28) and (29)

$$XAX = X,\ \mathcal{R}(A^k) = \mathcal{R}(X),\ \mathcal{N}((A^k)^*A) = \mathcal{R}(X). \tag{30}$$

The proof is finished by Lemma 6. □

In [12], Rao and Mitra showed that $A^{\|(B,C)} = B(CAB)^-C$, where $(CAB)^-$ stands for arbitrary inner inverse of CAB. Thus, by Theorem 10, we have the following theorem.

Theorem 11 *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$. Then the expression of the generalized constrained inverse of A is $A^-_{gRC} = A^k((A^k)^*A^{k+1})^-(A^k)^*A$ for arbitrary inner inverse of $(A^k)^*A^{k+1}$.*

In fact, we can use the Moore-Penrose inverse of $(A^k)^*A^{k+1}$ instead $((A^k)^*A^{k+1})^-$ in the formula $A^-_{gRC} = A^k((A^k)^*A^{k+1})^-(A^k)^*A$.

CONCLUSION

In this paper, two generalized constrained inverses were introduced by using the core part of the Core-EP decomposition of a complex matrix: the generalized column constrained inverse of A and the generalized row constrained inverse of A. We answer the question when a generalized column constrained inverse coincides with a generalized row constrained inverse, that is, if A^-_{gC} is a generalized column constrained inverse of A and A^-_{gR} is a generalized column constrained inverse of A, then $A^{\circledS} = A^-_{gC}A_1A^-_{gR}$. We obtained the "distance" between the generalized constrained inverse and the inverse along two matrices, that is, the generalized constrained inverse of A coincides with the $(A^k, (A^k)^*A)$-inverse of A for $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$.

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