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Gradient estimate for eigenfunctions of the operator $\mathfrak{L}$ on self-shrinkers

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ABSTRACT: In this paper, we study gradient estimates for eigenfunctions associated to the operator $\mathfrak{L}$ on self-shrinkers. As applications, we obtain a Harnack type inequality concerning those eigenfunctions. Besides, we obtain a gradient estimate of the higher eigenfunctions of the operator $\mathfrak{L}$ on self-shrinkers.

KEYWORDS: eigenfunction, self-shrinker, ∞-Bakry-Émery Ricci tensor, gradient estimate, Harnack inequality

MSC2020: 53C21 53C44

INTRODUCTION

Mean curvature flow is an evolution equation where a one-parameter family of $M_t \subset \mathbb{R}^{n+1}$ hypersurfaces flows by mean curvature, that is, it satisfies

$$(\partial_t X)^\perp = -HN, \tag{1}$$

where X is the position vector, H is the mean curvature and N is the outward unit normal. $(\cdot)^\perp$ denotes the projection on the normal bundle of M.

We call a hypersurface $M^n \subset \mathbb{R}^{n+1}$ a self-shrinker, if it satisfies

$$H = \frac{\langle X, N\rangle}{2}. \tag{2}$$

The self-shrinker plays an important role in the study of mean curvature flow. It appears as the rescaling limit of the Type I singularity of the mean curvature flow. For more information on self-shrinkers and singularities of mean curvature flow, we refer the readers to [1–4] and references therein.

In [1], Colding and Minicozzi introduced the following differential operator $\mathfrak{L}$ and used it to study self-shrinkers:

$$\mathfrak{L}(\cdot) = \Delta(\cdot) - \tfrac{1}{2}\langle X, \nabla(\cdot)\rangle, \tag{3}$$

where Δ, ∇ denote the Laplacian, the gradient operator on the self-shrinker, respectively, $\langle\cdot,\cdot\rangle$ stands for the standard inner product in $\mathbb{R}^{n+1}$.

In [5], Cheng and Peng investigated the closed eigenvalue problem of the differential operator $\mathfrak{L}$ on an n-dimensional compact self-shrinker, and obtained some universal inequalities for the eigenvalues of the drifting Laplacian. We refer the readers to [6–11] and references therein for more information about the eigenvalues of $\mathfrak{L}$ on self-shrinkers.

In this paper, we will deal with eigenfunctions of the operator $\mathfrak{L}$ on self-shrinkers. Our first result is the next theorem that presents a gradient estimate for eigenfunctions of $\mathfrak{L}$ on a compact self-shrinker with boundary, under Neumann boundary conditions, as well as on a closed self-shrinker.

Theorem 1 *Let $X: M^n \to \mathbb{R}^{n+1}$ $(n \geqslant 2)$ be an n-dimensional compact self-shrinker with convex boundary. Suppose $|A| \leqslant K_1$ and $|X^\top| \leqslant K_2$, where A and $X^\top$ denote the second fundamental form and the tangential projection of X, respectively, and both $K_1 \geqslant \sqrt{2}/2$ and K_2 are arbitrary nonnegative constants. Let u be a solution of $\mathfrak{L}u = -\lambda u$, bounded from below, satisfying the Neumann boundary condition $u_\nu = 0$ on ∂M, whenever $\partial M \neq \varnothing$. Then, for any $\alpha > 0$ and $\beta > 0$,*

$$|\nabla u| \leqslant C\big(u - \inf_M u\big), \tag{4}$$

where

$$C = \left\{\left[\sqrt{\left(\tfrac{K_2^2}{4\alpha(n-1)} + K_1^2 - \tfrac{1}{2}\right)^2 (1+\alpha)^2(1+\beta)^2(n-1)^2\beta^2 + 4\beta(1+\beta)\lambda^2} + \left(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2}\right)(1+\alpha)(1+\beta)(n-1)\beta\right]\cdot\frac{1}{2\beta}\right\}^{\frac{1}{2}}.$$

Moreover, if $K_1^2 + \frac{K_2^2}{4\alpha(n-1)} = \frac{1}{2}$ and taking the limit as β approaches to infinity, we can assume $C = \sqrt{|\lambda|}$.

Furthermore, we obtain a gradient estimate for eigenfunctions of $\mathfrak{L}u = -\lambda u$ on balls in complete self-shrinkers with $|A| \leqslant K_3 \,(\geqslant \sqrt{2}/2)$ and $|X^\top| \leqslant K_4$.

Theorem 2 *Let $X: M^n \to \mathbb{R}^{n+1}$ $(n \geqslant 2)$ be an n-dimensional complete self-shrinker. Fix a point $x \in M^n$, let $B(x,r)$ be a geodesic ball of radius r and centered at x. And for any $K_3 \geqslant \sqrt{2}/2$ and $K_4 \geqslant 0$, we assume $|A| \leqslant K_3$ and $|X^\top| \leqslant K_4$ on $B(x,r)$, where A and $X^\top$ denote the second fundamental form and the tangential projection of X, respectively. If u is a positive solution of $\mathfrak{L}u = -\lambda u$ on M, then, for any $\alpha > 0$ and $\beta > 0$,*

$$\sup_{B(x,r/2)} \frac{|\nabla u|}{u} \leqslant C, \tag{5}$$

where $C = C(\alpha, \beta, n, K_3, K_4, r, \lambda)$ is a positive constant depending on α, β, n, K_3, K_4, r and λ, and the supremum is taking over balls $B(x, r/2)$ in M centered at a point x with radius $r/2$.

As an application, we have the following Harnack type inequalities:

Corollary 1 *Let $X: M^n \to \mathbb{R}^{n+1}$ $(n \geqslant 2)$ be an n-dimensional complete self-shrinker. Fix a point $x \in M^n$, let $B(x, r)$ be a geodesic ball of radius r and centered at x. And for any $K_3 \geqslant \sqrt{2}/2$ and $K_4 \geqslant 0$, we assume $|A| \leqslant K_3$ and $|X^\top| \leqslant K_4$ on $B(x, r)$, where A and $X^\top$ denote the second fundamental form and the tangential projection of X, respectively.*

(i) *If u is a solution of $\mathfrak{L}u = -\lambda u$ on a geodesic ball $B(x, r)$, then*

$$\sup_{B(x,r/2)} |\nabla u| \leqslant 2C \sup_{B(x,r)} |u|.$$

(ii) *If u is a positive solution of $\mathfrak{L}u = -\lambda u$ on a geodesic ball $B(x, r)$, then*

$$\sup_{B(x,r/2)} u \leqslant \mathrm{e}^{2Cr} \inf_{B(x,r/2)} u.$$

In both cases $C = C(\alpha, \beta, n, K_3, K_4, r, \lambda)$ is a positive constant depending on α, β, n, K_3, K_4, r and λ.

We point out that the above theorems generalize some results due to Zhu and Chen [12] obtained for $\mathfrak{L}u = 0$. In the next sections we will present the proofs of them.

In [13], Wang and Zhou showed the lower bound for the higher eigenvalues of the Hodge Laplacian on a Riemannian manifold with Ricci curvature bounded from below. Following the ideas in the paper of Wang and Zhou [13], Dung, Le Hai and Thanh [14] showed a gradient estimate of the higher eigenfunctions of the weighted Laplacian on gradient steady Ricci soliton. Motivated by the above results, we will prove the following theorem.

Theorem 3 *Let $X: M^n \to \mathbb{R}^{n+1}$ be an n-dimensional compact self-shrinker. Suppose $|A| \leqslant \sqrt{2}/2$ and $|X^\top| \leqslant 2a$ for some constant $a > 0$, where A and $X^\top$ denote the second fundamental form and the tangential projection of X, respectively. Then*

(i) $|\nabla \phi_l| \leqslant c\lambda_l^{(n+2)/4}$, $|\phi_l| \leqslant c\lambda_l^{n/4}$;

(ii) $\lambda_l \geqslant c^{-1} l^{n/2}$.

Here ϕ_l be an eigenfunction of the $\mathfrak{L}$ with respect to the eigenvalue λ_l and $\|\phi_l\|_\varphi^2 := \int_M \phi_l^2 \mathrm{e}^{-\varphi}\,\mathrm{d}v = 1$.

GRADIENT ESTIMATE ON COMPACT SELF-SHRINKERS WITH BOUNDARY

Let $X: M^n \to \mathbb{R}^{n+1}$ be an n-dimensional compact hypersurface with boundary ∂M in the Euclidean space $\mathbb{R}^{n+1}$. We choose a local orthonormal frame field $\{e_\alpha\}_{\alpha=1}^{n+1}$ in $\mathbb{R}^{n+1}$ with dual coframe field $\{\omega_\alpha\}_{\alpha=1}^{n+1}$, such that, at any $x \in M^n$, $e_1, \ldots, e_n$ are the unit tangent vectors and $e_{n+1} = N$ is the unit normal vector to M^n, and $\underline{e_n} = \nu$ is the unit normal vector to ∂M. Let $\langle \cdot, \cdot \rangle$ and $\overline{\nabla}$ denote the standard inner product and Levi-Civita connection of $\mathbb{R}^{n+1}$. The coefficients of second fundamental form A of M^n are defined to be $A_{ij} = -\langle \overline{\nabla}_{e_i} e_j, N \rangle$. The mean curvature of M^n is expressed by $H = \sum_{i=1}^n A_{ii}$.

Let $\varphi = |X|^2/4$, and denote by $\mathrm{d}V$ the corresponding weighted volume measure of M^n,

$$\mathrm{d}V = \mathrm{e}^{-\varphi}\mathrm{d}v,$$

where $\mathrm{d}v$ is the volume form on M^n. Let g and ∇ be the Riemannian metric on M^n induced by $\langle \cdot, \cdot \rangle$ and the Levi-Civita connection induced $\overline{\nabla}$, respectively. Then $M^n = (M^n, g, \mathrm{d}V)$ is a smooth weighted metric measure space, and the drifting Laplacian operator

$$\mathfrak{L}(\cdot) = \Delta(\cdot) - g(\nabla\varphi, \nabla(\cdot)) = \Delta(\cdot) - \tfrac{1}{2}\langle X, \nabla(\cdot)\rangle$$

is a self-adjoint operator with respect to the weighted measure $\mathrm{d}V$, where ∇ and Δ be the gradient and the Laplacian on M^n, respectively. The ∞-Bakry-Émery Ricci tensor Ric_φ of M^n is defined by

$$\mathrm{Ric}_\varphi = \mathrm{Ric} + \mathrm{Hess}(\varphi).$$

From [15] (see also [12]), we get the following lower bound for the ∞-Bakry-Émery Ricci tensor Ric_φ of self-shrinkers,

$$\mathrm{Ric}_\varphi \geqslant \tfrac{1}{2} - |A|^2. \tag{6}$$

The next algebraic estimate will be useful: for any a, b real numbers and α strictly positive, we have

$$(a+b)^2 \geqslant \frac{a^2}{1+\alpha} - \frac{b^2}{\alpha}, \tag{7}$$

and equality holds if and only if $b = -\frac{\alpha}{1+\alpha}a$. Applying (6) and (7) we first deduce the following proposition.

Proposition 1 *Let $X: M^n \to \mathbb{R}^{n+1}$ $(n \geqslant 2)$ be an n-dimensional compact self-shrinker with $|A| \leqslant K_1$ and $|X^\top| \leqslant K_2$, where A and $X^\top$ denote the second fundamental form and the tangential projection of X, respectively, and both K_1 and K_2 are arbitrary nonnegative constants. Let u be a solution of $\mathfrak{L}u = -\lambda u$ with λ constant. Then, for any $\alpha > 0$ and $\beta > 0$,*

$$|\nabla u| \mathfrak{L} |\nabla u| \geqslant \frac{|\nabla(|\nabla u|)|^2}{(1+\alpha)(1+\beta)(n-1)} - \frac{(\lambda u)^2}{(1+\alpha)\beta(n-1)} - \left(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2} + \lambda\right)|\nabla u|^2. \tag{8}$$

Proof: We start using that $\mathfrak{L}|\nabla u|^2 = 2|\nabla u|\mathfrak{L}|\nabla u| + 2|\nabla(|\nabla u|)|^2$ and the Bochner formula

$$\tfrac{1}{2}\mathfrak{L}|\nabla u|^2 = |\nabla^2 u|^2 + \mathrm{Ric}_\varphi(\nabla u, \nabla u) + \langle \nabla u, \nabla \mathfrak{L}u \rangle, \tag{9}$$

to arrive at the following identity

$$\begin{aligned}|\nabla u|\mathfrak{L}|\nabla u| &= \tfrac{1}{2}\mathfrak{L}|\nabla u|^2 - |\nabla(|\nabla u|)|^2\\ &= |\nabla^2 u|^2 + \mathrm{Ric}_\varphi(\nabla u,\nabla u) + \langle \nabla u, \nabla \mathfrak{L}u\rangle - |\nabla(|\nabla u|)|^2\\ &= |\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2 + \mathrm{Ric}_\varphi(\nabla u,\nabla u) - \lambda|\nabla u|^2,\\ &= |\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2 + \mathrm{Ric}_\varphi(\nabla u,\nabla u) - \lambda|\nabla u|^2. \qquad (10)\end{aligned}$$

Note that $\mathrm{Ric}_\varphi \geqslant \frac{1}{2} - |A|^2 \geqslant \frac{1}{2} - K_1^2$, we have

$$|\nabla u|\mathfrak{L}|\nabla u| \geqslant |\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2 + \big(\tfrac{1}{2} - K_1^2 - \lambda\big)|\nabla u|^2. \quad (11)$$

Proceeding, given $p \in M$ we choose an orthonormal frame $\{e_1, \cdots, e_n\}$ around p so that $u_1(p) = |\nabla u|(p)$ and $u_i(p) = 0$, for $2 \leqslant i \leqslant n$, where $u_i := e_i(u)$. Thus,

$$|\nabla(|\nabla u|)|^2 = |\nabla u_1|^2 = \sum_{1\leqslant j\leqslant n} u_{1j}^2 \qquad (12)$$

and

$$\begin{aligned}-\sum_{2\leqslant i\leqslant n} u_{ii} &= -\Delta u + u_{11} = -\mathfrak{L}u + u_{11} - \langle \nabla\varphi, u_1 e_1\rangle\\ &= \lambda u + u_{11} - \varphi_1 u_1. \qquad (13)\end{aligned}$$

Therefore,

$$\begin{aligned}|\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2 &= \sum_{1\leqslant i,j\leqslant n} u_{ij}^2 - \sum_{1\leqslant j\leqslant n} u_{1j}^2\\ &= \sum_{i\neq 1, 1\leqslant j\leqslant n} u_{ij}^2\\ &\geqslant \sum_{2\leqslant i\leqslant n} u_{i1}^2 + \sum_{2\leqslant i\leqslant n} u_{ii}^2\\ &\geqslant \sum_{2\leqslant i\leqslant n} u_{i1}^2 + \frac{1}{n-1}\Big(\sum_{2\leqslant i\leqslant n} u_{ii}\Big)^2\\ &= \sum_{2\leqslant i\leqslant n} u_{i1}^2 + \frac{1}{n-1}(\lambda u + u_{11} - \varphi_1 u_1)^2.\end{aligned}$$

Using twice inequality (7) we obtain, for any α, β, both strictly positive, the following inequality

$$\begin{aligned}(\lambda u + u_{11} - \varphi_1 u_1)^2 &\geqslant \frac{(\lambda u + u_{11})^2}{1+\alpha} - \frac{(\varphi_1 u_1)^2}{\alpha}\\ &\geqslant \frac{1}{1+\alpha}\Big(\frac{u_{11}^2}{1+\beta} - \frac{(\lambda u)^2}{\beta}\Big) - \frac{(\varphi_1 u_1)^2}{\alpha}\\ &= \frac{u_{11}^2}{(1+\alpha)(1+\beta)} - \frac{(\lambda u)^2}{(1+\alpha)\beta} - \frac{(\varphi_1 u_1)^2}{\alpha}.\end{aligned}$$

Hence, for any $\alpha > 0$ and $\beta > 0$, we have

$$\begin{aligned}&|\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2\\ &\geqslant \sum_{2\leqslant i\leqslant n} u_{i1}^2 + \frac{1}{n-1}\Big(\frac{u_{11}^2}{(1+\alpha)(1+\beta)} - \frac{(\lambda u)^2}{(1+\alpha)\beta} - \frac{(\varphi_1 u_1)^2}{\alpha}\Big)\\ &= \Big(\sum_{2\leqslant i\leqslant n} u_{i1}^2 + \frac{u_{11}^2}{(1+\alpha)(1+\beta)(n-1)}\Big) - \frac{(\lambda u)^2}{(1+\alpha)\beta(n-1)} - \frac{(\varphi_1 u_1)^2}{\alpha(n-1)}\\ &\geqslant \frac{1}{(1+\alpha)(1+\beta)(n-1)}\sum_{1\leqslant i\leqslant n} u_{i1}^2 - \frac{(\lambda u)^2}{(1+\alpha)\beta(n-1)} - \frac{(\varphi_1 u_1)^2}{\alpha(n-1)}\\ &= \frac{|\nabla(|\nabla u|)|^2}{(1+\alpha)(1+\beta)(n-1)} - \frac{(\lambda u)^2}{(1+\alpha)\beta(n-1)} - \frac{\langle\nabla\varphi, \nabla u\rangle^2}{\alpha(n-1)}\\ &\geqslant \frac{|\nabla(|\nabla u|)|^2}{(1+\alpha)(1+\beta)(n-1)} - \frac{(\lambda u)^2}{(1+\alpha)\beta(n-1)} - \frac{|\nabla\varphi|^2|\nabla u|^2}{\alpha(n-1)}.\end{aligned}$$

Since $|\nabla\varphi| = |X^\top/2| \leqslant K_2/2$, we have

$$\begin{aligned}|\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2 \geqslant{}& \frac{|\nabla(|\nabla u|)|^2}{(1+\alpha)(1+\beta)(n-1)}\\ &- \frac{(\lambda u)^2}{(1+\alpha)\beta(n-1)} - \frac{K_2^2}{4\alpha(n-1)}|\nabla u|^2. \qquad (14)\end{aligned}$$

From inequalities (11) and (14) we arrive at

$$\begin{aligned}|\nabla u|\mathfrak{L}|\nabla u| &\geqslant |\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2 + \Big(\frac{1}{2} - K_1^2 - \lambda\Big)|\nabla u|^2\\ &\geqslant \frac{|\nabla(|\nabla u|)|^2}{(1+\alpha)(1+\beta)(n-1)} - \frac{(\lambda u)^2}{(1+\alpha)\beta(n-1)}\\ &\quad - \frac{K_2^2}{4\alpha(n-1)}|\nabla u|^2 + \Big(\frac{1}{2} - K_1^2 - \lambda\Big)|\nabla u|^2\\ &= \frac{|\nabla(|\nabla u|)|^2}{(1+\alpha)(1+\beta)(n-1)} - \frac{(\lambda u)^2}{(1+\alpha)\beta(n-1)}\\ &\quad - \Big(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2} + \lambda\Big)|\nabla u|^2.\end{aligned}$$

We complete the proof of Proposition 1. □

PROOF OF THEOREMS 1 AND 2

We will start with the proof of Theorem 1.

Proof: We can suppose u positive, otherwise, we replace u by $u - \inf_M u$. With this choice we can define $\phi := |\nabla u|/u = |\nabla \ln u|$. Then, we infer

$$\nabla\phi = \frac{\nabla|\nabla u|}{u} - \frac{|\nabla u|\nabla u}{u^2}. \qquad (15)$$

At any point where $|\nabla u| \neq 0$, we have

$$\begin{aligned}\mathfrak{L}|\nabla u| &= u\mathfrak{L}\phi + \phi\,\mathfrak{L}u + 2\langle\nabla\phi, \nabla u\rangle\\ &= u\mathfrak{L}\phi - \lambda|\nabla u| + 2\langle\nabla\phi, \nabla u\rangle.\end{aligned}$$

Using Proposition 1, we deduce for any $\alpha > 0$ and $\beta > 0$,

$$\begin{aligned}\mathfrak{L}\phi &= \frac{\mathfrak{L}|\nabla u|}{u} + \frac{\lambda|\nabla u|}{u} - \frac{2\langle\nabla\phi,\nabla u\rangle}{u}\\ &\geqslant \frac{1}{u|\nabla u|}\bigg\{\frac{|\nabla(|\nabla u|)|^2}{(1+\alpha)(1+\beta)(n-1)} - \frac{(\lambda u)^2}{(1+\alpha)\beta(n-1)}\\ &\quad -\bigg(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2} + \lambda\bigg)|\nabla u|^2\bigg\} + \frac{\lambda|\nabla u|}{u}\\ &\quad - \frac{2\langle\nabla\phi,\nabla u\rangle}{u}\\ &= \frac{1}{u|\nabla u|}\bigg\{\frac{|\nabla(|\nabla u|)|^2}{(1+\alpha)(1+\beta)(n-1)} - \frac{(\lambda u)^2}{(1+\alpha)\beta(n-1)}\\ &\quad -\bigg(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2}\bigg)|\nabla u|^2\bigg\} - \frac{2\langle\nabla\phi,\nabla u\rangle}{u}\\ &= \frac{1}{(1+\alpha)(1+\beta)(n-1)}\frac{|\nabla(|\nabla u|)|^2}{u|\nabla u|}\\ &\quad -\bigg(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2}\bigg)\phi - \frac{1}{(1+\alpha)\beta(n-1)}\frac{\lambda^2}{\phi}\\ &\quad - \frac{2\langle\nabla\phi,\nabla u\rangle}{u}.\end{aligned}$$

We have for any $\varepsilon > 0$,

$$\begin{aligned}&\frac{2\langle\nabla\phi,\nabla u\rangle}{u}\\ &\quad = (2-\varepsilon)\frac{\langle\nabla\phi,\nabla u\rangle}{u} + \varepsilon\frac{\langle\nabla(|\nabla u|),\nabla u\rangle}{u^2} - \varepsilon\frac{|\nabla u|^3}{u^3}\\ &\quad \leqslant (2-\varepsilon)\frac{\langle\nabla\phi,\nabla u\rangle}{u} + \varepsilon\frac{|\nabla(|\nabla u|)||\nabla u|}{u^2} - \varepsilon\phi^3\end{aligned}$$

and

$$\varepsilon\frac{|\nabla(|\nabla u|)||\nabla u|}{u^2} \leqslant \frac{\varepsilon}{2}\bigg(\frac{|\nabla(|\nabla u|)|^2}{|\nabla u|u} + \frac{|\nabla u|^3}{u^3}\bigg).$$

Therefore

$$\begin{aligned}\mathfrak{L}\phi &\geqslant \frac{1}{(1+\alpha)(1+\beta)(n-1)}\frac{|\nabla(|\nabla u|)|^2}{u|\nabla u|}\\ &\quad -\bigg(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2}\bigg)\phi - \frac{1}{(1+\alpha)\beta(n-1)}\frac{\lambda^2}{\phi}\\ &\quad - \frac{2\langle\nabla\phi,\nabla u\rangle}{u}\\ &\geqslant \frac{2}{(1+\alpha)(1+\beta)(n-1)}\frac{|\nabla(|\nabla u|)||\nabla u|}{u^2}\\ &\quad - \frac{1}{(1+\alpha)(1+\beta)(n-1)}\phi^3 - \bigg(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2}\bigg)\phi\\ &\quad - \frac{1}{(1+\alpha)\beta(n-1)}\frac{\lambda^2}{\phi} - (2-\varepsilon)\frac{\langle\nabla\phi,\nabla u\rangle}{u}\\ &\quad - \varepsilon\frac{|\nabla(|\nabla u|)||\nabla u|}{u^2} + \varepsilon\phi^3.\end{aligned}$$

Taking $\varepsilon = 2/(1+\alpha)(1+\beta)(n-1)$, we conclude that

$$\begin{aligned}\mathfrak{L}\phi &\geqslant -\bigg(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2}\bigg)\phi - \frac{1}{(1+\alpha)\beta(n-1)}\frac{\lambda^2}{\phi}\\ &\quad -\bigg(2 - \frac{2}{(1+\alpha)(1+\beta)(n-1)}\bigg)\frac{\langle\nabla\phi,\nabla u\rangle}{u}\\ &\quad + \frac{1}{(1+\alpha)(1+\beta)(n-1)}\phi^3. \qquad (16)\end{aligned}$$

Suppose that ϕ attains its maximum at a point $x_0 \in M$. We claim that x_0 is an interior point of M. Otherwise, by the strong maximum principle, $\phi_\nu(x_0) > 0$. Indeed, suppose that $x_0 \in \partial M$. Proceeding, we choose an orthonormal frame $\{e_1, \dots, e_n = \nu\}$ on TM. Then, at x_0,

$$u^2|\nabla u|\phi_\nu = u\bigg(\sum_{j=1}^{n-1} u_j u_{j\nu} + u_\nu u_{\nu\nu}\bigg) - |\nabla u|^2 u_\nu.$$

Let us denote by a_{jk} the components of the second fundamental form of ∂M to deduce, from Neumann condition, the following identity

$$u^2|\nabla u|\phi_\nu = u\sum_{j=1}^{n-1} u_j u_{j\nu} = -u\sum_{j,k=1}^{n-1} a_{jk}u_j u_k.$$

From the convexity boundary condition, we obtain $\phi_\nu(x_0) \leqslant 0$, which is a contradiction. Thus, x_0 lies in the interior of M. Moreover, $\nabla\phi(x_0) = 0$ and $\mathfrak{L}\phi(x_0) \leqslant 0$. Whence, using inequality (16), we deduce

$$\begin{aligned}0 &\geqslant -\bigg(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2}\bigg)\phi(x_0)\\ &\quad - \frac{1}{(1+\alpha)\beta(n-1)}\frac{\lambda^2}{\phi(x_0)} + \frac{1}{(1+\alpha)(1+\beta)(n-1)}\phi^3(x_0).\end{aligned}$$

That is,

$$\begin{aligned}&\beta\phi^4(x_0) - \bigg(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2}\bigg)(1+\alpha)(1+\beta)(n-1)\beta\phi^2(x_0)\\ &\qquad - (1+\beta)\lambda^2 \leqslant 0. \qquad (17)\end{aligned}$$

Therefore, there is a constant $C = C(n, K_1, K_2, \lambda) > 0$ such that, $\phi(x_0) \leqslant C$ and hence, $|\nabla u| \leqslant Cu$ on M. It is easy to verify that $C = \sqrt{|\lambda|}$, when $K_1^2 + \frac{K_2^2}{4\alpha(n-1)} = \frac{1}{2}$ and taking the limit as β approaches to infinity. On the other hand, if $K_1^2 + \frac{K_2^2}{4\alpha(n-1)} \neq \frac{1}{2}$, we obtain, solving inequality (17),

$$\begin{aligned}C = \Bigg\{\Bigg[&\sqrt{\bigg(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2}\bigg)^2(1+\alpha)^2(1+\beta)^2(n-1)^2\beta^2 + 4\beta(1+\beta)\lambda^2}\\ &+ \bigg(\frac{K_2^2}{4\alpha(n-1)} + K_1^2 - \frac{1}{2}\bigg)(1+\alpha)(1+\beta)(n-1)\beta\Bigg]\cdot\frac{1}{2\beta}\Bigg\}^{\frac{1}{2}} > 0,\end{aligned}$$

which completes the proof of Theorem 1. □

Remark 1 If we assume $\lambda = 0$ in Theorem 1, we can take the limit on C when $\beta \to 0$ to obtain the same estimate of Theorem 1.1 due to Zhu and Chen [12].

In order to present the proof of Theorem 2 we will need a generalized Laplacian comparison theorem obtained by Zhu and Chen [12] for $\mathfrak{L}d$, where d is a distance function on self-shrinkers.

Proposition 2 (Zhu and Chen) *Let $X\colon M^n \to \mathbb{R}^{n+1}$ $(n \geqslant 2)$ be an n-dimensional complete self-shrinker. Fix a point $x \in M^n$, let $B(x,r)$ be a geodesic ball of radius r and centered at x. And for any $K_3 \geqslant 0$ and $K_4 \geqslant 0$, we assume $|A| \leqslant K_3$ and $|X^\top| \leqslant K_4$ on $B(x,r)$, where A and $X^\top$ denote the second fundamental form and the tangential projection of X, respectively. Let $d(y) = d(y,x)$ be the distance function with respect to the fixed point x, then*

$$\mathfrak{L}d \leqslant n\frac{G'(d)}{G(d)} \qquad \text{on} \quad B(x,r), \tag{18}$$

where $G\colon [0,r) \to R^+$ is the solution of the equation

$$\begin{cases} G''(t) - \frac{K_3^2 + \frac{K_4^2}{4} - \frac{1}{2}}{n} G(t) = 0, \\ G(0) = 0, \qquad G(d) = 1. \end{cases} \tag{19}$$

Now we begin the proof of Theorem 2.
Proof: We start using inequality (16) to deduce

$$\begin{aligned} \mathfrak{L}\phi \geqslant & -\left(\frac{K_4^2}{4\alpha(n-1)} + K_3^2 - \frac{1}{2}\right)(n-1)\phi \\ & - \frac{1}{(1+\alpha)\beta(n-1)}\frac{\lambda^2}{\phi} \\ & - \left(2 - \frac{2}{(1+\alpha)(1+\beta)(n-1)}\right)\frac{\langle\nabla\phi,\nabla u\rangle}{u} \\ & + \frac{1}{(1+\alpha)(1+\beta)(n-1)}\phi^3. \end{aligned} \tag{20}$$

Given $r > 0$, let us define a function F as follows

$$F(y) = (r^2 - d^2(x,y))\phi(y), \qquad y \in B(x,r).$$

First we notice that

$$\nabla F = -\phi\nabla(d^2) + (r^2 - d^2)\nabla\phi,$$
$$\mathfrak{L}F = (r^2 - d^2)\mathfrak{L}\phi - \phi\,\mathfrak{L}(d^2) - 2\langle\nabla(d^2),\nabla\phi\rangle.$$

Suppose $|\nabla u| \neq 0$. Since $F = 0$ on $\partial B(x,r)$ and $F > 0$ in $B(x,r)$, F achieves its maximum at some point $x_0 \in B(x,r)$. By Calabi's argument used in [16, p 21], we can suppose that x_0 is not a cut point of x. Therefore, F is smooth near x_0 and $\nabla F = 0$ and $\Delta F \leqslant 0$ at x_0.

Thus, at x_0, we have

$$\mathfrak{L}F = \Delta F - \langle\nabla\varphi,\nabla F\rangle \leqslant 0,$$

$$\frac{\nabla\phi}{\phi} = \frac{\nabla(d^2)}{r^2 - d^2};$$

hence

$$\begin{aligned} \frac{\mathfrak{L}\phi}{\phi} &\geqslant \frac{\mathfrak{L}(d^2)}{r^2-d^2} + \frac{2\langle\nabla(d^2),\nabla\phi\rangle}{(r^2-d^2)\phi} \\ &= \frac{\mathfrak{L}(d^2)}{r^2-d^2} + \frac{2|\nabla(d^2)|^2}{(r^2-d^2)^2}. \end{aligned}$$

Note that $K_3 \geqslant \sqrt{2}/2$, $K_4 \geqslant 0$ and $|\nabla d| = 1$, by (18) and (19), we can get

$$\begin{aligned} \mathfrak{L}d &\leqslant n\sqrt{\frac{K_3^2 + \frac{K_4^2}{4} - \frac{1}{2}}{n}}\coth\left(\sqrt{\frac{K_3^2 + \frac{K_4^2}{4} - \frac{1}{2}}{n}}d\right) \\ &\leqslant \frac{n}{d}\left(1 + \sqrt{\frac{K_3^2 + \frac{K_4^2}{4} - \frac{1}{2}}{n}}d\right) \end{aligned} \tag{21}$$

and

$$\begin{aligned} \mathfrak{L}(d^2) &= 2d\,\mathfrak{L}d + 2|\nabla d|^2 \\ &\leqslant 2n\left(1 + \sqrt{\frac{K_3^2 + \frac{K_4^2}{4} - \frac{1}{2}}{n}}d\right) + 2. \end{aligned} \tag{22}$$

Since $|\nabla(d^2)|^2 = 4d^2$, by inequalities (20) and (22), we obtain, at x_0,

$$\begin{aligned} 0 \geqslant & \frac{\mathfrak{L}F}{(r^2-d^2)\phi} = \frac{\mathfrak{L}\phi}{\phi} - \frac{\mathfrak{L}(d^2)}{r^2-d^2} - \frac{8d^2}{(r^2-d^2)^2} \\ \geqslant & -\left(\frac{K_4^2}{4\alpha(n-1)} + K_3^2 - \frac{1}{2}\right)(n-1) \\ & - \frac{1}{(1+\alpha)\beta(n-1)}\frac{\lambda^2}{\phi^2} \\ & - \left(2 - \frac{2}{(1+\alpha)(1+\beta)(n-1)}\right)\frac{\langle\nabla\phi,\nabla u\rangle}{\phi u} \\ & + \frac{1}{(1+\alpha)(1+\beta)(n-1)}\phi^2 - \frac{8d^2}{(r^2-d^2)^2} \\ & - \frac{1}{r^2-d^2}\left[2n\left(1 + \sqrt{\frac{K_3^2 + \frac{K_4^2}{4} - \frac{1}{2}}{n}}d\right) + 2\right]. \end{aligned}$$

On the other hand, by the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} \frac{\langle\nabla\phi,\nabla u\rangle}{\phi u} &= \frac{\langle\nabla(d^2),\nabla u\rangle}{u(r^2-d^2)} \\ &= \frac{2d\langle\nabla d,\nabla u\rangle}{u(r^2-d^2)} \leqslant \frac{2d\phi}{r^2-d^2}. \end{aligned}$$

Then,

$$0 \geqslant -\left(\frac{K_4^2}{4\alpha(n-1)} + K_3^2 - \frac{1}{2}\right)(n-1)(r^2-d^2)^2$$
$$- \frac{1}{(1+\alpha)\beta(n-1)}\frac{\lambda^2}{F^2}(r^2-d^2)^4$$
$$-4\left(\frac{(1+\alpha)(1+\beta)(n-1)-1}{(1+\alpha)(1+\beta)(n-1)}\right)dF$$
$$+\frac{1}{(1+\alpha)(1+\beta)(n-1)}F^2 - 8d^2$$
$$-\left[2n\left(1+\sqrt{\frac{K_3^2+\frac{K_4^2}{4}-\frac{1}{2}}{n}}d\right)+2\right](r^2-d^2).$$

Note that $r^2 - d^2 \leqslant r^2$ and $d^2 \leqslant r^2$, we have

$$0 \geqslant \beta F^4 - 4\beta[(1+\alpha)(1+\beta)(n-1)-1]dF^3$$
$$-(1+\beta)\lambda^2 r^8 - (1+\alpha)(1+\beta)(n-1)\beta$$
$$\times\left\{\left(K_1^2 + \frac{K_2^2}{4\alpha(n-1)} - \frac{1}{2}\right)(n-1)r^4\right.$$
$$\left. + (10+2n)r^2 + 2n\sqrt{\frac{1}{n}(K_3^2 + \frac{1}{4}K_4^2 - \frac{1}{2})r^3}\right\}F^2.$$

Proceeding, we define

$$\rho(y) = \beta y^4 - 4\beta[(1+\alpha)(1+\beta)(n-1)-1]dy^3$$
$$-(1+\beta)\lambda^2 r^8 - (1+\alpha)(1+\beta)(n-1)\beta$$
$$\times\left\{\left(K_1^2 + \frac{K_2^2}{4\alpha(n-1)} - \frac{1}{2}\right)(n-1)r^4\right.$$
$$\left. + (10+2n)r^2 + 2n\sqrt{\frac{1}{n}(K_3^2 + \frac{1}{4}K_4^2 - \frac{1}{2})r^3}\right\}y^2. \quad (23)$$

Note that $\rho(0) = -(1+\beta)\lambda^2 r^8 < 0$ and hence the polynomial ρ just has two roots, with different signs. Thus, there is a positive constant C, depending on α, β, n, K_3, K_4, r and λ, such that $\rho \leqslant C$, when $\rho(y) \leqslant 0$. Then, we have $F \leqslant C$ on $B(x,r)$, and the following estimate holds

$$\frac{3}{4}r^2 \sup_{B(x,r/2)} \frac{|\nabla u|}{u} \leqslant \sup_{B(x,r/2)} F \leqslant C,$$

that is,

$$\sup_{B(x,r/2)} \frac{|\nabla u|}{u} \leqslant \frac{4}{3}Cr^{-2}. \quad (24)$$

Therefore, we obtain the desired estimate and this finishes the proof of Theorem 2. □

PROOF OF COROLLARY 1

This section is devoted to the proof of Corollary 1.

Proof: To prove the first assertion we consider $\mathscr{U} = \sup_{B(x,r)}|u|$. For any $\epsilon > 0$, we set $v := u + \mathscr{U} + \epsilon > 0$ on $B(x,r)$. Using Theorem 2 we infer

$$\sup_{B(x,r/2)} |\nabla u| = \sup_{B(x,r/2)} |\nabla v| \leqslant C \sup_{B(x,r/2)} (u + \mathscr{U} + \epsilon)$$
$$\leqslant C\left(2\sup_{B(x,r)}|u| + \epsilon\right).$$

Now making $\epsilon \to 0$ we conclude the claim of the first assertion.

Finally, we choose x_1, x_2 in $B(x,r/2)$ satisfying $u(x_1) = \sup_{B(x,r/2)} u$ and $u(x_2) = \inf_{B(x,r/2)} u$. Let $\gamma \subset B(x,r)$ be a minimal geodesic connecting x_1 to x_2. Since γ is contained in $B(x,r)$, we obtain from Theorem 2 and triangle inequality,

$$\log\frac{u(x_1)}{u(x_2)} = \left|\int_\gamma \frac{\mathrm{d}\log u}{\mathrm{d}s}\right| \leqslant \int_\gamma \frac{|\nabla u|}{u}\mathrm{d}s \leqslant \int_\gamma C\mathrm{d}s \leqslant 2Cr.$$

Therefore, $u(x_1) \leqslant \mathrm{e}^{2Cr}u(x_2)$, which ends the proof of Corollary 1. □

PROOF OF THEOREM 3

In this section, we will give a gradient estimate of the higher eigenfunctions of the $\mathfrak{L}$ on compact self-shrinkers. Let $X : M^n \to \mathbb{R}^{n+1}$ be an n-dimensional compact self-shrinkers. Suppose $|A| \leqslant \sqrt{2}/2$ and $|X^\top| \leqslant a$ for some constant $a > 0$, where A and $X^\top$ denote the second fundamental form and the tangential projection of X, respectively.

First, we consider the eigenfunctions ϕ_i ($i = 0, 1, 2, \dots$) of the $\mathfrak{L}$. Since the differential operator $\mathfrak{L}$ is self-adjoint with respect to volume measure $\mathrm{d}V = \mathrm{e}^{-\varphi}\mathrm{d}v$, then the closed eigenvalue problem:

$$\mathfrak{L}\phi_i = -\lambda_i\phi_i, \qquad \int_M \phi_i\phi_j\,\mathrm{d}V = \delta_{ij}$$

for the differential operator $\mathfrak{L}$ on compact self-shrinkers M has a real and discrete spectrum:

$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_l \leqslant \cdots \to \infty,$$

where each eigenvalue is repeated according to its multiplicity. For a given constant c, consider the function

$$P(x) = |\nabla\phi|^2 + c\phi^2,$$

where $\phi = \sum_{i=1}^l b_i\phi_i$ with $b_i \in \mathbb{R}$ and $\sum_{i=1}^l b_i^2 = 1$. Let

$$\psi(b_1, \dots, b_l) := \max_{x\in M} P(x).$$

Assume that $\psi(b_1, \dots, b_l)$ achieves its maximum at some point $a_1, \dots, a_l$.

Lemma 1 *Let $u = \sum_{i=1}^l a_i\phi_i$, then*

$$|\nabla u|^2 + Lu^2 \leqslant L\max_M u^2,$$

where $L = \left(2\lambda_l + a^2 + a\sqrt{4\lambda_l + a^2}\right)/2$.

Proof: We follow the arguments in [14]. Define

$$F(b_1,\dots,b_l,x,\theta)=P(x)-\theta\Big(\sum_{i=1}^{l}b_i^2-1\Big).$$

Then, subject to the constrain $\sum_{i=1}^{l}b_i^2=1$, F achieves its maximum value at some point $(a_1,\dots,a_k,x_0,\alpha)$. We now show

$$|\nabla u|^2(x_0)+cu^2(x_0)\leqslant c\max_M u^2,$$

for $c>\left(2\lambda_l+a^2+a\sqrt{4\lambda_l+a^2}\right)/2$.

At the point $(a_1,\dots,a_k,x_0,\alpha)$, F satisfies

$$\begin{cases}\nabla F(a_1,\dots,a_k,x_0,\alpha)=0\\ \Delta F(a_1,\dots,a_k,x_0,\alpha)\leqslant 0\\ \frac{\partial F}{\partial b_i}=0\\ \sum_{i=1}^{l}a_i^2=1.\end{cases}\tag{25}$$

From the third equation of (25), we have

$$\sum_{j=1}^{l}(2a_j\langle\nabla\phi_i,\nabla\phi_j\rangle+2ca_j\phi_i\phi_j)-2\alpha a_i=0.$$

After multiplying by a_i and summing over i, one sees that

$$\alpha=P(u,x_0)=|\nabla u|^2(x_0)+cu^2(x_0).$$

Suppose now that

$$|\nabla u|^2(x_0)+cu^2(x_0)>c\max_M u^2.$$

Then $\nabla u(x_0)\neq 0$ and one can choose an orthonormal frame $\{e_1,\dots,e_n\}$ at x_0 so that

$$\nabla u(x_0)=u_1(x_0)e_1.$$

Now the first equation of (25) becomes

$$2u_1u_{1i}+2cuu_i=0$$

for $i=1,\dots,n$. This in particular implies

$$|\nabla^2u|^2\geqslant u_{11}^2=c^2u^2.\tag{26}$$

On the other hand, at the maximum point $(a_1,\dots,a_k,x_0,\alpha)$,

$$\Delta F(a_1,\dots,a_k,x_0,\alpha)\leqslant 0$$

or equivalently,

$$\Delta|\nabla u|^2+c\Delta u^2\leqslant 0.\tag{27}$$

Note that $\mathfrak{L}(\cdot)=\Delta(\cdot)-g(\nabla\varphi,\nabla(\cdot))=\Delta(\cdot)-\frac{1}{2}\langle X,\nabla(\cdot)\rangle$. By the Bochner formula, we have

$$\tfrac{1}{2}\mathfrak{L}|\nabla u|^2=|\nabla^2u|^2+\mathrm{Ric}_\varphi(\nabla u,\nabla u)+\langle\nabla u,\nabla\mathfrak{L}u\rangle.\tag{28}$$

From (27) and (28), we obtain

$$|\nabla^2u|^2+\mathrm{Ric}_\varphi(\nabla u,\nabla u)+\langle\nabla u,\nabla\mathfrak{L}u\rangle+\tfrac{1}{2}\langle\nabla\varphi,\nabla|\nabla u|^2\rangle+cu\mathfrak{L}u+c|\nabla u|^2+\tfrac{c}{2}\langle\nabla\varphi,\nabla u^2\rangle\leqslant 0.\tag{29}$$

Since the Cauchy-Schwarz inequality, the Kato inequality ($|\nabla|\nabla u||\leqslant|\nabla^2u|$) and $|\nabla\varphi|=|X^\top/2|\leqslant a$, we have

$$|\langle\nabla\varphi,\nabla|\nabla u|^2\rangle|\leqslant 2|\nabla u||\nabla\varphi||\nabla|\nabla u||\leqslant 2a|\nabla u||\nabla^2u|$$

and

$$|\langle\nabla\varphi,\nabla u^2\rangle|\leqslant 2|u||\nabla u||\nabla\varphi|\leqslant 2a|u||\nabla u|.$$

Since $\mathrm{Ric}_\varphi\geqslant\frac{1}{2}-|A|^2\geqslant 0$, from (26) and (29) we have

$$|\nabla^2u|^2+\langle\nabla u,\nabla\mathfrak{L}u\rangle+cu\mathfrak{L}u+c|\nabla u|^2-a|\nabla u||\nabla^2u|-ca|u||\nabla u|\leqslant 0.$$

Using the inequality $xy\leqslant\frac{x^2}{4\varepsilon}+\varepsilon y^2$ for any $\varepsilon>0$, the above inequality implies

$$(1-a\beta)|\nabla^2u|^2+\langle\nabla u,\nabla\mathfrak{L}u\rangle+cu\mathfrak{L}u+c|\nabla u|^2-\frac{a}{4\beta}|\nabla u|^2-ca\gamma u^2-\frac{ca}{4\gamma}\leqslant 0$$

for any $\beta,\gamma>0$. Since $\mathfrak{L}u=-\sum_{i=1}^{l}\lambda_ia_i\phi_i$, we can compute

$$\begin{aligned}\langle\nabla u,\nabla\mathfrak{L}u\rangle+cu\mathfrak{L}u&=-\sum_{i,j=1}^{l}\lambda_ia_ia_j\langle\nabla\phi_i,\nabla\phi_j\rangle-c\sum_{i,j=1}^{l}\lambda_ia_ia_j\phi_i\phi_j\\&=-\sum_{i=1}^{l}\lambda_ia_i\sum_{j=1}^{l}\left(a_j\langle\nabla\phi_i,\nabla\phi_j\rangle+ca_j\phi_i\phi_j\right)\\&=-\alpha\sum_{i=1}^{l}\lambda_ia_i^2.\end{aligned}$$

Hence, in the view of the inequality (26), if β is small, we have

$$(1-a\beta)c^2u^2-\alpha\sum_{i=1}^{l}\lambda_ia_i^2-ca\gamma u^2+\left(c-\frac{a}{4\beta}-\frac{ca}{4\gamma}\right)|\nabla u|^2\leqslant 0.$$

By Dung, Le Hai and Thanh's arguments used to prove Lemma 2.1 in [14], the inequality reduces to

$$\left(c-\lambda_l-\frac{a^2c}{c-\lambda_l}\right)|\nabla u|^2(x_0)\leqslant 0.$$

This is impossible if $c>\left(2\lambda_l+a^2+a\sqrt{4\lambda_l+a^2}\right)/2$.

The proof is complete by letting c approach $\left(2\lambda_l+a^2+a\sqrt{4\lambda_l+a^2}\right)/2$. □

To prove Theorem 3, we need the following volume comparison theorem for compact self-shrinkers already proved in [17].

Lemma 2 *Let* $X: M^n \to \overline{\mathscr{B}}_k^{n+1}(0) \subset \mathbb{R}^{n+1}$ *be an n-dimensional compact self-shrinkers with* $|A| \leqslant \sqrt{3}/3$*, and* $\overline{\mathscr{B}}_k^{n+1}(0)$ *denotes the Euclidean closed ball with center* 0 *and radius k. Then for any* $p \in M^n$*,* $0 < R_1 \leqslant R_2$*, we have*

$$\frac{Vol(B(p,R_2))}{Vol(B(p,R_2))} \leqslant \mathrm{e}^{3k^2/4} \frac{V(R_2)}{V(R_1)},$$

where $B(p,R)$ *is a geodesic ball of* M^n *with radius R centered at p, and* $V(r)$ *is the volume of the ball with radius r in Euclidean space* $\mathbb{R}^n$.

Proof of Theorem 3: The proof is similar to the proof of Theorem 2.2 in [14] with note that the Bishop volume comparison theorem in [14] is now replaced by the volume comparison in Lemma 2. Since the proof is essentially the same as in Theorem 2.2 in [14], we omit it here.

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