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Average trapping time on the weighted hierarchical triangle network with primary node iteration

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ABSTRACT: In this paper, we consider the average trapping time (ATT) on the weighted hierarchical triangle network with primary node iteration. Firstly, the structure with primary node iteration is shown, then ATT is studied and the exact expression is obtained based on the self-similarity of the network. The results illustrate that ATT grows linearly or sublinearly with the iterative times at different weights and with increasing weights. Besides, compared to the network without primary node iteration, the network with primary node iteration is more efficient in random walks and the difference is greater with increasing weights. When the weight is small, there is little difference.

KEYWORDS: random walk, average trapping time, weighted factor

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INTRODUCTION

Complex network is an emerging cross-discipline that attracted widespread attention and became a popular research topic, with researches involving many aspects of the natural world and social life. In the study of complex networks, the trapping problem has received a lot of attention, which is mainly concerned with the average trapping time (ATT) on networks. Importantly, ATT is defined as the mean of mean first passage times (MFPTs) when a random walker walks from all locations on the network to the trap node after determining a trap node, which can reflect information spreading rate. Therefore, it is of practical importance to consider this problem. For example, many scholars studied ATT on Sierpinski Gasket, Koch network [1, 2]; Wu and Zhang considered ATT on the disrupted network to compare the diffusion efficiency before and after the network is damaged [3]; Xing et al [4] considered mixed random walks. Similarly, there are some other important quantities in random walks, for example, Zhang studied the mean commute time and the results indicated the mean commute time of hierarchical scale-free networks grows in a power law with the iterative times [5].

Generally, only the existence of connected edges between nodes is frequently considered in complex networks, thus ignoring the tightness of the connections, i.e., the weights of edges. In traditional random walks, each path is chosen with the same probability on the network, which is termed as unbiased random walks. However, the weights affect the probability of choosing a path, which develops into biased random walks. Therefore, it has more practical significance to focus on biased random walks. Biased random walks were considered on different networks, such as hierarchical network, (u, v)-folwers (where each link is replaced by two parallel paths of u and v links long), and so on [6–10]. Further, Refs. [11, 12] presented different expansions of the weighted Koch network; Refs. [13, 14] studied ATT on the pseudofractal network by different ways. Besides, Wu et al [15] considered a particular weighted hierarchical graph $H(n, k)$ where n represents the iterative times and k represents the number of nodes of the complete graph, so the simplest iterated graph was got when $k = 3$.

In general cases, each node of the network possesses the same status and iterated capability. But different nodes may occupy different status in practice, which is why Ref. [16] introduced primary nodes and secondary nodes. The constructed network iterates only over the primary nodes, so primary nodes occupy a dominant position and have stronger iterated capability. For example, each iteration of the network models in [17, 18] was highly related to the initial two end vertices, which could be considered as primary nodes. In this paper, based on the idea above, we consider the iterated graph of hierarchical network in [15] and further take into account ATT on the weighted hierarchical triangle network with only primary node iteration, i.e., ignoring the iteration of secondary nodes.

STRUCTURE OF THE WEIGHTED HIERARCHICAL TRIANGLE NETWORK WITH PRIMARY NODE ITERATION

The weighted hierarchical triangle network with primary node iteration is a particular fractal network, which can be generated by iteration and the iteration process is as follows:

(1) The initial graph $G(0)$ is a triangle that each edge has unit weight (written as ω_0). Its nodes are denoted as 1, 2, 3, where 1, 2 are the primary nodes and 3 is the secondary node.

(2) Copy two G_0 and let the weight of each edge be $\omega_1 = r\omega_0 = r(0 < r \leqslant 1)$. Denote the two parts with scale ratio r as G_0^1 and G_0^2. They have two primary nodes respectively, denoted as 4, 5, 6, 7. Then their primary nodes are combined with primary nodes 1 and 2 on G_0. Thus we get the first generation network G_1.

(3) To get the nth generation network G_n, we copy two G_{n-1} and let the weight of each edge be $\omega_n = r\omega_{n-1}$. Similarly, denote the two replications with scale ratio r as G_{n-1}^1, G_{n-1}^2. Then their primary nodes are combined with the primary nodes 1 and 2 on G_0, respectively.

Fig. 1 shows the first three generations of the network. According to the structure of the network, the n-th generation network G_n is divided into three parts: $G_0, G_{n-1}^1, G_{n-1}^2$, where G_0 is initial graph and G_{n-1}^1, G_{n-1}^2 are two replications of the (n–1)-th generation network with scale ratio r. Denote the number of nodes on $G(n)$ as N_n, so we have $N_n = 2^{n+1} - 1$.

ATT ON THE WEIGHTED HIERARCHICAL TRIANGLE NETWORK WITH PRIMARY NODE ITERATION

First, some notations are given: let node 1 be the trap node and $< T >_n$ be the average trapping time (ATT) on G_n, where $< T >_n$ is the mean of MFPTs of all nodes to the trap node,

$$< T >_n = \frac{T_{\text{total}}(n)}{N_n - 1}. \tag{1}$$

$T_{\text{total}}(n)$ is the sum of MFPTs of all nodes to the trap node on G_n,

$$T_{\text{total}}(n) = \sum_{i=2}^{N_n} T_i(n), \tag{2}$$

where $T_i(n)$ is MFPT from node i to the trap node on G_n. Let the probability of node i choosing the path of its neighbor node j be p_{ij}, then

$$p_{ij} = \frac{\omega_{ij}}{S_i} = \frac{\omega_{ij}}{\sum_{j\in V_i} \omega_{ij}},$$

where S_i is the strength of node i, i.e., the sum of the weights between node i and all its neighbors, and V_i denotes the set of all the neighbors. Therefore,

$$T_i(n) = 1 + \sum_{j\in V_i} p_{ij} T_j(n).$$

It has the matrix form

$$T = e + PT.$$

Then

$$< T >_n = \frac{T_{\text{total}}(n)}{N_n - 1} = \frac{\sum_{i=2}^{N_n} T_i(n)}{N_n - 1} = \frac{\sum_{i=2}^{N_n}\sum_{j=2}^{N_n} a_{ij}}{N_n - 1}, \tag{3}$$

where a_{ij} is the ij-th element of matrix $(I-P)^{-1}$. The exact value of ATT can calculate from the matrix, but it is difficult when n is large. Usually we use it to check the analytical expression of ATT.

Next, we give the following results.

Proposition 1 *Let $T_2(n)$ be MFPT from node 2 to the trap node on G_n where n is the iterative number, and the weight factor be r $(0 < r \leqslant 1)$. Then*

$$T_2(n) = T_2(n-1) + 2^{n+1} r^n. \tag{4}$$

Proof: The proposition is proved by mathematical induction. Firstly the calculation yields

$$\begin{aligned} &T_2(0) = 2,\\ &T_2(1) = 2 + 4r,\\ &T_2(2) = 2 + 4r + 8r^2,\\ &T_2(3) = 2 + 4r + 8r^2 + 16r^3,\\ &\qquad\cdots \end{aligned}$$

so there is the conjecture of (4).

Then consider the n-th generation network G_n. Divide the part G_{n-1}^2 into n smaller parts which are $A_0, A_1, \dots, A_{n-1}$ (shown as in Fig. 2). So

$$\begin{cases} T_4(n) = T_2(0) + T_2(n), \quad T_5(n) = T_3(0) + T_2(n),\\ T_6(n) = T_2(1) + T_2(n), \quad T_7(n) = T_3(1) + T_2(n),\\ T_8(n) = T_2(2) + T_2(n), \quad T_9(n) = T_3(2) + T_2(n),\\ \qquad\vdots\\ T_{2n+2}(n) = T_2(n-1) + T_2(n),\\ T_{2n+3}(n) = T_3(n-1) + T_2(n), \end{cases} \tag{5}$$

where $T_3(n) = 1 + \frac{1}{2}T_2(n)$ holds for any n.

Further, let the strength of node 2 on G_n be S, i.e., $S = 2 + 2r + 2r^2 + \cdots + 2r^n$. By the structure of the network, it follows that

$$\begin{aligned} T_2(n) = 1 + \frac{1}{S}T_2(n) + \frac{r^n}{S}(T_4(n) + T_5(n))\\ + \frac{r^{n-1}}{S}(T_6(n) + T_7(n)) + \cdots + \frac{r}{S}(T_{2n+2}(n) + T_{2n+3}(n)). \end{aligned} \tag{6}$$

The simplification gives

$$T_2(n) = S + r^n T_2(0) + r^{n-1}T_2(1) + \cdots + rT_2(n-1). \tag{7}$$

Suppose that $T_2(n-1) = T_2(n-2) + 2^n r^{n-1}$ holds for any $n-1$, then

$$T_2(n-1) = 2 + 2\times 2r + 2\times 2^2 r^2 + \cdots + 2\times 2^{n-1} r^{n-1}.$$

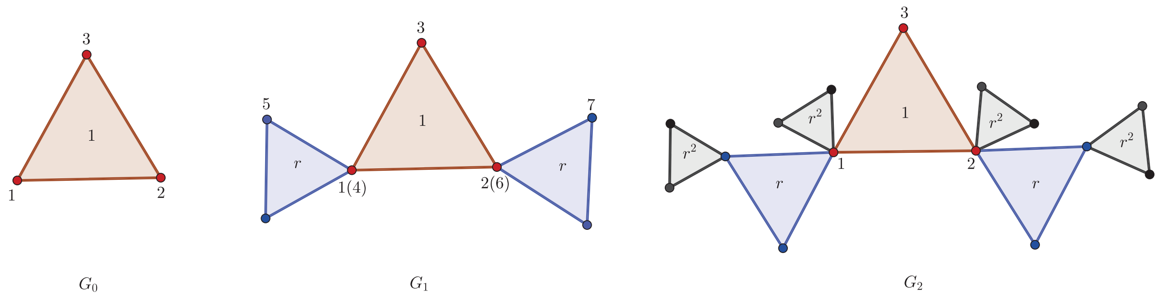

Fig. 1 Iterative process of the weighted hierarchical triangle network with primary node iteration.

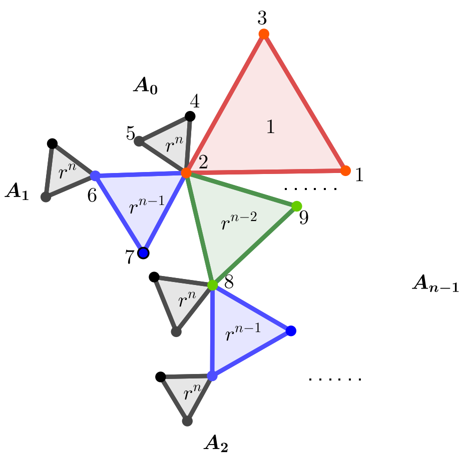

Fig. 2 The block diagram of node 2 on G_n.

Substitute the above equation into (7), we get

$$\begin{aligned}T_2(n) &= 2+2\times 2r+\cdots+2\times 2^{n-1}r^{n-1}+2\times 2^n r^n\\ &= T_2(n-1)+2\times 2^n r^n,\end{aligned}$$

so finally the conjecture is valid. □

Proposition 2 *If $r\neq \frac{1}{2}, 1$, two equations are known respectively:*

$$T_2(n)=2+2\times 2r+\cdots+2\times 2^{n-1}r^{n-1}+2\times 2^n r^n, \quad (8)$$

and

$$\begin{aligned}T_{\text{total}}(n) &= 2^n\times 4+2^{n+1}[T_2(1)+T_2(2)+\cdots+T_2(n)]\\ &\quad -\tfrac{1}{2}[2^{n-1}T_2(1)+2^{n-2}T_2(2)+\cdots+T_2(n)]+2^n-1. \quad (9)\end{aligned}$$

So simplify them yields

$$\begin{aligned}T_{\text{total}}(n) &= 2^{n+1}\left[\frac{2n}{1-2r}+2-\frac{4r^2}{(1-2r)^2}-\frac{r}{1-r}\right]\\ &\quad +\frac{2^{2n+3}r^{n+2}}{(1-2r)^2}-\frac{2^{n+1}r^{n+2}}{(1-r)(1-2r)}+\frac{2r}{1-2r}. \quad (10)\end{aligned}$$

Proof: First simplify the second and third terms of (9).

According to (8), there is

$$T_2(n)=2\cdot\frac{1-2^{n+1}r^{n+1}}{1-2r},$$

if $r\neq\frac{1}{2}$. So the second term is

$$\begin{aligned}&T_2(1)+T_2(2)+\cdots+T_2(n)\\ &\quad=2\left[\frac{1-(2r)^2}{1-2r}+\frac{1-(2r)^3}{1-2r}+\cdots+\frac{1-(2r)^{n+1}}{1-2r}\right]\\ &\quad=\frac{2}{1-2r}\left[n-\left((2r)^2+(2r)^3+\cdots+(2r)^{n+1}\right)\right]\\ &\quad=\frac{2n}{1-2r}-\frac{4r^2(1-2^n r^n)}{(1-2r)^2}, \quad (11)\end{aligned}$$

and the third term is

$$\begin{aligned}&\tfrac{1}{2}[2^{n-1}T_2(1)+2^{n-2}T_2(2)+\cdots+T_2(n)]\\ &\quad=2^{n-1}(1+2r)+2^{n-2}[1+2r+(2r)^2]+\cdots\\ &\qquad+2[1+2r+\cdots+(2r)^{n-1}]+[1+2r+\cdots+(2r)^n]\\ &\quad=(2r)^n+(2r)^{n-1}(1+2)+\cdots\\ &\qquad+2r(1+2+\cdots+2^{n-1})+(1+2+\cdots+2^{n-1})\\ &\quad=(2r)^n(2^1-1)+(2r)^{n-1}(2^2-1)+\cdots\\ &\qquad+2r(2^n-1)+(2^n-1)\\ &\quad=2^{(n+1)}(r^n+r^{n-1}+\cdots+r)\\ &\qquad-[(2r)^n+(2r)^{n-1}+\cdots+(2r)]+2^n-1\\ &\quad=\frac{2^{n+1}r(1-r^n)}{1-r}-\frac{2r(1-2^n r^n)}{1-2r}+2^n-1, \quad (12)\end{aligned}$$

if $r\neq\frac{1}{2},1$.

Then the two terms after simplification are substituted into (9), the proposition is proved. □

In order to solve for ATT, we deal with the analytic formula of $T_{\text{total}}(n)$. Based on the division of the network and the definition, we obtain the following relationship:

$$T_{\text{total}}(n)=2T_{\text{total}}(n-1)+N_{n-1}\cdot T_2(n)+T_3(n), \quad (13)$$

where $T_{\text{total}}(n-1)$ is the sum of MFPTs of all nodes on G^1_{n-1} to the trap node (i.e., node 1), $T_{\text{total}}(n-1)+N_{n-1}\cdot$

$T_2(n)$ is the sum of MFPTs of all nodes on G^2_{n-1} to the trap node, $T_3(n)$ is MFPT from node 3 to the trap node.

Easily see

$$T_3(n) = 1 + \tfrac{1}{2}T_2(n)$$

so $T_{\text{total}}(n)$ is only related to $T_2(n)$.

Further, by Proposition 1, we have

$$T_2(n) = T_2(n-1) + 2^{n+1}r^n,$$

then

$$T_2(n) = 2 + 2\times 2r + \cdots + 2\times 2^{n-1}r^{n-1} + 2\times 2^n r^n, \quad (14)$$

Besides, from (13), the relationship is obtained as follow:

$$T_{\text{total}}(n) = 2^n T_{\text{total}}(0) + 2^{n+1}[T_2(1) + T_2(2) + \cdots + T_2(n)] - \tfrac{1}{2}[2^{n-1}T_2(1) + 2^{n-2}T_2(2) + \cdots + T_2(n)] + 2^n - 1, \quad (15)$$

where $T_{\text{total}}(0) = 4$.

Therefore, by Proposition 2, we have

$$T_{\text{total}}(n) = 2^{n+1}\left[\frac{2n}{1-2r} + 2 - \frac{4r^2}{(1-2r)^2} - \frac{r}{1-r}\right] + \frac{2^{2n+3}r^{n+2}}{(1-2r)^2} - \frac{2^{n+1}r^{n+2}}{(1-r)(1-2r)} + \frac{2r}{1-2r},$$

if $r \neq \frac{1}{2}, 1$.

After $T_{\text{total}}(n)$ is found, by (1), it follows that

$$< T >_n = \frac{2^n}{2^{n+1}-1}\left[\frac{2n}{1-2r} + 2 - \frac{4r^2}{(1-2r)^2} - \frac{r}{1-r}\right] + \frac{2^{2n+2}r^{n+2}}{(2^{n+1}-1)(1-2r)^2} - \frac{2^n r^{n+2}}{(2^{n+1}-1)(1-r)(1-2r)} + \frac{r}{(2^{n+1}-1)(1-2r)}, \quad (16)$$

if $r \neq \frac{1}{2}, 1$.

If $r = \frac{1}{2}$, $T_2(n) = 2(1+n)$ is obtained from (14). Substitute it into (15), we have

$$T_{\text{total}}(n) = 2^{n+2}(n^2 + 3n + 1) + n + 2.$$

Then it follows that

$$< T >_n = \frac{2^n}{2^{n+1}-1}(n^2 + 3n + 1) + \frac{n+2}{2^{n+2}-2}. \quad (17)$$

Similarly, if $r = 1$, there is

$$< T >_n = \frac{2^{2n+2}}{2^{n+1}-1} - \frac{2^n}{2^{n+1}-1}(3n+1) + \frac{1}{2^{n+1}-1}. \quad (18)$$

Eventually, integrating (16) to (18), and letting $n \to \infty$, we obtain the asymptotic expression of ATT (shown in Fig. 3):

$$< T >_n \approx \begin{cases} \dfrac{n}{1-2r}, & 0 < r < \dfrac{1}{2} \\ \dfrac{1}{2}n^2 + \dfrac{3}{2}n, & r = \dfrac{1}{2} \\ \dfrac{r}{(1-2r)^2}(2r)^{n+1}, & \dfrac{1}{2} < r < 1 \\ 2^{n+1}. & r = 1 \end{cases} \quad (19)$$

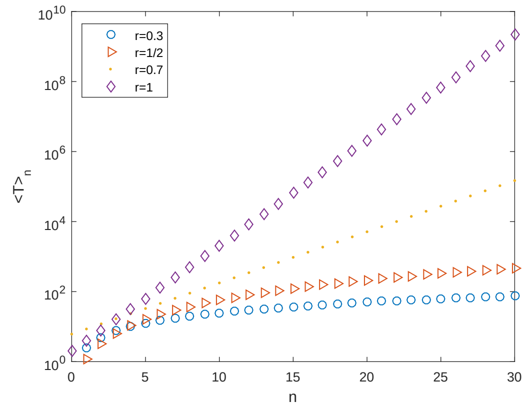

Fig. 3 The trends of ATT with n in semi-logarithmic scale at different weights.

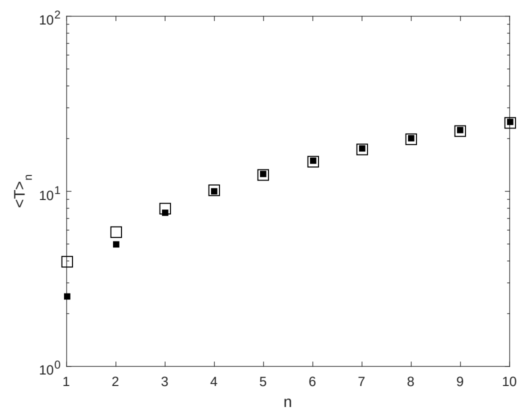

Fig. 4 Numerical simulation when $r = 0.3$.

In Fig. 4, we compare the asymptotic results and the numerical values from (3) when $r = 0.3$. The hollow symbols are the numerical values and the solid symbols are the asymptotic results. These two results are in good agreement, so the asymptotic results are verified.

CONCLUSION

In this paper, we mainly study the ATT on the weighted hierarchical triangle network with primary node iteration. We find that ATT is related to the MFPT of the

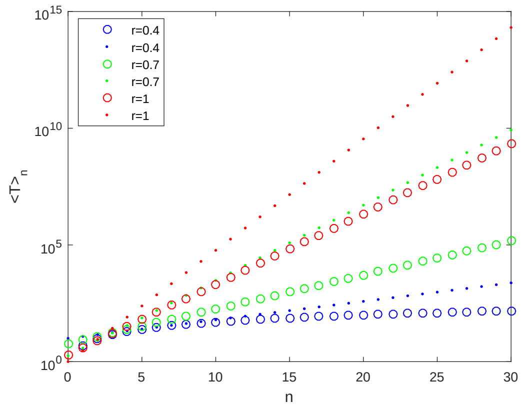

Fig. 5 A comparison for two networks constructed by different iterative ways.

primary node on initial graph (i.e., node 2), which is solved by using hypothetical induction method. Then the exact expression is obtained based on the self-similar structure of the network. Combining the asymptotic formula and Fig. 3, it can be seen that ATT is longer as the network size and the weight increase, which indicates the efficiency of random walks is lower. Further, ATT grows linearly with the iterative times when $\frac{1}{2} < r \leqslant 1$ and sublinearly with the iteration times when $0 < r \leqslant \frac{1}{2}$ in semi-logarithmic scale.

In connection with the results of [15], we get a comparison plot shown in Fig. 5. The hollow circles represent the data of the network with primary node iteration, while the solid dots represent the data of the network without primary node iteration. It can be seen that ATT of the network with primary node iteration is shorter, i.e., the efficiency of random walks is higher. The difference is greater as the weight increases, and the difference is little when the weight is small.

Moreover, there are some interesting open issues that need to be highlighted. For example, since we only consider the average trapping time involving one node, it is better to study the trapping time of multiple nodes. For the network we constructed, it is an interesting topic to study the other fractal properties as well. On the other hand, since our construction was performed on a simple structured network, it is more challenging to explore the primary node iteration approach for a more complex structured network. Another future direction may be to explore the effect of the number of primary nodes on the efficiency of random walks.

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