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Extremal properties of moment for generalized gamma distribution

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ABSTRACT: In this paper, we consider the asymptotic behaviors of moment for normalized extreme of the generalized gamma distribution. Under optimal norming constants, we establish higher-order expansion of moment for the maximum. The expansion is used to deduce the rate of convergence of the moment for normalized partial maximum to the moment of the associating extreme value limit. Numerical simulations are given to sustain the results of our findings.

KEYWORDS: convergence rate, extreme value distribution, generalized gamma distribution, maximum

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INTRODUCTION

Recent work of Cordeiro [1], Suksaengrakcharoen and Bodhisuwan [2] and Agarwal and Kalla [3] has revealed that lifetime data can be moderately fitted by the three-parameter generalized gamma distribution due to Stacy [4]. We say that a random variable X has a generalized gamma distribution with scale parameter $\lambda \in \mathbb{R}^+$ and shape parameters $\beta, c \in \mathbb{R}^+$ (written as $X \sim \mathrm{GGD}(\beta, c, \lambda)$) if its probability density function (pdf) is

$$f(x) = \frac{c\lambda^{c\beta}}{\Gamma(\beta)} x^{c\beta-1} \exp\{-(\lambda x)^c\}, \quad x \in \mathbb{R}^+, \qquad (1)$$

where $\Gamma(\cdot)$ stands for the gamma function [5]. It is known that $\mathrm{GGD}(\beta, 1, \lambda)$ is the gamma distribution, $\mathrm{GGD}(1, c, \lambda)$ is Weibull distribution, and $\mathrm{GGD}(1/2, 2, \lambda)$ is the half-normal distribution. Let $F(x)$ stand for the cumulative distribution function (cdf) associating with (1).

The generalized gamma distribution has lots of applications, ranging from actuarial science, survival analysis to machine learning. Some recent examples of its application include: many problems of diffraction theory and corrosion problems [6], lifetime data analysis and reliability [7], modeling and analysis of lifetimes [8], modeling right censored survival data [9], representing the full rain drop size distribution spectra [10], medical image retrieval system [11], modeling lifetime distribution [12], modeling ultra wideband indoor channel [13], estimating and comparing the reliability of two Operating Systems (Windows and Linux) of DDL MYSQL server [14], generating independent component analysis (ICA) algorithm [15].

It is of great significance to study the properties of given distributions. In this paper, the goal is to establish asymptotic properties for moment of normalized extreme for generalized gamma samples. Let $M_n = \max_{1\leqslant k\leqslant n} X_k$ stand for the partial maximum of an independent and identically (iid) random samples from generalized gamma population $\mathrm{GGD}(\beta, c, \lambda)$. Castro [16] has derived the uniform convergence rate of distribution of extreme from $\mathrm{GGD}(\beta, 1, \lambda)$ to its extreme value limit. Recently, Du and Chen [17] showed that with suitable normalizing constants $a_n \in \mathbb{R}^+$ and $b_n \in R$, the normalized maximum $(M_n - b_n)/a_n$ tends to the Gumbel extreme value distribution $\Lambda(x) = \exp(-\mathrm{e}^{-x})$, i.e.,

$$\lim_{n\to\infty} \mathbb{P}(M_n \leqslant a_n x + b_n) = \Lambda(x). \qquad (2)$$

They also gave the higher-order expansions for the distribution and density of maximum from $\mathrm{GGD}(\beta, c, \lambda)$. However, distribution and density convergence do not necessarily lead to moment convergence, see, e.g., Resnick [18]. Therefore, the natural problems are how about the convergence and higher-order expansions of moments of the normalized extremes, separately. Pickands [19] studied moments convergence of general maxima under some appropriate conditions. Nair [20] obtained asymptotic expansions for the moments of extreme of standard normal distribution. For more related work, see, e.g. Refs. [21–24].

MAIN RESULTS

In order to give the main results, we cite the following results due to Du and Chen [17]:

$$\lim_{n\to\infty} b_n^c \left\{ b_n^c (F^n(a_n x + b_n) - \Lambda(x)) - k_1(x)\Lambda(x) \right\} = \left[k_2(x) + \frac{1}{2}k_1^2(x) \right], \qquad (3)$$

where the norming constants a_n and b_n are given by

$$1-F(b_n)=n^{-1},\quad a_n=c^{-1}\lambda^{-c}b_n^{1-c}, \tag{4}$$

$$k_1(x)=\lambda^{-c}\Big\{(1-c^{-1})\frac{x^2}{2}-(\beta-1)x\Big\}\mathrm{e}^{-x}, \tag{5}$$

and

$$\begin{aligned}k_2(x)=&\lambda^{-2c}\Big\{-(1-c^{-1})^2\frac{x^4}{8}+(1-c^{-1})\\&\times(-2c^{-1}+3\beta-2)\frac{x^3}{6}\\&+(\beta-1)(c^{-1}-\beta+1)\frac{x^2}{2}+(\beta-1)x\Big\}\mathrm{e}^{-x}.\end{aligned} \tag{6}$$

In this section, the asymptotic expansion of moment of extreme for GGD(β,c,λ) is established. For convenience, with norming constants a_n and b_n determined by (4), let, for $r\in\mathbb{N}^+$,

$$m_r(n)=E(\frac{M_n-b_n}{a_n})^r=\int_{-\infty}^{+\infty}x^r(F^n(a_nx+b_n))'\,\mathrm{d}x$$

and

$$m_r=E\xi^r=\int_{-\infty}^{+\infty}x^r(\Lambda(x))'\,\mathrm{d}x \tag{7}$$

separately represent the r-th moments of normalized extreme and Gumel extreme value distribution, where ξ follows Gumel extreme value distribution $\Lambda(x)$. We present the main results as follows.

Theorem 1 *For the norming constant b_n defined by (4), and $m_r(n)$ and m_r given by (7), we have*

$$\begin{aligned}&\lim_{n\to\infty}b_n^c\Big\{b_n^c[m_r(n)-m_r]\\&\quad+r\lambda^{-c}\Big[\frac{1}{2}(1-c^{-1})m_{r+1}+(1-\beta)m_r\Big]\Big\}\\&=r\lambda^{-2c}\Big\{\Big[-\frac{1}{6}(1-c^{-1})(-2c^{-1}+\beta)\\&\quad+\frac{1}{8}(1-c^{-1})^2(r+3)\Big]m_{r+2}\\&\quad+\frac{1}{6}(1-\beta)[3c^{-1}+2(1-c^{-1})(r+2)]m_{r+1}\\&\quad+(1-\beta)\Big[1+\frac{1}{2}(1-\beta)(r+1)\Big]m_r\Big\}.\end{aligned} \tag{8}$$

Observing that $b_n^c\sim\lambda^{-c}\log n$ from (4), by Theorem 1 the convergence rate of moment of normalized extreme can be obtained, which is described as follows.

Corollary 1 *For the norming constant b_n defined by (4), we have, for large n,*

$$m_r(n)-m_r\sim\frac{r\left[(1-c^{-1})m_{r+1}+2(1-\beta)m_r\right]}{2\log n}.$$

The main results in this paper are of practical value. One evident field is the statistical modeling of extreme values based on the Gumbel distribution. One of the common methods used in statistical estimation is the method of moments [25]. This method includes moments, so it is central to know the specific expressions for the moments of the maximum distribution. Moreover, the obtained results can be used to estimate the accuracy of replacing the exact moments of maximum distribution by the moments of the extreme limit distribution. When independent observations are made, this result can be used to determine the sample size when applying asymptotic theory; for more details, refer to the literature [26].

AUXILIARY LEMMAS

In order to prove the main results, the following lemmas will be used. The following Mills-type inequalities of the GGD(β,c,λ) is due to Du and Chen [17].

Lemma 1 *Let $F(x)$ and $f(x)$ separately represent the cdf and pdf of GGD(β,c,λ). For $c\in\mathbb{R}^+$ and $\lambda\in\mathbb{R}^+$, we have*
(i) for $\beta\in(0,1]$ and $x\in\big(\lambda^{-1}[(\beta-1)(\beta-2)]^{1/2c},\infty\big)$,

$$\begin{aligned}\frac{x^{1-c}}{c\lambda^c}\left(1+\frac{\beta-1}{(\lambda x)^c}\right)&\leqslant\frac{1-F(x)}{f(x)}\\&\leqslant\frac{x^{1-c}}{c\lambda^c}\left(1-\frac{(\beta-1)(\beta-2)}{(\lambda x)^{2c}}\right)^{-1};\end{aligned} \tag{9}$$

(ii) for $\beta\in(1,2)$ and $x\in(0,\infty)$,

$$\begin{aligned}&\frac{x^{1-c}}{c\lambda^c}\left(1+\frac{\beta-1}{(\lambda x)^c}\right)\left(1-\frac{(\beta-1)(\beta-2)}{(\lambda x)^{2c}}\right)^{-1}\\&\qquad\leqslant\frac{1-F(x)}{f(x)}\leqslant\frac{x^{1-c}}{c\lambda^c}\left(1+\frac{\beta-1}{(\lambda x)^c}\right);\end{aligned} \tag{10}$$

(iii) for $\beta\in[2,\infty)$ and $x\in(\lambda^{-1}[(\beta-1)(\beta-2)]^{1/2c},\infty)$,

$$\begin{aligned}\frac{x^{1-c}}{c\lambda^c}\left(1+\frac{\beta-1}{(\lambda x)^c}\right)&\leqslant\frac{1-F(x)}{f(x)}\\&\leqslant\frac{x^{1-c}}{c\lambda^c}\left(1-\frac{(\beta-1)(\beta-2)}{(\lambda x)^{2c}}\right)^{-1}.\end{aligned} \tag{11}$$

Lemma 2 *Suppose that the normalizing constant b_n is defined by (4). For $c_1\in(0,1)$ and $i,j\in(0,\infty)$, we have, as $n\to\infty$,*

$$\begin{aligned}&\int_{c_1b_n^{c/3}}^{\infty}b_n^ix^j\Lambda'(x)\,\mathrm{d}x\to0\quad\textit{and}\\&\int_{c_1b_n^{c/3}}^{\infty}b_n^ix^j(1-\Lambda(x))\,\mathrm{d}x\to0.\end{aligned} \tag{12}$$

Proof: By using the inequalities $1-x < e^{-x} < 1$, $x \in (0,\infty)$, we get as $n \to \infty$,

$$\int_{c_1 b_n^{c/3}}^{\infty} b_n^i x^j \Lambda'(x)\,dx \leqslant \int_{c_1 b_n^{c/3}}^{\infty} b_n^i x^j e^{-x}\,dx \leqslant \int_{c_1 b_n^{c/3}}^{\infty} \exp\{-(1-p)c_1 b_n^{c/3}\} b_n^i x^j e^{-px}\,dx \to 0$$

with $p \in (0,1)$. By analogous arguments, as $n \to \infty$,

$$\int_{c_1 b_n^{c/3}}^{\infty} b_n^i x^j (1-\Lambda(x))\,dx \leqslant \int_{c_1 b_n^{c/3}}^{\infty} b_n^i x^j e^{-x}\,dx \to 0.$$

The proof is complete. □

Lemma 3 *For $d \in (0,1)$ and $i, j \in [0,\infty)$, we have, as $n \to \infty$,*

$$\int_{-\infty}^{-d\log b_n} b_n^i |x|^j \Lambda'(x)\,dx \to 0,$$
$$\int_{-\infty}^{-d\log b_n} b_n^i |x|^j \Lambda(x)\,dx \to 0, \qquad (13)$$

and

$$\int_{-\infty}^{-d\log b_n} b_n^i |x|^j F^n(a_n x + b_n)\,dx \to 0. \qquad (14)$$

Proof: It follows from $1-F(b_n) = n^{-1}$ that $b_n \to \infty$ as $n \to \infty$. For $d, p \in (0,1)$, we get as $n \to \infty$,

$$\int_{-\infty}^{-d\log b_n} b_n^i |x|^j \Lambda'(x)\,dx \leqslant \int_{-\infty}^{-1} b_n^i \exp\{-(1-p)b_n^d\}|x|^j \exp\{-p\,e^{-x}\}\,e^{-x}\,dx = \int_{1}^{\infty} b_n^i \exp\{-(1-p)b_n^d\} x^j \exp\{-p\,e^{x}\}\,e^{x}\,dx \to 0,$$

and

$$\int_{-\infty}^{-d\log b_n} b_n^i |x|^j \Lambda(x)\,dx \leqslant \int_{1}^{\infty} b_n^i \exp\{-(1-p)b_n^d\} x^j \exp\{-p\,e^{x}\}\,dx \to 0$$

because $\int_1^\infty x^j \exp\{-p\,e^x\}\,dx < \infty$.

By (4), we obtain $b_n - d a_n \log n \to \infty$ as $n \to \infty$. By utilizing the following inequalities:

$$-x - \frac{x^2}{2(1-x)} < \log(1-x) < -x \text{ for } x \in (0,\ 1),$$

$$(1-x)^c < 1-cx+\frac{c(c-1)}{2}x^2 \text{ for } x \in (0,\frac{1}{2}),\ c \in (1,\infty),$$

$$(1-x)^c < 1-cx \text{ for } x \in (0,1),\ c \in (0,1),$$

and Lemma 1, we get as $n \to \infty$,

$$\begin{aligned} b_n^k F^n(b_n - d a_n \log b_n) &< b_n^k \exp\{-n(1-F(b_n - d a_n \log b_n))\} \\ &< b_n^k \exp\Bigg\{-\frac{(1-dc^{-1}\lambda^{-c}b_n^{-c})^{c(\beta-1)}}{\left(1-\frac{(\beta-1)(\beta-2)}{(\lambda b_n)^{2c}}\right)^{-1}} \times \left(1+\frac{\beta-1}{\lambda^c(b_n - dc^{-1}\lambda^{-c}b_n^{1-c}\log b_n)^c}\right) \\ &\quad \times \exp\{-\lambda^c b_n^c[(1-d a_n b_n^{-1}\log b_n)^2 - 1]\}\Bigg\} \\ &< b_n^k \exp\Bigg\{-\frac{(1-dc^{-1}\lambda^{-c}b_n^{-c})^{c(\beta-1)}}{\left(1-\frac{(\beta-1)(\beta-2)}{(\lambda b_n)^{2c}}\right)^{-1}} \times \left(1+\frac{\beta-1}{\lambda^c(b_n - dc^{-1}\lambda^{-c}b_n^{1-c}\log b_n)^c}\right) b_n^d\Bigg\} \to 0. \end{aligned} \qquad (15)$$

Therefore, as $n \to \infty$,

$$\begin{aligned} &\int_{-\infty}^{-d\log b_n} b_n^i |x|^j F^n(a_n x + b_n)\,dx \\ &\leqslant b_n^i F^{n-1}(b_n - d a_n \log b_n) \int_{-\infty}^{-d\log b_n} |y-b_n|^j F(y)\,dy \\ &\leqslant b_n^i a_n^{-j-1} F^{n-1}(b_n - d a_n \log b_n) \int_{-\infty}^{0} |y-b_n|^j F(y)\,dy \\ &\quad + b_n^{i+j-1} a_n^{-j-1} F^n(b_n - d a_n \log b_n) \int_{0}^{1-d a_n b_n^{-1}\log b_n} |y-1|^j\,dy \\ &\leqslant \sum_{s=0}^{j} \binom{j}{s} b_n^{i+s} a_n^{-j-1} F^{n-1}(b_n - d a_n \log b_n) \int_{-\infty}^{0} y^{j-s} F(-y)\,dy \\ &\quad + b_n^{i+j-1} a_n^{-j-1} F^n(b_n - d a_n \log b_n) \int_{0}^{1} (1-y)^j\,dy \to 0, \end{aligned}$$

since $\int_0^\infty y^r F(-y)\,dy < \infty$ for any $r \in \mathbb{R}^+$. We complete the proof. □

Lemma 4 *For $c_1 \in (0,1)$ and $i, j \in [0,\infty)$, we have, as $n \to \infty$,*

$$\int_{c_1 b_n^{c_1/3}}^{\infty} b_n^i x^j (1-F^n(a_n x + b_n))\,dx \to 0, \qquad (16)$$

and, as $x \to \infty$,

$$x^i(1-F^n(a_n x + b_n)) \to 0. \qquad (17)$$

Proof: The proof is similar to that of Lemma 3.5 in Jia et al [22]. We omit it here, for more details see [22]. □

Lemma 5 *Let $I(b_n;x)=n\log F(a_n x+b_n)+\mathrm{e}^{-x}$ with norming constants a_n and b_n determined by (4). Then, for large n, we have*

$$|I(b_n;x)|<C \tag{18}$$

uniformly for all $x\in(-d\log b_n, c_1 b_n^{c_1/3})$ with $c_1\in(0,1)$ and $d\in(0,\min\{1,c\})$, where C is a positive constant.

Proof: By employing Lemma 4.1 of [27], we get

$$\begin{aligned}1-F(x)&=\frac{\lambda^{c\beta-c}}{\Gamma(\beta)}x^{c\beta-c}\exp\{-(\lambda x)^c\}-r(x)\\&=\frac{\lambda^{c\beta-c}}{\Gamma(\beta)}x^{c\beta-c}\exp\{-(\lambda x)^c\}\\&\quad\times\left[1-(1-\beta)\lambda^{-c}x^{-c}\right]+s(x),\end{aligned} \tag{19}$$

for large $x\in\mathbb{R}^+$ and for $\beta\in(0,1)$, where

$$0<r(x)<\frac{(1-\beta)\lambda^{c\beta-2c}}{\Gamma(\beta)}x^{c\beta-2c}\exp\{-(\lambda x)^c\}\ \text{and}\quad s(x)>0. \tag{20}$$

Let $\Phi_n(x)=1-F(a_n x+b_n)$ and $n\log F(a_n x+b_n)=-n\Phi_n(x)-R_n(x)$, where

$$0<R_n(x)<\frac{n\Phi_n^2(x)}{2(1-\Phi_n(x))}$$

due to $-x-\frac{x^2}{2(1-x)}<\log(1-x)<-x$ for $x\in(0,\ 1)$. Thus,

$$\begin{aligned}|I(b_n;x)|&=|-n\Phi_n(x)-R_n(x)+\mathrm{e}^{-x}|\\&\leqslant|-n\Phi_n(x)+\mathrm{e}^{-x}|+R_n(x).\end{aligned} \tag{21}$$

For $x\in(-d\log b_n, c_1 b_n^{c_1/3})$ and large n, we get

$$\begin{aligned}\Phi_n(x)&<\Phi_n(d\log b_n)\\&=1-F\left(b_n\left(1-\frac{d}{c\lambda^c}b_n^{-c}\log b_n\right)\right)<C_3<1\end{aligned}$$

and by (19) and $1+cx\leqslant(1+x)^c$ as $-1<x<1$ for $c>1$,

$$\begin{aligned}0<R_n(x)&<\frac{1}{2(1-C_3)}\frac{(1-F(a_n x+b_n))^2}{1-F(b_n)}\\&<\frac{1}{2(1-C_3)}\frac{\lambda^{c\beta-c}}{\Gamma(\beta)}\frac{b_n^{c\beta-c}(1+a_n b_n^{-1}x)^{2c\beta-2c}}{1-(c-\beta)\lambda^{-c}b_n^{-c}}\\&\quad\times\exp\{-2\lambda^c b_n^c(1+a_n b_n^{-1}x)^c+\lambda^c b_n^c\}\\&<\frac{1}{2(1-C_3)}\frac{\lambda^{c\beta-c}}{\Gamma(\beta)}\frac{b_n^{c\beta-c}(1+a_n b_n^{-1}x)^{2c\beta-2c}}{1-(c-\beta)\lambda^{-c}b_n^{-c}}\\&\quad\times\exp\{-\lambda^c b_n^c-2x\}\\&<\frac{1}{2(1-C_3)}\frac{\lambda^{c\beta-c}}{\Gamma(\beta)}\frac{b_n^{c\beta-c}(1+a_n b_n^{-1}x)^{2c\beta-2c}}{1-(c-\beta)\lambda^{-c}b_n^{-c}}\\&\quad\times\exp\{-\lambda^c b_n^c+2d\log b_n\}\\&<C_4.\end{aligned} \tag{22}$$

For the case of $x\in[0,\infty)$, it follows that

$$\begin{aligned}|-n\Phi_n(x)+\mathrm{e}^{-x}|&\leqslant n\Phi_n(x)+\mathrm{e}^{-x}\\&\leqslant n(1-F(b_n))+1=2.\end{aligned} \tag{23}$$

Combining with (22) and (23), $|I(b_n;x)|<C_4+2:=C_5$ for $x\in(0,c_1 b_n^{c_1/3})$.

In the following, we take into account the case of $x\in(-d\log b_n,0)$. By (19) and (20), we get

$$\begin{aligned}&-n\Phi_n(x)+\mathrm{e}^{-x}\\&=-(1+a_n b_n^{-1}x)^{c\beta-c}\exp\{-\lambda^c[(a_n x+b_n)^c-b_n^c]\}\\&\quad\times\frac{1-\lambda^{c-c\beta}\Gamma(\beta)(a_n x+b_n)^{c-c\beta}r(a_n x+b_n)}{1-\lambda^{c-c\beta}\Gamma(\beta)b_n^{c-c\beta}r(b_n)\exp\{\lambda^c b_n^c\}}\\&\quad\times\exp\{\lambda^c(a_n x+b_n)^c\}+\mathrm{e}^{-x}\\&=\mathrm{e}^{-x}(1+a_n b_n^{-1}x)^{c\beta-c}D_n(x),\end{aligned}$$

where

$$\begin{aligned}D_n(x)&=(1+a_n b_n^{-1}x)^{c-c\beta}-\frac{1-\mu(a_n x+b_n)}{1-\mu(b_n)}\\&\quad\times\exp\left\{-\lambda^c b_n^c\sum_{k=2}^{+\infty}\binom{c}{k}(a_n b_n^{-1}x)^k\right\}\end{aligned}$$

and

$$\mu(x)=\lambda^{c-c\beta}\Gamma(\beta)x^{c-c\beta}r(x)\exp\{\lambda^c x^c\}$$

with $0<\mu(x)<(1-\beta)\lambda^{-c}x^{-2c}$ for large x. Let

$$G_n(x)=\sum_{k=2}^{\infty}\binom{c}{k}(a_n b_n^{-1}x)^k\quad\text{and}$$

$$H_n(x)=\sum_{k=1}^{\infty}\binom{c-c\beta}{k}(a_n b_n^{-1}x)^k$$

for $x\in(-d\log b_n,0)$. Observing that $1+cx<(1+x)^c<1$ as $c>1$ and $-1<x<0$, it leads to $G_n(x)>0$ and $H_n(x)<0$. By exploiting $1-x<\mathrm{e}^{-x}<1$ for $x>0$, it results in

$$\begin{aligned}D_n(x)&<(1+a_n b_n^{-1}x)^{c-c\beta}-\frac{1-\mu(a_n x+b_n)}{1-\mu(b_n)}(1-\lambda^c b_n^c G_n(x))\\&<1-(1-\lambda^c b_n^c G_n(x))(1-\mu(a_n x+b_n))\\&<(1-\beta)\lambda^{-c}(a_n x+b_n)^{-2c}+\lambda^c b_n^c G_n(x),\end{aligned}$$

and

$$\begin{aligned}D_n(x)&>(1+a_n b_n^{-1}x)^{c-c\beta}-\frac{1-\mu(a_n x+b_n)}{1-\mu(b_n)}\\&>1+H_n(x)-\frac{1}{1-\mu(b_n)}\\&>2H_n(x)-2(1-\beta)\lambda^{-c}b_n^{-2c}.\end{aligned}$$

Consequently, for large n, we get

$$|D_n(x)| < \lambda^c b_n^c G_n(x) + 2|H_n(x)| + 3(1-\beta)\lambda^{-c}(a_n x + b_n)^{-2c}$$

for $x \in (-d\log b_n, 0)$. Since for large n

$$\lambda^c b_n^c G_n(x) \leqslant \frac{1}{2}\lambda^{-c} d^2 b_n^{-c}(\log b_n)^2,$$
$$|H_n(x)| \leqslant (1-\beta) d\lambda^{-c} b_n^{-c}\log b_n$$

holds uniformly for $x \in (-d\log b_n, 0)$, and

$$(a_n x + b_n)^{-2c} \leqslant b_n^{-2c}(1 - dc^{-1}\lambda^{-c} b_n^{-c}\log b_n)^{-2c},$$

there exists a positive constant C_6 such that

$$|D_n(x)| < C_6 b_n^{-c}(\log b_n)^2.$$

Thereby, for large n,

$$\begin{aligned}|-\Phi_n(x) + \mathrm{e}^{-x}| &= \mathrm{e}^{-x}(1 + a_n b_n^{-1} x)^{c\beta - c}|D_n(x)|\\ &= C_6(1 - dc^{-1}\lambda^{-c} b_n^{-c}\log b_n)^{c\beta-c} b_n^{d-c}(\log b_n)^2\\ &< C_7 \qquad (24)\end{aligned}$$

uniformly for $x \in (-d\log b_n, 0)$. A combination of (22) and (24) implies that (18) holds uniformly for $x \in (-d\log b_n, 0)$. The proof is finished. □

Lemma 6 *For large n, $r \in \mathbb{R}^+$ and $x \in (-d\log b_n, c_1 b_n^{c_1/3})$ with $c_1 \in (0,1)$ and $d \in (0, \min\{1, c\})$, $x^r b_n^c \{b_n^c(F^n(a_n x + b_n) - \Lambda(x)) - k_1(x)\Lambda(x)\}$ is controlled by integrable functions independent of n, where norming constants a_n and b_n are determined by (4), and $k_1(x)$ is provided by (5).*

Proof: By applying Lemma 5, for large n it brings about

$$\begin{aligned}&|b_n^c\left\{b_n^c(F^n(a_n x + b_n) - \Lambda(x)) - k_1(x)\Lambda(x)\right\}|\\ &\quad< \left|b_n^c[b_n^c I(b_n; x) - k_1(x)]\Lambda(x)\right|\\ &\qquad+ b_n^{2c} I^2(b_n; x)\left[2^{-1} + \exp(|I(b_n; x)|)\right]\Lambda(x)\\ &\quad< \left|b_n^c[b_n^c I(b_n; x) - k_1(x)]\Lambda(x)\right|\\ &\qquad+ b_n^{2c} I^2(b_n; x)\left[2^{-1} + \mathrm{e}^C\right]\Lambda(x),\end{aligned}$$

with $I(b_n; x) = n\log F(a_n x + b_n) + \mathrm{e}^{-x}$.

Observe that for $s \in \mathbb{R}^+$ and $i \in \mathbb{Z}^+$, $\int_{-\infty}^{+\infty} x^i \mathrm{e}^{-sx}\exp\{-\mathrm{e}^{-x}\}\,\mathrm{d}x = (-1)^i\Gamma^{(i)}(s) < \infty$. Next, both $b_n^c[b_n^c I(b_n; x) - k_1(x)]$ and $b_n^{2c} I^2(b_n; x)$ are controlled by $q(x)\mathrm{e}^{-x}$ will be proved with $q(x)$ being a polynomial about x. Because of the proof of $b_n^{2c} I^2(b_n; x)$ is similar to that of $b_n^c[b_n^c I(b_n; x) - k_1(x)]$, we just prove that $b_n^c[b_n^c I(b_n; x) - k_1(x)]$ is controlled by $q(x)\mathrm{e}^{-x}$.

Adapt

$$\begin{aligned}&b_n^c[b_n^c I(b_n; x) - k_1(x)]\\ &\quad= b_n^{2c}(n\log F(a_n x + b_n) + e^{-x} - b_n^{-c} k_1(x))\\ &\quad= b_n^{2c}(-n\Phi_n(x) + \mathrm{e}^{-x} - b_n^{-c} k_1(x)) - b_n^{-c} R_n(x). \qquad (25)\end{aligned}$$

By making use of (22), for large n, we get

$$\begin{aligned}&b_n^{2c} R_n(x)\\ &< \frac{1}{2(1-C_3)}\frac{\lambda^{c\beta-c}}{\Gamma(\beta)}\frac{b_n^{c\beta-c}(1 - dc^{-1}\lambda^{-c} b_n^{-c}\log b_n)^{2c\beta-2c}}{1 - (c-\beta)\lambda^{-c} b_n^{-c}}\\ &\quad\times \mathrm{e}^{-x}\exp\{-\lambda^c b_n^c + d\log b_n\}\\ &< \frac{1}{2(1-C_3)}\frac{\lambda^{c\beta-c}}{\Gamma(\beta)}\frac{b_n^{c\beta-c+d}(1 - dc^{-1}\lambda^{-c} b_n^{-c}\log b_n)^{2c\beta-2c}}{1 - (c-\beta)\lambda^{-c} b_n^{-c}}\\ &\quad\times \mathrm{e}^{-x}\exp\{-\lambda^c b_n^c\}\\ &< \mathrm{e}^{-x} \qquad (26)\end{aligned}$$

for $x \in (-d\log b_n, c_1 b_n^{c_1/3})$. It is easy to see that $a_n x + b_n > 0$ as $x \in (-d\log b_n, c_1 b_n^{c_1/3})$ for large n. Note that $1 + cx < (1+x)^c$ for $c > 1$ and $-1 < x < 0$. By (19) and (20), it implies that

$$\begin{aligned}\frac{1 - F(a_n x + b_n)}{1 - F(b_n)} &< \frac{\exp\{-\lambda^c b_n^c[(1 + a_n b_n^{-1} x)^c - 1]\}}{(1 + a_n b_n^{-1} x)^{c\beta-c}[1 - (1-\beta)\lambda^{-c} b_n^{-c}]}\\ &< \frac{\mathrm{e}^{-x}}{(1 + a_n b_n^{-1} x)^{c\beta-c}[1 - (1-\beta)\lambda^{-c} b_n^{-c}]} < C_8\,\mathrm{e}^{-x} \qquad (27)\end{aligned}$$

holds for all $x \in (-d\log b_n, c_1 b_n^{c_1/3})$. It follows from Lemma 4.2 of [17] that

$$\frac{1 - F(a_n x + b_n)}{1 - F(b_n)}\,\mathrm{e}^{-x} = A_n(x)\exp\left\{\int_0^x B_n(t)\,\mathrm{d}t\right\},$$

where

$$A_n(x) = \frac{\overline{A}_n(x)}{\underline{A}_n(x)}$$

and

$$B_n(t) = c\lambda^c a_n(a_n t + b_n)^{c-1} - \frac{c(\beta-1)a_n}{a_n t + b_n} - 1$$

with $A_n(x) \to 1$ as $n \to \infty$ uniformly for $x \in (-d\log b_n, c_1 b_n^{c_1/3})$, where

$$\overline{A}_n(x) = 1 + (\beta-1)\lambda^{-c} b_n^{-c} + (\beta-1)(\beta-2)\lambda^{-2c} b_n^{-2c} + O(b_n^{-3c})$$

and

$$\begin{aligned}\underline{A}_n(x) &= 1 + (\beta-1)\lambda^{-c}(a_n x + b_n)^{-c}\\ &\quad+ (\beta-1)(\beta-2)\lambda^{-2c}(a_n x + b_n)^{-2c} + O((a_n x + b_n)^{-3c}).\end{aligned}$$

Thereupon,

$$b_n^{2c}\left[-n\Phi_n(x)+e^{-x}-b_n^{-c}k_1(x)\right]=\frac{1-F(a_nx+b_n)}{1-F(b_n)}b_n^{2c}\times\left\{-1+\frac{1-F(b_n)}{1-F(a_nx+b_n)}e^{-x}[1-k_1(x)e^x b_n^{-c}]\right\}=:\frac{1-F(a_nx+b_n)}{1-F(b_n)}[P_n(x)+Q_n(x)-T_n(x)+S_n(x)],\quad(28)$$

where

$$P_n(x)=b_n^{2c}(A_n(x)-1)$$
$$Q_n(x)=b_n^{2c}A_n(x)\left\{\int_0^x B_n(t)\,dt-\lambda^{-c}\left[(1-c^{-1})\frac{x^2}{2}-(\beta-1)x\right]b_n^{-c}\right\}$$
$$T_n(x)=b_n^cA_n(x)\int_0^x B_n(t)\,dt\,\lambda^{-c}\left[(1-c^{-1})\frac{x^2}{2}-(\beta-1)x\right]$$
$$S_n(x)=b_n^{2c}A_n(x)\sum_{k=2}^{\infty}\frac{\left(\int_0^x B_n(t)\,dt\right)^k}{k!}\times\left\{1-\lambda^{-c}\left[(1-c^{-1})\frac{x^2}{2}-(\beta-1)x\right]b_n^{-c}\right\}.$$

For $x\in(-0,c_1b_n^{c_1/3})$, by employing $1-cx<(1+x)^{-c}<1$ for $c>1$ and $-1<x<0$, we get

$$\begin{aligned}|P_n(x)|&<b_n^{2c}[1-(1-\beta)(\lambda b_n)^{-c}(1+c^{-1}\lambda^{-c}b_n^{-c}x)^{-c}]^{-1}\\&\quad\times\Big|(\beta-1)(\lambda b_n)^{-c}[1-(1+c^{-1}\lambda^{-c}b_n^{-c}x)^{-c}]\\&\quad+(\beta-1)(\beta-2)(\lambda b_n)^{-2c}\\&\quad\times[1-(1+c^{-1}\lambda^{-c}b_n^{-c}x)^{-2c}]+O(b_n^{-3c})\Big|\\&<\left[1-(1-\beta)\lambda^{-c}b_n^{-c}\right]^{-1}[(\beta-1)\lambda^{-2c}x\\&\quad+2(\beta-1)(\beta-2)\lambda^{-3c}b_n^{-c}x+O(b_n^{-2c})]\\&<2\lambda^{-2c}(x+4\lambda^{-c}x).\end{aligned}\quad(29)$$

For $x\in(-d\log b_n,0)$ and large n, it leads to

$$\begin{aligned}&|P_n(x)|\\&<b_n^{2c}[1-(1-\beta)(\lambda b_n)^{-c}(1-dc^{-1}\lambda^{-c}b_n^{-c}\log b_n)^{-c}]^{-1}\\&\times\Big|(\beta-1)(\lambda b_n)^{-c}(1-dc^{-1}\lambda^{-c}b_n^{-c}\log b_n)^{-c}\\&\times[(1+c^{-1}\lambda^{-c}b_n^{-c}x)^c-1]\\&+(\beta-1)(\beta-2)(\lambda b_n)^{-2c}(1-dc^{-1}\lambda^{-c}b_n^{-c}\log b_n)^{-2c}\\&\times[(1+c^{-1}\lambda^{-c}b_n^{-c}x)^{2c}-1]+O(b_n^{-3c})\Big|\\&<4(1-\beta)\lambda^{-2c}[1+2(2-\beta)\lambda^{-c}]|x|\end{aligned}\quad(30)$$

which follows from $1+cx<(1+x)^c<1$ for $c>1$ and $-1<x<0$.

For $Q_n(x)$, $T_n(x)$ and $S_n(x)$, we get, for large n,

$$|Q_n(x)|<2\left\{\frac{1}{6}\lambda^{-2c}|(1-c^{-1})(1-2c^{-1})||x|^3+\frac{1}{2}(1-\beta)\lambda^{-c}|c\lambda^c-d|^{-1}x^2\right\}\quad(31)$$

$$|T_n(x)|<2\lambda^{-c}\left\{(1-c^{-1})\frac{x^2}{2}+(1-\beta)|x|\right\}\times\left\{\frac{c(1-\beta)}{|c\lambda^c-d|}|x|+\frac{1}{2}(1-c^{-1})\lambda^{-c}x^2+\frac{1}{6}\lambda^{-2c}|(1-c^{-1})(1-2c^{-1})||x|^3\right\}\quad(32)$$

$$|T_n(x)|<\left\{1+\lambda^{-c}(1-c^{-1})\frac{x^2}{2}+(1-\beta)|x|\right\}\times\left\{\frac{c(1-\beta)}{|c\lambda^c-d|}|x|+\frac{1}{2}(1-c^{-1})\lambda^{-c}x^2+\tfrac{1}{6}\lambda^{-2c}|(1-c^{-1})(1-2c^{-1})||x|^3\right\}^2\exp\left\{\frac{2(1-\beta)d}{1-d}\right\}\quad(33)$$

for $x\in(-d\log b_n,c_1b_n^{c_1/3})$.

Combining with (27)–(33) and (26), the wanted result is obtained. □

PROOF OF MAIN RESULT

From [18, Proposition 2.1(iii)] and $\int_{-\infty}^0|x|^rf(x)\,dx<\infty$ for all $r\in\mathbb{Z}^+$, it results in, as $n\to\infty$,

$$m_r(n)=\mathbb{E}\Big(\frac{M_n-b_n}{a_n}\Big)^r\to m_r=\int_{-\infty}^{+\infty}x^r\,d\Lambda(x)=(-1)^r\Gamma(r)(1)$$

with $\Gamma(r)(1)$ indicating the r-th derivative of the gamma function at $x=1$. Accordingly, for large n, $m_r(n)<\infty$ and

$$\begin{aligned}m_r(n)-m_r&=\int_{-\infty}^{+\infty}x^r\,[F^n(a_nx+b_n)-\Lambda(x)]'\,dx\\&=\int_{-\infty}^{+\infty}x^r\,d\,[F^n(a_nx+b_n)-\Lambda(x)].\end{aligned}$$

Since $\int_{-\infty}^0|x|^rf(x)\,dx<\infty$, we have $|x|^rF(x)\to0$ as $x\to-\infty$. By using the C_r-inequality, it implies that

$$\begin{aligned}0&\leqslant\lim_{x\to-\infty}|x|^rF(a_nx+b_n)\\&\leqslant\lim_{x\to-\infty}2^{r-1}a_n^{-r}(|y|^r+|b_n|^r)F(y)=0.\end{aligned}$$

Thus, as $x\to-\infty$,

$$|x|^rF^n(a_nx+b_n)\to0.\quad(34)$$

Hence, from (17) and (34), we get

$$\lim_{x\to\infty} x^r(F^n(a_nx+b_n)-\Lambda(x)) = \lim_{x\to\infty} x^r(1-\Lambda(x)) - \lim_{x\to\infty} x^r(1-F^n(a_nx+b_n)) = 0 \quad (35)$$

and

$$\lim_{x\to-\infty} x^r(F^n(a_nx+b_n)-\Lambda(x)) = \lim_{x\to-\infty} x^r(1-\Lambda(x)) - \lim_{x\to-\infty} x^r(1-F^n(a_nx+b_n)) = 0. \quad (36)$$

Consequently, by taking advantage of integration by parts, and (35), (36), we get, for large n,

$$\begin{aligned} m_r(n)-m_r &= \int_{-\infty}^{+\infty} x^r \,\mathrm{d}\left[F^n(a_nx+b_n)-\Lambda(x)\right] \\ &= -r\int_{-\infty}^{+\infty} x^{r-1}\left[F^n(a_nx+b_n)-\Lambda(x)\right]\mathrm{d}x \quad (37) \end{aligned}$$

Observing that

$$\int_{-\infty}^{+\infty} x^k e^{-2x}\Lambda(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} x^k e^{-x}\Lambda'(x)\,\mathrm{d}x = -km_{k-1}+m_k,$$

and by (3), (37), Lemmas 2–6, and the dominated convergence theorem, we get

$$\begin{aligned} &\lim_{n\to\infty} b_n^c\left\{b_n^c[m_r(n)-m_r]+r\lambda^{-c}\left[\tfrac{1}{2}(1-c^{-1})m_{r+1}+(1-\beta)m_r\right]\right\} \\ &= \lim_{n\to\infty} -r\int_{-\infty}^{+\infty} b_n^c\left\{b_n^c[F^n(a_nx+b_n)-\Lambda(x)]-x^{r-1}k_1(x)\Lambda(x)\right\}\mathrm{d}x \\ &= \lim_{n\to\infty}\left\{-r\int_{c_1 b_n^{c_1/3}}^{\infty} b_n^{2c}x^{r-1}[(1-\Lambda(x))-(1-F^n(a_nx+b_n))]\,\mathrm{d}x\right. \\ &+ r\int_{c_1 b_n^{c_1/3}}^{\infty} b_n^c x^{r-1}k_1(x)\Lambda(x)\,\mathrm{d}x \\ &- r\int_{-d\log b_n}^{c_1 b_n^{c_1/3}} b_n^c x^{r-1} b_n^c\left\{[F^n(a_nx+b_n)-\Lambda(x)]-x^{r-1}k_1(x)\Lambda(x)\right\}\mathrm{d}x \\ &- r\int_{-\infty}^{-d\log b_n} b_n^{2c}x^{r-1}[F^n(a_nx+b_n)-\Lambda(x)]\,\mathrm{d}x \\ &\left.+ r\int_{-\infty}^{-d\log b_n} b_n^c x^{r-1}k_1(x)\Lambda(x)\,\mathrm{d}x\right\} \\ &= -r\int_{-\infty}^{+\infty} x^{r-1}\Lambda(x)\left[k_2(x)+\tfrac{1}{2}k_1^2(x)\right]\mathrm{d}x \\ &= r\lambda^{-2c}\left\{\left[-\tfrac{1}{6}(1-c^{-1})(-2c^{-1}+\beta)+\tfrac{1}{8}(1-c^{-1})^2(r+3)\right]m_{r+2}\right. \\ &\quad + \tfrac{1}{6}(1-\beta)[3c^{-1}+2(1-c^{-1})(r+2)]m_{r+1} \\ &\quad \left. + (1-\beta)\left[1+\tfrac{1}{2}(1-\beta)(r+1)\right]m_r\right\}. \end{aligned}$$

The proof is complete.

NUMERICAL ANALYSIS

In this section, numerical studies are given to illustrate the precision of the higher-order expansions of the moment of the normalized maximum $(M_n-b_n)/a_n$. Let E_1, E_2, and E_3 stand for the first-order, second-order and third-order asymptotics of the moment of $(M_n-b_n)/a_n$, separately. Notice that the second and third order asymptotics are connected with the sample size n. By Theorem 1, we get

$$\begin{aligned} E_1 &= m_r, \\ E_2 &= m_r - b_n^{-c}r\lambda^{-c}\left[\tfrac{1}{2}(1-c^{-1})m_{r+1}+(1-\beta)m_r\right], \\ E_3 &= m_r - b_n^{-c}r\lambda^{-c}\left[\tfrac{1}{2}(1-c^{-1})m_{r+1}+(1-\beta)m_r\right] \\ &+ b_n^{-2c}r\lambda^{-2c}\left\{\left[-\tfrac{1}{6}(1-c^{-1})(-2c^{-1}+\beta)+\tfrac{1}{8}(1-c^{-1})^2(r+3)\right]m_{r+2}\right. \\ &\quad + \tfrac{1}{6}(1-\beta)[3c^{-1}+2(1-c^{-1})(r+2)]m_{r+1} \\ &\quad \left. + (1-\beta)\left[1+\tfrac{1}{2}(1-\beta)(r+1)\right]m_r\right\}. \end{aligned}$$

To show the accuracy of all asymptotics as the sample size n varies, for fixed parameters β, λ, c, and $r=2$, images of the true values and their asymptotics are drawn. Figs. 1 and 2 demonstrate the following facts: (i) except for some special cases, all approximations get closer to the true values as the sample size n increases; (ii) for sufficiently large n, the third-order approximation is closer to the true value.

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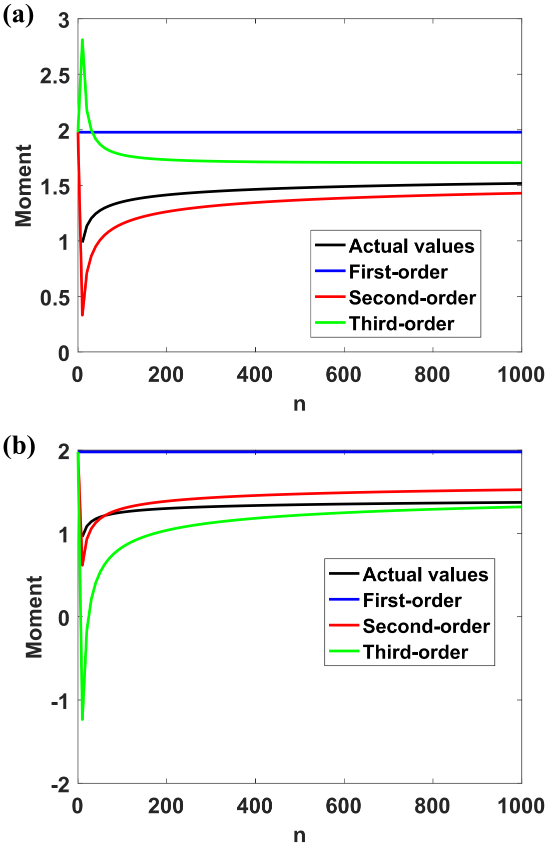

Fig. 1 Actual values and its corresponding asymptotics of the moment of M_n with $r = 2$. (a) $c = 3/2$, $\beta = 1/2$, and $\lambda = 1$; (b) $c = 3/2$, $\beta = 2/3$, and $\lambda = 1$.

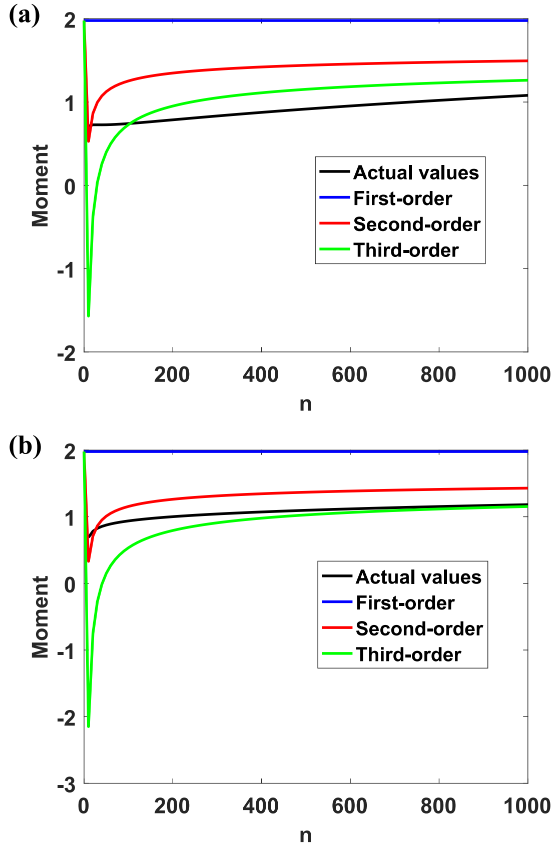

Fig. 2 Actual values and its associating asymptotics of the moment of M_n with $r = 2$. (a) $c = 4/3$, $\beta = 1/2$, and $\lambda = 1$; (b) $c = 3/2$, $\beta = 1/2$, and $\lambda = 0.9$.

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