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On existence of meromorphic solutions for nonlinear q-difference equation

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ABSTRACT: In this paper, we mainly consider the existence of meromorphic solutions of nonlinear q-difference equation of type

$$f(qz)+f(z/q)=\frac{P(z,f(z))}{Q(z,f(z))},$$

where the right-hand side is irreducible, $P(z,f(z))$ and $Q(z,f(z))$ are polynomials in f with rational coefficients, and q is a nonzero complex constant. We obtain that such equation has no transcendental meromorphic solution when $|q|=1$ and $m=\deg_f(P)-\deg_f(Q)>1$. And we investigate the growth of transcendental meromorphic solutions of nonlinear q-difference equation and find lower bounds for their characteristic functions for transcendental meromorphic solutions of such equation for the case $|q|\neq 1$.

KEYWORDS: nonlinear q-difference equation, difference Painlevé equation, the existence of transcendental meromorphic solution

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INTRODUCTION AND MAIN RESULTS

A function $f(z)$ is called meromorphic if it is analytic in the complex plane $\mathbb{C}$ except at isolated poles. In what follows, we use standard notations in the Nevanlinna's value distribution theory of meromorphic functions, see [1, 2]. Let $f(z)$ be a meromorphic function. We also use notations $\sigma(f)$, $\mu(f)$, $\lambda(f)$, $\lambda(1/f)$ for the order, the lower order, the exponents of convergence of zeros and poles of f, respectively.

Recently, some papers focus on complex difference equations, see [3–6]. There are also papers focusing on the existence and the growth of meromorphic solutions of q-difference equations, see [7–10].

Zhang and Korhonen [11] studied the existence of zero order transcendental meromorphic solutions of the certain q-difference equation, and showed the following theorem.

Theorem 1 ([11]) *Let $q_1,\dots,q_n\in\mathbb{C}\backslash\{0\}$, and let $a_0(z),\dots,a_p(z)$, $b_0(z),\dots,b_d(z)$ be rational functions. If the q-difference equation*

$$\sum_{j=1}^{n} f(q_j z)=\frac{P(z,f(z))}{Q(z,f(z))}=\frac{a_0(z)+a_1(z)f(z)+\cdots+a_p(z)f(z)^p}{b_0(z)+b_1(z)f(z)+\cdots+b_d(z)f(z)^d},\qquad (1)$$

where $P(z,f(z))$ and $Q(z,f(z))$ do not have any common factors in $f(z)$, admits a transcendental meromorphic solution of zero order, then $\max\{p,d\}\leqslant n$.

Peng and Huang [12] considered the growth problem for transcendental meromorphic solutions of q-difference Painlevé IV equation, and obtained the following result.

Theorem 2 ([12]) *Consider q-difference equation*

$$(f(qz)+f(z))(f(z)+f(z/q))=\frac{P(z,f(z))}{Q(z,f(z))},\qquad (2)$$

where $P(z,f(z))=a_0(z)+a_1(z)f(z)+\cdots+a_p(z)f(z)^p$ and $Q(z,f(z))=b_0(z)+b_1(z)f(z)+\cdots+b_d(z)f(z)^d$ are relatively prime polynomials in f, and $a_0(z),\dots,a_p(z)$, $b_0(z),\dots,b_d(z)$ are polynomials with $a_p(z)b_d(z)\not\equiv 0$, $q\in\mathbb{C}\backslash\{0\}$. Let $m=p-d\geqslant 3$.

(i) *Suppose that $|q|=1$. Then (2) has no transcendental meromorphic solution.*
(ii) *Suppose that $|q|\neq 1$ and f is a transcendental meromorphic solution of (2).*

① *If f is entire or has finitely many poles, then there exist constants $K>0$ and $r_0>0$ such that*

$$\log M(r,f)\geqslant K\left(\frac{m}{2}\right)^{\log r/|\log|q||}$$

holds for all $r\geqslant r_0$. Thus, the lower order of f satisfies $\mu(f)\geqslant\log(\frac{m}{2})/|\log|q||$.

② *If f has infinitely many poles, then there exist constants $K>0$ and $r_0>0$ such that*

$$n(r,f)\geqslant K(m-1)^{\log r/|\log|q||}$$

holds for all $r \geqslant r_0$. *Thus, the lower order of* f *satisfies* $\mu(f) \geqslant \log(m-1)/|\log|q||$.

③ *Thus, the lower order of* f *satisfies* $\mu(f) \geqslant \log(m-1)/|\log|q||$ *when* $|q| \neq 1$.

Qi and Yang [13] considered the properties of transcendental meromorphic solutions of q-difference equation, and obtained the following result.

Theorem 3 ([13]) *Let* $|q| \neq 1$ *and* $n \geqslant 2$, *let* $f(z)$ *be a meromorphic solution of*

$$f(qz)+f(z/q)=a(z)f(z)^n+b(z)f(z)+c(z)$$

with meromorphic coefficients satisfying $T(r,a)=S(r,f)$, $T(r,b)=S(r,f)$ *and* $T(r,c)=S(r,f)$. *Then* $f(z)$ *is of positive order of growth.*

By Theorem 2 and Theorem 3, if we replace the left-hand side of (2) by $f(qz)+f(z/q)$, then we obtain Theorem 4 as show below.

Theorem 4 *Let* $a_0(z),\dots,a_p(z)$, $b_0(z),\dots,b_d(z)$ *be rational functions with* $a_p(z)b_d(z) \not\equiv 0$. *Consider q-difference equation*

$$\begin{aligned} f(qz)+f(z/q) &= \frac{P(z,f(z))}{Q(z,f(z))} \\ &= \frac{a_0(z)+a_1(z)f(z)+\cdots+a_p(z)f(z)^p}{b_0(z)+b_1(z)f(z)+\cdots+b_d(z)f(z)^d}, \end{aligned} \quad (3)$$

where $P(z,f(z))$ *and* $Q(z,f(z))$ *are relatively prime polynomials in* f, $q \in \mathbb{C}\backslash\{0\}$. *Let* $m=p-d \geqslant 2$.

(i) *Suppose that* $|q|=1$. *Then* (3) *has no transcendental meromorphic solution.*

(ii) *Suppose that* $|q| \neq 1$ *and* f *is a transcendental meromorphic solution of* (3).

① *If* f *is entire or has finitely many poles, then there exist constants* $K>0$ *and* $r_0>0$ *such that for all* $r \geqslant r_0$

$$\log M(r,f) \geqslant K m^{\log r/|\log|q||}.$$

② *If* f *has infinitely many poles, then there exist constants* $K>0$ *and* $r_0>0$ *such that for all* $r \geqslant r_0$

$$n(r,f) \geqslant K m^{\log r/|\log|q||}.$$

③ *Thus, the lower order of* f *satisfies* $\mu(f) \geqslant \log m/|\log|q||$ *when* $|q| \neq 1$.

From Theorem 4, we see that Theorem 3 is extended into more general type.

By Theorem 1 and Theorem 4, we can get that if (3) admits a transcendental meromorphic solution of zero order, then $\max\{p,d\} \leqslant 2$ and $p-d \leqslant 1$.

In fact, many authors studied special forms of Eq. (3) when $\max\{p,d\} \leqslant 2$ and $p–d \leqslant 1$. In particular, they mainly considered three types of equations as shown below.

$$f(qz)+f(z/q)=\frac{A(z)}{f(z)}+C(z), \quad (4)$$

$$f(qz)+f(z/q)=\frac{A(z)}{f(z)}+\frac{C(z)}{f^2(z)}, \quad (5)$$

$$f(qz)+f(z/q)=\frac{A(z)f(z)+C(z)}{1-f^2(z)}, \quad (6)$$

where $A(z)$, $C(z)$ are polynomials. These equations are now known as the q-difference analogues of difference Painlevé equations I and II. Some results about transcendental meromorphic solutions of zero order to (4)–(6), can be found in [13–15].

From this, we see that (3) is an important class of q-difference equations. It will play an important role for research of q-difference Painlevé equations I and II.

By the same arguments as the proof of Theorem 4, we can obtain Corollary 1.

Corollary 1 *Suppose that the q-difference equation* (1) *satisfies the hypothesis of Theorem 1. If* $m=p-d \geqslant 2$ *and* $0<|q_j| \leqslant 1$ $(j=1,2,\dots,n)$, *then* (1) *has no transcendental entire solution.*

Remark 1 ([10]) We shall also use the observation that

$$\begin{aligned} M(r,f(qz)) &= M(|q|r,f), \\ N(r,f(qz)) &= N(|q|r,f)+O(1), \\ T(r,f(qz)) &= T(|q|r,f)+O(1) \end{aligned}$$

hold for any meromorphic function f and any non-zero constant q.

PROOFS OF THEOREM 4 AND COROLLARY 1

The proof of Theorem 4

Without loss of generality, suppose that the coefficients $a_i(z)$ $(i=0,1,\dots,p)$ and $b_n(z)$ $(n=0,1,\dots,d)$ in (3) are polynomials.

(i): On the contrary, suppose that (3) has a transcendental meromorphic solution f. Our conclusion holds for the cases.

Case 1: Suppose that f, the solution of (3), is transcendental entire.

Denote $l_n=\deg b_n$, $t=\deg a_p$. Note that $M(r,f(qz))=M(|q|r,f)$ for z satisfying $|z|=r$. Set $v=1+\max\{l_0,l_1,\dots,l_d\}$. It concludes that

$$\begin{aligned} M\left(r,\frac{P(z,f(z))}{Q(z,f(z))}\right) &= M(r,f(qz)+f(z/q)) \\ &\leqslant M(|q|r,2f(z)) \leqslant CM(|q|r,f(z)), \end{aligned} \quad (7)$$

when r is large enough and $|q| \geqslant 1$, where C is a positive constant. It follows that

$$\begin{aligned} &\left|\sum_{i=0}^{p} a_i(z)f(z)^i\right| \\ &\quad \geqslant |a_p(z)f(z)^p|-(|a_{p-1}(z)f(z)^{p-1}|+\cdots+|a_0(z)|) \\ &\quad \geqslant \frac{1}{2}|a_p(z)f(z)^p| = \frac{1}{2}r^t|f(z)|^p(1+o(1)), \end{aligned}$$

and

$$\left|\sum_{n=0}^{d} b_n(z)f(z)^n\right| \leqslant \sum_{n=0}^{d} |b_n(z)f(z)^n|$$
$$\leqslant \sum_{n=0}^{d} r^{\nu}|f(z)|^d = (d+1)r^{\nu}|f(z)|^d,$$

when r is sufficiently large. Thus, we have

$$\left|\frac{P(z,f(z))}{Q(z,f(z))}\right| = \left|\frac{\sum_{i=0}^{p} a_i(z)f(z)^i}{\sum_{n=0}^{d} b_n(z)f(z)^n}\right|$$
$$\geqslant \frac{|a_p(z)f(z)^p| - (|a_{p-1}(z)f(z)^{p-1}| + \cdots + |a_0(z)|)}{|b_d(z)f(z)^d| + \cdots + |b_1(z)f(z)| + |b_0(z)|}$$
$$\geqslant \frac{1}{2(d+1)} r^{(t-\nu)}|f(z)|^{(p-d)}(1+o(1)),$$

when r is large enough. Thus

$$M\left(r, \frac{P(z,f(z))}{Q(z,f(z))}\right) \geqslant \frac{r^{(t-\nu)}M(r,f(z))^m}{2(d+1)}, \qquad (8)$$

when r is large enough. We have by (7) and (8) that

$$\log M(|q|r, f(z)) \geqslant m \log M(r, f(z)) + g(r), \qquad (9)$$

where $|g(r)| < K \log r$ for some $K > 0$, when r is sufficiently large. By (9) and $|q| = 1$, we have

$$\log M(r,f) = \log M(|q|r,f) \geqslant m \log M(r,f) + g(r). \quad (10)$$

And (10) is a contradiction since $m \geqslant 2$.

Case 2: Suppose that f, the solution of (3), is transcendental meromorphic with finitely many poles. Then there exists a polynomial $P(z)$ such that $F(z) = P(z)f(z)$ is transcendental entire. Substituting $f(z) = F(z)/P(z)$ into (3) and multiplying away the denominators, we will obtain an equation similar to (3). Applying the same reasoning above to $F(z)$, we obtain that for sufficiently large r

$$\log M(r,f) = \log M(r,F) + O(1) \geqslant m \log M(r,F) + g(r).$$

It is a contradiction since $m \geqslant 2$.

Case 3: Suppose that f, the solution of (3), is a meromorphic function with infinitely many poles. Since $a_i(z)$ $(i = 0,1,\ldots,p)$, $b_n(z)$ $(n = 0,1,\ldots,d)$ are polynomials, there is a constant $R > 0$ such that all zeros of $a_i(z)$ $(i = 0,1,\ldots,p)$ and $b_n(z)$ $(n = 0,1,\ldots,d)$ are not in $D = \{z : |z| > R\}$. Since $f(z)$ has infinitely many poles, there exists a pole $z_0(\in D)$ of $f(z)$ having multiplicity $k_0 \geqslant 1$. Then the right-hand side of (3) has a pole of multiplicity mk_0 at z_0. Thus, there exists at least one index $l_1 \in \{q, 1/q\}$ such that $l_1 z_0$ is a pole of $f(z)$ of multiplicity $k_1 = mk_0$.

Without loss of generality, suppose that $l_1 = q$ since $|q| = |1/q| = 1$. Then qz_0 is a pole of $f(z)$ of multiplicity k_1 and $qz_0 \in D$. Substitute qz_0 for z in (3) to obtain

$$f(q^2 z_0) + f(z_0) = \frac{a_0(qz_0) + \cdots + a_p(qz_0)f^p(qz_0)}{b_0(qz_0) + \cdots + b_d(qz_0)f^d(qz_0)}. \quad (11)$$

By (11) and $m = p - d \geqslant 2$, we conclude that $q^2 z_0$ is a pole of $f(z)$ of multiplicity $k_2 = mk_1 = m^2 k_0$. Obviously $q^2 z_0 \in D$. Replacing z by $q^2 z_0$ in (3) to obtain

$$f(q^3 z_0) + f(qz_0) = \frac{a_0(q^2 z_0) + \cdots + a_p(q^2 z_0)f^p(q^2 z_0)}{b_0(q^2 z_0) + \cdots + b_d(q^2 z_0)f^d(q^2 z_0)}. \quad (12)$$

By (12) and $m = p - d \geqslant 2$, we conclude that $q^3 z_0$ is a pole of $f(z)$ of multiplicity $k_3 = mk_2 = m^3 k_0$. Obviously $q^3 z_0 \in D$.

Similarly, $q^l z_0(\in D)$ is a pole of $f(z)$ of multiplicity $k_l = m^l k_0$. Thus, there exists a sequence $\{q^l z_0,\ l = 1,2,\ldots\}$ which are the poles of $f(z)$. Since $k_l = m^l k_0 \to \infty$, as $l \to \infty$, and since $f(z)$ does not have essential singularities in the finite plane, we conclude $|q^l z_0| \to \infty$, as $l \to \infty$. In fact, $|q^l z_0| = |z_0| \nrightarrow \infty$ since $|q| = 1$. It is a contradiction.

Thus, part (i) is proved.

(ii) ①: Suppose that f, the solution of (3), is transcendental entire. Our conclusion holds for the cases.

Case 1: $|q| > 1$. By a similar method as Case 1 in (i), we have (9). Iterating (9), we have

$$\log M(|q|^j r, f(z)) \geqslant m^j \log M(r, f(z)) + E_j(r), \quad (13)$$

where

$$|E_j(r)| = \left|m^{j-1}g(r) + m^{j-2}g(|q|r) + \cdots + g(|q|^{j-1}r)\right|$$
$$\leqslant Km^{j-1}\sum_{k=0}^{j-1}\frac{\log(|q|^k r)}{m^k} \leqslant Km^{j-1}\sum_{k=0}^{\infty}\frac{\log(|q|^k r)}{m^k}.$$

Since $\log(|q|^k r) = \log|q|^k + \log r \leqslant (\log r)(\log|q|^k)$ for sufficiently large r and k, we have

$$\sum_{k=0}^{\infty}\frac{\log(|q|^k r)}{m^k} \leqslant \sum_{k=0}^{\infty}\frac{(\log r)(\log|q|^k)}{m^k} = \log r \log|q| \sum_{k=0}^{\infty}\frac{k}{m^k}.$$

Obviously, the series $\sum_{k=0}^{\infty}\frac{k}{m^k}$ is convergent. Suppose that $\sum_{k=0}^{\infty}\frac{k}{m^k}$ converges to I. It follows that $|\sum_{k=0}^{n_1}\frac{k}{m^k} - I| < 1$ for sufficiently large n_1. So, $\sum_{k=0}^{\infty}\frac{k}{m^k} \leqslant |I| + 1$. Hence

$$|E_j(r)| \leqslant Km^{j-1}\log r \log|q|(|I|+1) = K' m^j \log r, \quad (14)$$

where $K' = K(|I|+1)\log|q|/m$. Since f is transcendental entire, we get $\log M(r,f) \geqslant 2K' \log r$ for large enough r. By (13) and (14), there exists $r_0 \geqslant \mathrm{e}$ such that for $r \geqslant r_0$,

$$\log M(|q|^j r, f(z)) \geqslant K' m^j \log r. \qquad (15)$$

Thus, for each sufficiently large s, there exists a $j \in \mathbb{N}$ such that $s \in [|q|^j r_0, |q|^{j+1} r_0)$, i.e., $j > \frac{\log s - \log(|q| r_0)}{\log |q|}$. Therefore, (15) implies

$$\log M(s, f(z)) \geqslant \log M(|q|^j r_0, f(z)) \geqslant K' m^j \log r_0 \geqslant K'' m^{\log s / \log |q|},$$

where $K'' = K' \log r_0 m^{-\log(|q| r_0)/\log |q|}$.

Suppose now that f, the solution of (3), is meromorphic with finitely many poles. Then there exists a polynomial $P(z)$ such that $F(z) = P(z) f(z)$ is entire. Using the same reasoning as above and Case 2 in (i), we obtain that for sufficiently large r, $\log M(r, f) = \log M(r, F) + O(1) \geqslant (K'' - \varepsilon) m^{\log r / \log |q|} = K''' m^{\log r / \log |q|}$, where $K'''(> 0)$ is some constant.

Case 2: $|q| < 1$. Set $q_1 = 1/q$. Then $|q_1| > 1$. (3) yields

$$f(z/q_1) + f(q_1 z) = \frac{P(z, f(z))}{Q(z, f(z))}.$$

By the same reasoning as Case 1, we obtain

$$\log M(r, f) \geqslant K m^{\log r / \log |q_1|} = K m^{\log r / |\log |q||}.$$

From Case 1 and Case 2, we have

$$\log M(r, f) \geqslant K m^{\log r / |\log |q||}.$$

Finally, since $K m^{\log r / |\log |q||} \leqslant \log M(r, f) \leqslant 3T(2r, f)$ for all $r \geqslant r_0$, we get $\mu(f) \geqslant \log m / |\log |q||$.

Thus, part ① is proved.

②: Suppose that f, the solution of (3), is meromorphic with infinitely many poles. Since $a_i(z)$ ($i = 0, 1, \ldots, p$), $b_n(z)$ ($n = 0, 1, \ldots, d$) are polynomials, there are two constants $R > 0$ and $M > 0$ such that all nonzero zeros of $a_i(z)$ ($i = 0, 1, \ldots, p$) and $b_n(z)$ ($n = 0, 1, \ldots, d$) are in $D_1 = \{z : M \leqslant |z| \leqslant R\}$. Set $D = \{z : |z| > R\}$.

Since $f(z)$ has infinitely many poles, there exists a pole $z_0 (\in D)$ of $f(z)$ having multiplicity $k_0 \geqslant 1$. Then the right-hand side of (3) has a pole of multiplicity mk_0 at z_0. Thus, there exists at least one index $l_1 \in \{q, 1/q\}$ such that $l_1 z_0$ is a pole of $f(z)$ of multiplicity $k_1 = mk_0$.

Without loss of generality, suppose that $|q| > 1$. We need to discuss the following two cases.

Case 1: If $l_1 = q$, then qz_0 is a pole of $f(z)$ of multiplicity k_1 and $qz_0 \in D$. Substitute qz_0 for z in (3) to obtain (11). By (11) and $m = p - d \geqslant 2$, we conclude that $q^2 z_0$ is a pole of $f(z)$ of multiplicity $k_2 = m^2 k_0$. By a similar method as Case 3 in (i), we obtain that $q^l z_0 (\in D)$ is a pole of $f(z)$ of multiplicity $k_l = m^l k_0$. Thus, we find a sequence $\{q^j z_0 \in D,\ j = 0, 1, 2, \ldots\}$ which are the poles of $f(z)$. Since $k_j = m^j k_0 \to \infty$, as $j \to \infty$, and since $f(z)$ does not have essential singularities in the finite plane, we conclude $|q^j z_0| \to \infty$, as $j \to \infty$. For sufficiently large j, say $j > j_0$, we obtain

$$m^j k_0 \leqslant k_0(1 + m + \cdots + m^j) \leqslant n(|q^j z_0|, f) = n(|q|^j |z_0|, f). \tag{16}$$

Thus, for each large enough r, there exists a $j \in \mathbb{N}$ such that $r \in [|q|^j |z_0|, |q|^{j+1} |z_0|)$. We obtain by (16) that

$$n(r, f) \geqslant m^j k_0 \geqslant k_0 m^{(\log r - \log |q z_0|)/\log |q|} = K m^{\log r / \log |q|},$$

where $K = k_0 m^{-\log |q z_0| / \log |q|}$.

Case 2: We can affirm that $l_1 = 1/q$ is impossible. On the contrary, suppose that $l_1 = 1/q$. Set $q_1 = 1/q$ and $\deg a_p = A (\geqslant 0)$. Since $z_0 \in D$, we know that $z_0/q = q_1 z_0$ has two possibilities:

(a): If $q_1 z_0 \in D_1$, this process will be terminated and we have to choose another pole z_0 of $f(z)$ in the way we did above.

(b): If $q_1 z_0 \notin D_1$, then $q_1 z_0$ is a pole of $f(z)$ of multiplicity $k_1 = mk_0$, since the right-hand side of (3) has a pole of multiplicity mk_0 at z_0.

If $q_1 z_0 \notin D \cup D_1$, that is $0 < |q_1 z_0| < M$, then we choose pole z_0 of $f(z)$ and substitute $q_1 z_0$ for z in (3).

If $q_1 z_0 \in D$, that is $|q_1 z_0| > R$, replacing z by $q_1 z_0$ in (3) to obtain

$$f(z_0) + f(q_1^2 z_0) = \frac{a_0(q_1 z_0) + \cdots + a_p(q_1 z_0) f^p(q_1 z_0)}{b_0(q_1 z_0) + \cdots + b_d(q_1 z_0) f^d(q_1 z_0)}.$$

By the above equality, it concludes that $q_1^2 z_0$ is a pole of $f(z)$ of multiplicity $k_2 = mk_1 = m^2 k_0$.

If $q_1^2 z_0 \in D_1$, this process will be terminated and we have to choose another pole z_0 of $f(z)$ in the way we did above.

If $q_1^2 z_0 \in D$, then the right-hand side of (3) has a pole of multiplicity mk_2 at $q_1^2 z_0$.

Replacing z by $q_1^2 z_0$ in (3), it concludes that $q_1^3 z_0$ is a pole of $f(z)$ of multiplicity $k_3 = mk_2 = m^3 k_0$.

We proceed to follow the steps (a) and (b) as above. Since there are infinitely many poles of $f(z)$ in D, we will find a pole $z_0 (\in D)$ of $f(z)$ such that $q_1^{n_1} z_0 (\in D)$ is a pole of $f(z)$ of multiplicity $k_{n_1} = m^{n_1} k_0$. And z_0 satisfies $q_1^{n_1+1} z_0 \in D_1$. By (3) and $m = p - d \geqslant 2$, we conclude that $q_1^{n_1+1} z_0$ is a pole of $f(z)$ of multiplicity $k_{(n_1+1)} = mk_{n_1} = m^{n_1+1} k_0$.

Replacing z by $q_1^{n_1+1} z_0$ in (3) to obtain

$$f\left(q_1^{n_1} z_0\right) + f\left(q_1^{n_1+2} z_0\right) = \frac{a_0\left(q_1^{n_1+1} z_0\right) + \cdots + a_p\left(q_1^{n_1+1} z_0\right) f^p\left(q_1^{n_1+1} z_0\right)}{b_0\left(q_1^{n_1+1} z_0\right) + \cdots + b_d\left(q_1^{n_1+1} z_0\right) f^d\left(q_1^{n_1+1} z_0\right)}. \tag{17}$$

The right-hand side of (17) has a pole of multiplicity at least $pk_{(n_1+1)} - A - dk_{(n_1+1)} = mk_{(n_1+1)} - A$ at $q_1^{n_1+1} z_0$. Without loss of generality, suppose that the right-hand side of (17) has a pole of multiplicity $mk_{(n_1+1)} - A$ at $q_1^{n_1+1} z_0$.

In the left-hand side of (17), $f(qz)$ has a pole of multiplicity $k_{n_1} = m^{n_1} k_0$ at $q_1^{n_1+1} z_0$. By $m \geqslant 2$, when $n_1 > \max\left\{\frac{\log A - \log(m^2 - 1) k_0}{\log m}, 1\right\}$, we have $mk_{(n_1+1)} - A = m^{n_1+2} k_0 - A > m^{n_1} k_0$. Thus $mk_{(n_1+1)} - A > k_{n_1}$.

Hence, by (17), it concludes that $q_1^{n_1+2}z_0(\in D_1)$ is a pole of $f(z)$ of multiplicity $k_{(n_1+2)} = mk_{(n_1+1)} - A = m^{n_1+2}k_0 - A$.

Replacing z by $q_1^{n_1+2}z_0$ in (3) to obtain

$$f\left(q_1^{n_1+1}z_0\right)+f\left(q_1^{n_1+3}z_0\right) = \frac{a_0\left(q_1^{n_1+2}z_0\right)+\cdots+a_p\left(q_1^{n_1+2}z_0\right)f^p\left(q_1^{n_1+2}z_0\right)}{b_0\left(q_1^{n_1+2}z_0\right)+\cdots+b_d\left(q_1^{n_1+2}z_0\right)f^d\left(q_1^{n_1+2}z_0\right)}. \quad (18)$$

The right-hand side of (18) has a pole of multiplicity at least $pk_{(n_1+2)} - A - dk_{(n_1+2)} = mk_{(n_1+2)} - A$ at $q_1^{n_1+2}z_0$. Without loss of generality, suppose that the right-hand side of (18) has a pole of multiplicity $mk_{(n_1+2)} - A$ at $q_1^{n_1+2}z_0$.

In the left-hand side of (18), $f(qz)$ has a pole of multiplicity $k_{(n_1+1)} = m^{n_1+1}k_0$ at $q_1^{n_1+2}z_0$. By $m \geqslant 2$, when $n_1 > \max\left\{\frac{\log A - \log(m-1)k_0}{\log m} - 1, 1\right\}$, we have $mk_{n_1+2} - A = m^{n_1+3}k_0 - A(m+1) > m^{n_1+1}k_0$. Thus $mk_{(n_1+2)} - A > k_{(n_1+1)}$.

Hence, by (18), it concludes that $q_1^{n_1+3}z_0$ is a pole of $f(z)$ of multiplicity $k_{(n_1+3)} = mk_{(n_1+2)} - A = m(m^{n_1+2}k_0 - A) - A = m^{n_1+3}k_0 - A(m+1)$.

We proceed to follow the step as above. We conclude that $q_1^{n_1+n_2}z_0$ is a pole of $f(z)$ of multiplicity $k_{(n_1+n_2)} = m^{n_1+n_2}k_0 - A(m^{n_2-2} + \cdots + m + 1)$ such that $0 < \left|q_1^{n_1+n_2}z_0\right| < M$, that is $q_1^{n_1+n_2}z_0 \notin D \cup D_1$.

Set $k := k_{(n_1+n_2)} = m^{n_1+n_2}k_0 - A(m^{n_2-2} + \cdots + m + 1)$. Then

$$k = \frac{m^{n_2-1}}{m-1}[(m-1)m^{n_1+1}k_0 - A] + \frac{A}{m-1}.$$

When $n_2 \geqslant 2$ and $n_1 > \max\left\{\frac{\log(A+1) - \log(m-1)k_0}{\log m} - 1, 1\right\}$, we get $(m-1)m^{n_1+1}k_0 > A+1$, that is $(m-1)m^{n_1+1}k_0 - A > 1$. Hence $k \geqslant 1$.

Set $z_1 := q_1^{n_1+n_2}z_0 (0 < |q_1^{n_1+n_2}z_0| < M)$. Then z_1 is a pole of $f(z)$ of multiplicity $k \geqslant 1$. In particular, when $n_1 = 1$ and $n_2 = 0$, then $z_1 = q_1 z_0$ is a pole of $f(z)$ of multiplicity $k = k_1 = mk_0$.

Applying the same reasoning as Case 1, we will find that $q_1^l z_1 (\notin D \cup D_1)$ is a pole of $f(z)$ of multiplicity $k_l = m^l k$. Thus, there exists a sequence $\{q_1^l z_1,\ l = 1, 2, \ldots\}$ which are the poles of $f(z)$. We conclude $q_1^l z_1 \to 0$ as $l \to \infty$ since $|q_1| < 1$. Therefore, $f(z)$ is not a meromorphic function. It is a contradiction.

From Case 1 and Case 2, when $|q| \neq 1$, we obtain

$$n(r, f) \geqslant Km^{\log r / |\log |q||}.$$

Finally, since $Km^{\log r/|\log|q||} \leqslant n(r, f) \leqslant \frac{1}{\log 2}N(2r, f) \leqslant \frac{1}{\log 2}T(2r, f)$ for all $r \geqslant r_0$, we immediately obtain $\mu(f) \geqslant \log m / |\log |q||$.

Thus, Theorem 4 is proved.

The proof of Corollary 1

Without loss of generality, suppose that the coefficients $a_i(z)$ $(i = 0, 1, \ldots, p)$ and $b_n(z)$ $(n = 0, 1, \ldots, d)$ in (1) are polynomials. On the contrary, suppose that (1) has a transcendental entire solution f.

Denote $|q| = \max\{|q_1|, \ldots, |q_n|\}$. Obviously $0 < |q| \leqslant 1$ since $0 < |q_j| \leqslant 1$ $(j = 1, \ldots, n)$. Note that $M(r, f(qz)) = M(|q|r, f)$ for z satisfying $|z| = r$. It concludes that

$$M\left(r, \frac{P(z, f(z))}{Q(z, f(z))}\right) = M\left(r, \sum_{j=1}^{n} f(q_j z)\right) \leqslant M(|q|r, nf(z)) \leqslant CM(|q|r, f(z)), \quad (19)$$

when r is large enough, where C is a positive constant. Applying the same reasoning as Case 1 in (i) of Theorem 4, we obtain (8). Thus, we have by (8) and (19) that

$$\log M(r, f(z)) \geqslant \log M(|q|r, f(z)) \geqslant m \log M(r, f(z)) + g(r),$$

where $|g(r)| < K \log r$ for some $K > 0$, when r is sufficiently large. It is a contradiction since $m \geqslant 2$.

Corollary 1 is proved.

THE EXISTENCE OF MEROMORPHIC SOLUTION OF LINEAR q-DIFFERENCE EQUATION

Bergweiler et al [16] studied the existence and properties of transcendental meromorphic solution of linear q-difference equation. They obtained the following results.

Theorem 5 ([16]) *Let $a_0(z), \ldots, a_{n+1}(z)$ be polynomials without common zeros and $0 < |q| < 1$. Suppose that the equation*

$$a_0(z)f(z) + a_1(z)f(qz) + \cdots + a_n(z)f(q^n z) = a_{n+1}(z) \quad (20)$$

possesses a transcendental entire solution $f(z)$. Then there is some j, $1 \leqslant j \leqslant n$, such that $\deg a_0(z) < \deg a_j(z)$.

Theorem 6 ([16]) *Suppose that the coefficients $a_0(z), \ldots, a_{n+1}(z)$ in (20) are meromorphic and of finite order $\leqslant \rho$ and $0 < |q| < 1$. Then the meromorphic solution $f(z)$ of (20) is of finite order $\sigma(f) \leqslant \rho$. In addition, if $\sigma(a_{n+1}) > \sigma(a_j)$ for all $j = 0, 1, \ldots, n$, then $\sigma(f) = \sigma(a_{n+1})$.*

Remark 2 ([10]) If the coefficients in (20) are constants, then (20) has no transcendental meromorphic solution.

In Theorem 3, we see that $n \geqslant 2$ is necessary. A natural question is: what is the result when $n = 1$ in Theorem 3? Corresponding to this question, we get Theorem 7.

Theorem 7 *Consider q-difference equation*

$$f(qz)+f(z/q)=b(z)f(z)+a(z), \tag{21}$$

where $q\in\mathbb{C}\backslash\{0\}$, $|q|\neq 1$.

(i) *If* $a(z)$ *and* $b(z)=M(z)/N(z)$ *are irreducible rational functions satisfying* $\deg M(z)\leqslant \deg N(z)$*, then* (21) *does not possess transcendental meromorphic solution with finitely many poles.*

(ii) *Suppose that* $a(z)$ *and* $b(z)=M(z)/N(z)$ *are nonconstant irreducible rational functions satisfying* $\deg M(z)\leqslant \deg N(z)$*. If* (21) *has a transcendental meromorphic solution* $f(z)$*, then* $f(z)$ *has infinitely many poles and* $\sigma(f)\geqslant 1$.

(iii) *Suppose that* $a(z)$ *and* $b(z)$ *are meromorphic and of finite order* $\leqslant \rho$*. Then the meromorphic solution* $f(z)$ *of* (21) *is of finite order* $\sigma(f)\leqslant\rho$*. In addition, if* $\sigma(a(z))>\sigma(b(z))$*, then* $\sigma(f)=\sigma(a(z))$.

Remark 3 In particular, if $a(z)$ and $b(z)$ are complex constants, then (21) has no transcendental meromorphic solution.

Proof: **(i):** Without loss of generality, suppose that $a(z)$ is a polynomial.

On the contrary, suppose that (21) possesses a transcendental meromorphic solution $f(z)$ with finitely many poles. Our conclusion holds for the cases.

Case 1: $0<|q|<1$. We only need to discuss the following two subcases.

Subcase 1: Suppose that $f(z)$ is transcendental entire. (21) yields

$$N(z)f(qz)+N(z)f(z/q)=M(z)f(z)+N(z)a(z).$$

Thus, we obtain

$$N(qz)f(q^2z)-M(qz)f(qz)+N(qz)f(z)=N(qz)a(qz). \tag{22}$$

Obviously, $\deg M(qz)\leqslant \deg N(qz)$. Without loss of generality, suppose that polynomials $M(qz)$, $N(qz)$ and $a(qz)$ have no common zeros. By Theorem 5 and (22), we conclude a contradiction.

Subcase 2: Suppose that $f(z)$ is meromorphic with finitely many poles. Then there is a polynomial $P(z)$ such that $g(z)=P(z)f(z)$ is entire. Substituting $f(z)=g(z)/P(z)$ into (22), we will get

$$a_2(z)g(q^2z)+a_1(z)g(qz)+a_0(z)g(z)=a_3(z),$$

where $a_0(z)=P(q^2z)P(qz)N(qz)$, $a_1(z)=-P(q^2z)P(z)M(qz)$, $a_2(z)=P(qz)P(z)N(qz)$, $a_3(z)=P(q^2z)P(qz)P(z)N(qz)a(qz)$. Obviously, $\deg a_0(z)=\deg a_2(z)\geqslant \deg a_1(z)$. Using the same reasoning above to $g(z)$, we conclude a contradiction.

Case 2: $|q|>1$. Set $q_1=1/q$. Then $0<|q_1|<1$. (21) shows

$$f(z/q_1)+f(q_1z)=b(z)f(z)+a(z). \tag{23}$$

Applying the same reasoning as Case 1, the result is obtained.

(ii): Without loss of generality, suppose that $a(z)$ is a polynomial.

Suppose that $f(z)$ is a meromorphic solution of (21). By (i), $f(z)$ has infinitely many poles. Similarly as (i), we can get (22). Since $M(qz)$, $N(qz)$ and $a(qz)$ are polynomials, there is a constant $R>0$ such that all zeros of $M(qz)$, $N(qz)$ and $a(qz)$ are not in $D=\{z:|z|>R\}$. Without loss of generality, suppose that $|q|>1$.

Since $f(z)$ has infinitely many poles, there is a pole $z_0(\in D)$ of $f(z)$ having multiplicity $k_0\geqslant 1$. Then the left-hand side of (22) has a pole of multiplicity k_0 at z_0. Hence, there exists at least one index $l_1\in\{1,2\}$ such that $q^{l_1}z_0$ is a pole of $f(z)$ of multiplicity k_0. Replacing z by $\hat{z}:=q^{l_1}z_0$ in (22), we obtain

$$N(q\hat{z})f(q^2\hat{z})-M(q\hat{z})f(q\hat{z})+N(q\hat{z})f(\hat{z})=N(q\hat{z})a(q\hat{z}). \tag{24}$$

Since $|q^{l_1}z_0|>|z_0|$, the all coefficients of (24) cannot have a zero at $\hat{z}=q^{l_1}z_0$. Thus, the left-hand side of (24) has a pole of $f(z)$ of multiplicity k_0 at $q^{l_1}z_0$. Hence, there exists at least one index $l_2\in\{1,2\}$ such that $q^{l_1+l_2}z_0$ is a pole of $f(z)$ of multiplicity k_0.

Similarly, $q^{l_1+l_2+\cdots+l_n}z_0(\in D)$ is a pole of $f(z)$ of multiplicity k_0. Thus, there exists a sequence $\{q^{l_1+l_2+\cdots+l_j}z_0\in D, j=1,2,\ldots\}$ which are the poles of $f(z)$. So, $\sigma(f)\geqslant\lambda(1/f)\geqslant 1$.

(iii): We only need to discuss the following two cases.

Case 1: $0<|q|<1$. Then $\sigma(b(qz))\leqslant\sigma(b(z))$ and $\sigma(a(qz))\leqslant\sigma(a(z))$. (21) yields

$$f(q^2z)-b(qz)f(qz)+f(z)=a(qz). \tag{25}$$

Applying Theorem 6 to (25), the results is proved.

Case 2: $|q|>1$. Set $q_1=1/q$. Then $0<|q_1|<1$. By (21), we have (23). Applying the same reasoning as Case 1, the result is obtained.

Thus, Theorem 7 is proved. □

THE GROWTH OF MEROMORPHIC SOLUTIONS OF q-DIFFERENCE PAINLEVÉ EQUATION I

Recently, some authors investigated zero order meromorphic solutions of q-difference equations [8, 11, 14, 15]. Qi and Yang [13] considered q-difference Painlevé equation I, and obtained the following Theorem 8.

Theorem 8 ([13]) *Let* $f(z)$ *be a transcendental meromorphic solution with zero order of equation*

$$f(qz)+f(z/q)=\frac{az+b}{f(z)}+c,$$

where a, b, c are three constants such that cannot vanish simultaneously. Then,

(i) $f(z)$ *has infinitely many poles;*

(ii) *if* $a\neq 0$*, then* $f(z)$ *has infinitely many finite values;*

(iii) *if $a=0$ and $f(z)$ takes a finite value A finitely often, then A is a solution of $2z^2-cz-b=0$.*

In Theorem 8, if $c=0$, what do we get? In the following, we will answer this question. We investigate the growth of transcendental meromorphic solutions of q-difference Painlevé equation $f(qz)+f(z/q)=A(z)/f(z)$ and find lower bounds for the order of transcendental meromorphic solutions for such equation. We obtain the following result.

Theorem 9 *Let $A(z)=t(z)/s(z)(\not\equiv 0)$ be an irreducible rational function. Suppose that $f(z)$ is a transcendental meromorphic solution of q-difference equation*

$$f(qz)+f(z/q)=\frac{A(z)}{f(z)}, \tag{26}$$

where $q\in\mathbb{C}\backslash\{0\}$, $|q|\neq 1$. Then $\sigma(f)\geqslant 1$.

From Theorem 9, we conclude that the (26) has no zero order transcendental meromorphic solution.

We need the following lemmas to prove Theorem 9.

Lemma 1 *Let $f(z)$ be a transcendental meromorphic function with $\sigma(f)<1$, and $q\in\mathbb{C}\backslash\{0\}$, $|q|\neq 1$. Then*

$$g(z)=f(qz)f(z) \tag{27}$$

is transcendental.

Proof: On the contrary, we suppose that $g(z)$ is a rational function. There is a constant $R>0$ such that all zeros and poles of $g(z)$ are not in $D=\{z:|z|>R\}$.

Without loss of generality, suppose that $|q|>1$. Since $\sigma(f)<1$, $f(z)$ has infinitely many poles or zeros. Our conclusion holds for the cases.

Case 1: If $f(z)$ has infinitely many poles, there exists pole $z_0(\in D)$ of $f(z)$ having multiplicity $k\geqslant 1$. By (27), qz_0 is a zero of $f(z)$ and $qz_0\in D$. Substitute qz_0 for z in (27) to obtain

$$g(qz_0)=f(q^2z_0)f(qz_0). \tag{28}$$

By (28) and $f(qz_0)=0$, we have $f(q^2z_0)=\infty$ and $q^2z_0\in D$.

Similarly, $q^{2n}z_0(\in D)$ is a pole of $f(z)$. Thus, there is a sequence $\{q^{2n}z_0\in D,\ n=0,1,2\ldots\}$ which are the poles of $f(z)$. Thus, $\lambda(1/f)\geqslant 1$. It is a contradiction.

Case 2: If $f(z)$ has infinitely many zeros, there is a zero $z_1(\in D)$ of $f(z)$. By (27), it concludes that qz_1 is a pole of $f(z)$ and $qz_1\in D$. Replacing z by qz_1 in (27) to obtain

$$g(qz_1)=f(q^2z_1)f(qz_1). \tag{29}$$

By (29) and $f(qz_1)=\infty$, we get $f(q^2z_1)=0$ and $q^2z_1\in D$.

Similarly, $\{q^{2m}z_1\in D,\ m=0,1,2,\ldots\}$ is a zero sequence of $f(z)$. Thus, $\lambda(f)\geqslant 1$. It is a contradiction.

Thus, $g(z)$ is transcendental. □

Lemma 2 *Let $g_1(z)$, $g_2(z)(\not\equiv 0)$ and $h(z)(\not\equiv 0)$ be rational functions, q_1, q_2 $(|q_1|\neq|q_2|)$ be nonzero complex constants. Suppose that $f(z)$ be a transcendental meromorphic solution with infinitely many poles of q-difference equation*

$$g_2(z)f(q_1z)+g_1(z)f(q_2z)=h(z). \tag{30}$$

Then $\sigma(f)\geqslant 1$.

Proof: Our conclusion holds for the cases.

Case 1: $|q_1|>|q_2|$. Set $q=q_1/q_2$. Then $|q|>1$. (30) yields

$$g_2\left(\frac{z}{q_2}\right)f(qz)+g_1\left(\frac{z}{q_2}\right)f(z)=h\left(\frac{z}{q_2}\right). \tag{31}$$

Since $h(z)$, $g_i(z)$ $(i=1,2)$ are rational, there is a constant $R>0$ such that all zeros and poles of $h(z/q_2)$, $g_i(z/q_2)$ $(i=1,2)$ are not in $D=\{z:|z|>R\}$.

Since $f(z)$ has infinitely many poles, there exists a pole $z_0(\in D)$ of $f(z)$ having multiplicity $k\geqslant 1$. By (31), we conclude that qz_0 is a pole of $f(z)$ of multiplicity k and $qz_0\in D$. Replacing z by qz_0 in (31) to obtain

$$g_2\left(\frac{qz_0}{q_2}\right)f\left(q^2z_0\right)+g_1\left(\frac{qz_0}{q_2}\right)f(qz_0)=h\left(\frac{qz_0}{q_2}\right). \tag{32}$$

By (32) and $f(qz_0)=\infty$, we conclude that q^2z_0 is a pole of $f(z)$ of multiplicity k and $q^2z_0\in D$.

Similarly, $q^nz_0(\in D)$ is a pole of $f(z)$ of multiplicity k. Thus, there is a sequence $\{q^jz_0\in D,\ j=0,1,2,\ldots\}$ which are the poles of $f(z)$. So, $\sigma(f)\geqslant\lambda(1/f)\geqslant 1$.

Case 2: $|q_1|<|q_2|$. Set $q=q_2/q_1$. Then $|q|>1$. (30) implies

$$g_2\left(\frac{z}{q_1}\right)f(z)+g_1\left(\frac{z}{q_1}\right)f(qz)=h\left(\frac{z}{q_1}\right). \tag{33}$$

Using the same method as Case 1, we get $\sigma(f)\geqslant 1$.

The proof of Theorem 9

On the contrary, we suppose that $f(z)$ is a transcendental meromorphic solution of (26) and $\sigma(f)<1$.

Without loss of generality, suppose that $0<|q|<1$. (26) implies

$$f(qz)f(z)+f(z)f(z/q)=\frac{t(z)}{s(z)}. \tag{34}$$

Set $y(z)=f(qz)f(z)$. From Remark 1, we get $\sigma(y)\leqslant\sigma(f)<1$. By Lemma 1, it concludes that $y(z)$ is transcendental. By (34), we obtain

$$s(z)y(z)+s(z)y(z/q)=t(z).$$

That is

$$s(qz)y(qz)+s(qz)y(z)=t(qz). \tag{35}$$

Similarly to the proof of Theorem 7, (35) has no transcendental meromorphic solution with finitely many poles. So, if $y(z)$ is a transcendental meromorphic solution of (35), then $y(z)$ has infinitely many poles. By Lemma 2 and (35), we get $\sigma(y) \geqslant 1$. This is a contradiction.

Thus, Theorem 9 is proved. □

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