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THE :cos$\alpha\psi$: INTERACTING SYSTEM ON A LATTICE: THE CLASSICAL SOLUTIONS

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Abstract

We present the most probable uniform configurations of the :cos $\alpha\psi$: interacting system on a lattice in d-dimension, where $d \leq 4$. We point out that several global and local minima can appear depending on the value of the parameters J, α, λ, m^2, d. For J = O, $\alpha = \lambda = 1$, we have calculated numerically the critical mass square. We show graphically the dependence of the amplitude of the solution on the variation of mass square.

I. Introduction

In this paper, we present our study on the classical solutions of a Euclidean Scalar Boson Field Theory on a d-dimensional lattice. We are interested in this model as it can be a representation of an Ising-Like Model on a continuous field. [1, 2] Here, the theory is restricted to an ultralocal interaction of the form

$$F^{\Lambda}(\psi) = \Sigma_{j\varepsilon\Lambda} :\cos\alpha\psi_j:_C \tag{I.1}$$

for $\Lambda \subset Z^d$, where $\psi_j \varepsilon$ R for all j ε Λ, $\alpha > 0$. [3] The lattice spacing is fixed to be unity. C is the kernel of $(m^2 - \Delta)^{-1}$, and can be represented by the Fourier transform;

$$C_{jj'} = (2\pi)^{-d} \int \ldots \int_{[-\pi,\pi]^d} dk_1 \ldots dk_d . e^{ik.(j-j')} . \{m^2 + 2\Sigma_{\ell=1}^{d} (1 - \cos k_\ell)\}^{-1} \tag{I.2}$$

The $:.:_C$ denotes the Wick ordering; i.e.,

$$:e^{i\alpha\psi_j}:_C = e^{i\alpha\psi_j + \frac{1}{2}\alpha^2 C_{jj}} \tag{I.3}$$

The Hamiltonian for the model is comprised of a free part H_o^Λ, an interacting term F^Λ, and the source term $J.\psi$,

$$H^\Lambda(\psi) = H_0^\Lambda(\psi) + F^\Lambda(\psi) + J.\psi \tag{I.4}$$

where the free part is

$$H_0^\Lambda(\psi) = \tfrac{1}{2}\Sigma_{ij\,\varepsilon\Lambda}\psi_i C_{ij}^{-1}\psi_j \tag{I.5}$$

and the source term is

$$J.\psi = \Sigma_{j\varepsilon\Lambda} J_j\psi_j \tag{I.6}$$

As with most of the models in statistical mechanics, the main feature of the problem are governed by the most probable configuration. For the canonical ensemble, the most probable configuration is that which minimizes the Hamiltonian of the system.

II. The Minima of the Hamiltonian

The most probable configuration is the one that leads to the maximum of the probability density $\exp\{-H^\Lambda\}$ having the probability measure

$$d\mu^\Lambda(\psi) = (W^\Lambda)^{-1}.\exp\{-H^\Lambda(\psi)\}\,d\psi \tag{II.1}$$

Let the configuration that minimizes the Hamiltonian H, eqn. (I.4), be $\{\psi_j\}_{j\varepsilon\Lambda}$. For large Λ, and uniform external field, $J_j = J$, the solution may be assumed to be uniform; i.e.,

$$\tilde{\psi}_j = \bar{\psi} \tag{II.2}$$

for all $j \varepsilon \Lambda$. This is what we expect from the physical consideration. When $\Lambda \to Z^d$, the bulk properties should be independent of where they are observed.

Substituting (II.2) into (I.1)-(I.6), we get

$$H^\Lambda(\bar{\psi}:\lambda, J) = \tfrac{1}{2}(\Sigma_{ij\,\varepsilon\Lambda} C_{ij}^{-1})\psi^{-2} + \lambda|\Lambda|:\cos\alpha\bar{\psi}:_C + |\Lambda|\,J\psi \tag{II.3}$$

Since C^{-1} is the $(m^2 - \Delta)$, we can verify that the Laplacian contributes very little to the sum in the bracket of the first term in (II.3);

So

$$\Sigma_{ij \varepsilon \Lambda} \Delta_{ij} = |\partial\Lambda| \tag{II.4}$$

Here, $|\partial\Lambda|$ denotes the surface area of Λ. As $|\Lambda| \to \infty$, we can ignore $|\partial\Lambda|/|\Lambda|$. (II.2) becomes

$$|\Lambda|^{-1} H^{\Lambda}(\bar{\psi} : \Lambda, J) = \tfrac{1}{2} m^2 \bar{\psi}^2 + \lambda \colon \cos\alpha\bar{\psi} \colon_C + J\bar{\psi} \tag{II.5}$$

The minimization of (II.5) leads to

$$m^2 \bar{\psi} + J - \lambda\alpha e^{\frac{1}{2}\alpha^2 C_{00}} \sin\alpha\bar{\psi} = 0 \tag{II.6}$$

$$m^2 - \lambda\alpha^2 e^{\frac{1}{2}\alpha^2 C_{00}} \cos\alpha\bar{\psi} > 0 \tag{II.7}$$

where we have used the definition of the Wick Ordering and the covariance C_{00} for $j = j'$. Note that $1/m^2 > C_{00} > 1/(m^2 + 4d)$. For $m^2 >> 4d$, the covariance C_{00} is approximately $1/m^2$.

The solution of (II.6) is the intersection of a straight line $m^2\bar{\psi} + J$ and the sin-curve $\lambda\alpha \exp\{\tfrac{1}{2}\alpha^2 C_{00}\} \sin\alpha\bar{\psi}$. They can have a singular or multiple solutions depending on the value of the parameters m^2, λ, J, α.

Inequality (II.7) tells us that the minimum can occur only when the slope of the straight line is greater than the tangent of the sin-curve at the intersection points.

The Case of J = 0

Setting J to zero, the straight line $m^2\bar{\psi}$ passes through the origin. Thus, $\bar{\psi} = 0$ is always a solution of (II.6).

For large m^2, we have only $\bar{\psi} = 0$ as a unique solution of (II.6). It is minimum due to (II.7). Slowly lowering the value of m^2 until the line $m^2\bar{\psi}$ is tangent to the sin-curve at the origin,we still have $\bar{\psi} = 0$ as a unique solution of (II.6) but it does not, however, satisfy (II.7). In fact, the inequality sign becomes an equality, i.e., $\bar{\psi} = 0$ is a critical point. In decreasing m^2 further, the straight line will cut the sin-curve at $\bar{\psi} = 0$ and two other points, meaning that three solutions exist. The solution $\bar{\psi} = 0$ does not give a minimum, but a maximum. The minimums are obtained with the other two solutions.

The critical points of (II.6) can occur only when the straight line $m^2\bar{\psi}$ is itself a tangent line to the sin-curve, i.e., the slope

$$m_*^2 = \alpha^2\lambda:\cos\alpha\bar{\psi}^*:_C = \frac{\alpha\lambda:\sin\alpha\bar{\psi}^*:_C}{\bar{\psi}^*} \qquad \text{(II.8)}$$

(II.8) can be simplified to give

$$\tan\alpha\bar{\psi}^* = \alpha\bar{\psi}^* \qquad \text{(II.9)}$$

The solution of this equation can lead to undesirable results, i.e., a negative slope. Note that in the solution of (II.9) $\bar{\psi}^*$ is independent of the mass, m^2 and the coupling constant λ. It is clear that the first solution of (II.9) is at $\bar{\psi}^* = 0$.

Letting $\bar{\psi}_n^*$ be the n^{th} -solution of (II.9), we find it to lie between $2(n-1)\pi$ and $2(n-1)\pi + \pi/2$. $\bar{\psi}_n^*$ approaches $2(n-1)\pi+\pi/2$ for large n. Therefore, critical points occur when

$$m^2 = \alpha^2\lambda e^{\frac{1}{2}\alpha^2 C_{00}(m^2)} \cos\alpha\bar{\psi}_n^* \qquad \text{(II.10)}$$

$$= \frac{\alpha^2\lambda e^{\frac{1}{2}\alpha^2 C_{00}(m^2)}}{\sqrt{1 + (\alpha\bar{\psi}_n^*)^2}}$$

We label the solution of (II.10) as the critical mass, m_n^2. We find m_n^2 to decrease monotonic to zero.

For $$m^2 = m_n^2 \qquad \text{(II.11)}$$

It is immediately seen that equation (II.6) gives 4n-3 solutions, of which 2 are critical, 2n-2 are minimum, and 2n-3 are maximum.

For $$m_{n-1}^2 > m^2 > m_n^2 \qquad \text{(II.12)}$$

(II.6) has 4n-1 solutions, of which 2n are minimum, and 2n-1 maximum, where $n \geq 2$. The case of $n = 1$ is a special one. A unique solution exists at $\bar{\psi}_1^* = 0$, if

$$m^2 \geq m_1^2 \qquad \text{(II.13)}$$

The solution leads to a minimum when $m^2 > m_1^2$, and to a critical point when $m^2 = m_1^2$.

For the increasing dimensionality d, one sees that the kernel C_{00} of the operator $(m^2 - \Delta)^{-1}$ decreases. This has an effect on the variation of $\bar{\psi}$ with respect to m^2; $\bar{\psi}$ decreases pointwise as d increases. The general features of the graph for arbitrary d is similar to the case of $d = 1$ shown in Fig. I, but the critical mass square m_n^2 is lower for each n (see table I).

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TABLE I.

THE FIRST FIVE CRITICAL MASS SQUARES FOR

$d \leqq 4, \ J = 0, \ \lambda = \alpha = 1$

d	m_1^2	m_2^2	m_3^2	m_4^2	m_5^2
1	1.2192	0.2166	0.1375	0.1048	0.0865
2	1.1292	0.1580	0.0894	0.0627	0.0483
3	1.0872	0.1434	0.0795	0.0551	0.0421
4	1.0649	0.1383	0.0765	0.0529	0.0404

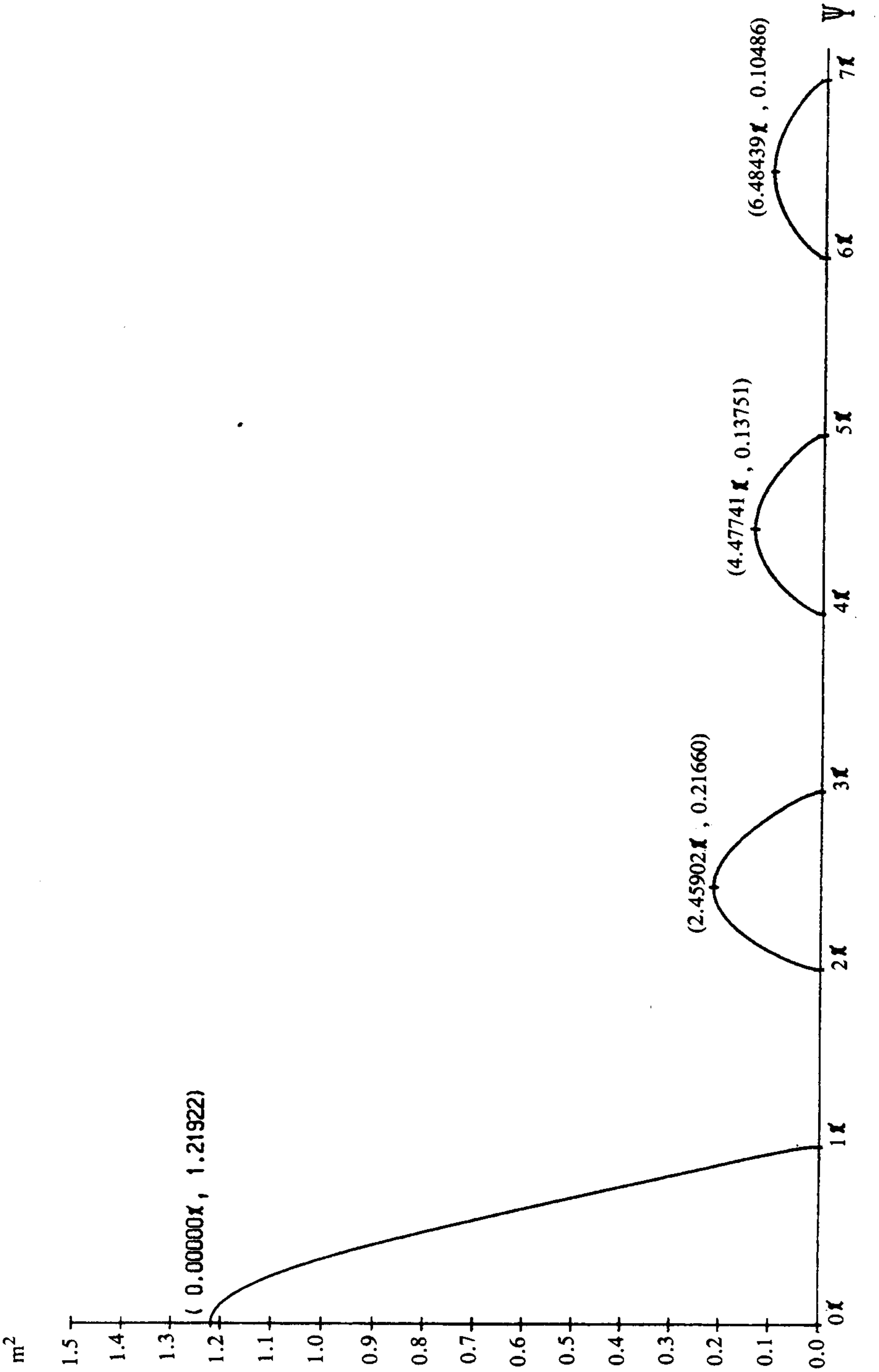

Fig. 1. The uniform most probable configuration versus the mass square for d = 1, J = 0, $\lambda = \alpha = 1$. The critical mass squares are indicated as the maximums of the curve.