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A note on an analogue of Hadamard's theorem for determining the radii of m-meromorphy

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ABSTRACT: In this paper, we prove an extension of Hadamard's classical theorem for determining the radius of m-meromorphy of an analytic function in terms of its Taylor coefficients. Our extension is expressed in terms of Fourier coefficients with respect to an orthonormal polynomial system on the unit circle. Our main result confirms a conjecture posed in [*Dolomit Res Notes Approx* **17** (2024):12–21].

KEYWORDS: meromorphic function, holomorphic function, pole, radius of meromorphy, Cauchy-Hadamard formula

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INTRODUCTION

Suppose that

$$F(z)=\sum_{k=0}^{\infty} f_k z^k \tag{1}$$

is the power series expansion of a function holomorphic in a neighborhood of zero. One of the classical problems in complex analysis is to describe analytic properties of F in terms of the sequence $\{f_k\}_{k\in\mathbb{N}_0}$ (here and in the rest of this paper, we set $\mathbb{N}:=\{1,2,3,\ldots\}$ and $\mathbb{N}_0:=\{0,1,2,3,\ldots\}$). Hadamard's theorem for determining the radius of m-meromorphy of F is a central result in this problem (see [1]).

For F as in (1), we denote by $R_m(F)$ the radius of the largest disk centered at the origin to which F can be extended meromorphically with at most m poles counting their multiplicity. The constant $R_m(F)$ is commonly known as the radius of m-meromorphy of F. We write R_m when it is clear to which function the notation refers.

Theorem 1 (Hadamard's theorem [1]) *Let F be defined as in (1). Then, for each $m\in\mathbb{N}_0$, we have*

$$R_m=\frac{\hat{l}_m}{\hat{l}_{m+1}}, \tag{2}$$

where $\hat{l}_0:=1$ and $\hat{l}_m:=\limsup_{n\to\infty}|H_{n,m}|^{1/n}$,

$$H_{n,m}:=\det\Big(\big[f_{n-m+i+j-1}\big]_{1\leqslant i,j\leqslant m}\Big), \tag{3}$$

for $m\in\mathbb{N}$ and $n\geqslant m-1$. The equality (2) comes with the convention that $0/0=\infty$.

In (3) and the rest of this paper, $\big[a_{i,j}\big]_{1\leqslant i,j\leqslant m}$ denotes an $m\times m$ matrix such that its entry on the i-th row and the j-th column is $a_{i,j}$. The reader can also see the proof of Theorem 1 in [2].

In 2003, Rolanía et al [3] generalized Theorem 1 using orthogonal polynomials on the unit circle defined as follows. Let μ be a finite positive Borel measure with infinite support $\operatorname{supp}(\mu)$ contained in the unit circle $\mathbb{T}:=\{z\in\mathbb{C}:|z|=1\}$. We write $\mu\in\mathscr{M}$ and define the associated inner product,

$$\langle g,h\rangle:=\int g(\zeta)\overline{h(\zeta)}\,\mathrm{d}\mu(\zeta),\quad g,h\in L_2(\mu).$$

Let

$$\varphi_n(z):=\kappa_n z^n+\cdots,\quad \kappa_n>0,\quad n\in\mathbb{N}_0$$

be the orthonormal polynomial of degree n with respect to μ having positive leading coefficient; that is, $\langle\varphi_n,\varphi_m\rangle=\delta_{n,m}$. Let

$$\mathbb{B}_R:=\{z\in\mathbb{C}:|z|<R\}$$

and

$$\mathbb{B}:=\mathbb{B}_1=\{z\in\mathbb{C}:|z|<1\}$$

be the disk centered at 0 of radius R and the disk centered at 0 of radius 1, respectively. Denote by $\mathscr{H}(\overline{\mathbb{B}})$ the space of all functions holomorphic on some neighborhood of $\overline{\mathbb{B}}$. From now on, we will only consider $F\in\mathscr{H}(\overline{\mathbb{B}})$.

Now, let us define subclasses of $\mathscr{M}$. We say that $\mu\in\mathbf{S}$ if and only if μ satisfies the Szegő condition, namely

$$\int_{\mathbb{T}}\log\mu'(\zeta)|\,\mathrm{d}\zeta|>-\infty,$$

where μ' denotes the Radon-Nikodym derivative of μ with respect to the arc length on $\mathbb{T}$. We denote by $\hat{\mathbf{S}}$ the class of all $\mu \in \mathscr{M}$ such that

$$\rho(\mu) := \left(\limsup_{n\to\infty} |\varphi_n(0)|^{1/n}\right)^{-1} > 1. \tag{4}$$

It is well-known that $\hat{\mathbf{S}}$ is the class of all measures meeting the Szegő condition such that the radius of 0-meromorphy (holomorphy) of the reciprocal of the corresponding interior Szegő function is $\rho(\mu) > 1$ (see (2.1), (2.5), and Theorems 6.2 and 7.4 in [4] for more details).

The main result in [3] is the following theorem.

Theorem 2 *Let $\mu \in \hat{\mathbf{S}}$ and $F \in \mathscr{H}(\overline{\mathbb{B}})$. Then, for all $m \in \mathbb{N}_0$, we have*

$$R_m = \frac{\tilde{l}_m}{\tilde{l}_{m+1}}, \tag{5}$$

where $\tilde{l}_0 := 1$ and $\tilde{l}_m := \limsup_{n\to\infty} |\tilde{\Delta}_{n,m}|^{1/n}$,

$$\tilde{\Delta}_{n,m} := \det\left(\left[\langle z^{m-j}F, \varphi_{n+i-1}\rangle\right]_{1\leqslant i,j\leqslant m}\right),$$

for $m \in \mathbb{N}$ (the equation (5) comes with the same convention as in (3)).

In [3], the authors proved Theorem 2 using a result (see [5]) concerning the convergence of row sequences of standard orthogonal Padé approximants (sometimes called Fourier-Padé approximants or Frobenius-Padé approximants). Moreover, they employed Theorem 2 to find the location of poles of the reciprocal of interior Szegő functions. Note that there is a result in [6] similar to (5) when $m = 1$ but the measure μ is supported on the interval $[-1, 1]$.

In [7], the author proposed a new way to compute R_m using a different determinant defined as follows. For a given function $F \in \mathscr{H}(\overline{\mathbb{B}})$, we define the following determinant:

$$\Delta_{n,m} := \det\left(\left[\langle z^{m+i-j-1}F, \varphi_n\rangle\right]_{1\leqslant i,j\leqslant m}\right). \tag{6}$$

Set

$$l_0 := 1 \qquad \text{and} \qquad l_m := \limsup_{n\to\infty} |\Delta_{n,m}|^{1/n},$$

for all $m \in \mathbb{N}$.

Our main result in this paper is

Theorem 3 *Let $\mu \in \hat{\mathbf{S}}$ and $F \in \mathscr{H}(\overline{\mathbb{B}})$. Then, for all $m \in \mathbb{N}_0$, we have*

$$R_m = \frac{l_m}{l_{m+1}}, \tag{7}$$

where by the same convention $0/0 = \infty$.

Theorem 3 is an affirmative answer to Conjecture 1.2 in [7]. The determinant $\Delta_{n,m}$ used in the calculation (7) was motivated by modified orthogonal Padé approximation (see, for example, Definition 1.2 in [8] for its definition). Indeed, the formula (7) when $m = 1$ was proved in Theorem 1.1 in [7] using a convergence of the first row sequence of modified orthogonal Padé approximants (see Theorem 1.2 in [9]). However, we find that using our main lemma (Lemma 4 below), we can relate the determinants $\tilde{\Delta}_{n,m}$ and $\Delta_{n,m}$. Therefore, the formula (7) can be deduced from the formula (5).

LEMMAS AND AUXILIARY RESULTS

In order to state some auxiliary results, we need another class of measures which is a subclass of $\mathscr{M}$. We say that $\mu \in \mathbf{Reg}$ if and only if $\operatorname{supp}(\mu) = \mathbb{T}$ and

$$\lim_{n\to\infty} |\varphi_n(z)|^{1/n} = |z|, \tag{8}$$

uniformly on compact subsets of $\mathbb{C}\backslash\overline{\mathbb{B}}$. When $\operatorname{supp}(\mu) = \mathbb{T}$, it was shown in Theorem 3.1.1 in [10] that the condition (8) is equivalent to the condition

$$\lim_{n\to\infty} \kappa_n^{1/n} = 1. \tag{9}$$

The following lemma (see Theorems 6.2 and 7.4 in [4] or Theorem 6.6.1 in [10]) is equivalent to Theorem 2 or Theorem 3 when $m = 0$. Furthermore, it also serves as an analogue of the Cauchy-Hadamard formula.

Lemma 1 *Let $F \in \mathscr{H}(\overline{\mathbb{B}})$ and $\mu \in \mathbf{Reg}$. Then,*

$$l_1 = \limsup_{n\to\infty} |\langle F, \varphi_n\rangle|^{1/n} = \frac{1}{R_0}. \tag{10}$$

Moroever, the series $\sum_{n=0}^{\infty} \langle F, \varphi_n\rangle \varphi_n$ converges to F uniformly on compact subsets of $\mathbb{B}_{R_0}$ and diverges pointwise for all $z \in \mathbb{C}\backslash\overline{\mathbb{B}_{R_0}}$.

It can also be proved that the partial sum of the series in Lemma 1 converges to F in the $L_2(\mu)$ space with the following rate of convergence (see Theorem 6.6.1 in [10]).

Lemma 2 *Let $F \in \mathscr{H}(\overline{\mathbb{B}})$ and $\mu \in \mathbf{Reg}$. Then,*

$$\limsup_{n\to\infty} \|F - S_n\|_2^{1/n} \leqslant \frac{1}{R_0}, \tag{11}$$

where $\|\cdot\|_2$ denotes the $L_2(\mu)$-norm and

$$S_n(z) := \sum_{k=0}^{n} \langle F, \varphi_k\rangle \varphi_k(z) \tag{12}$$

is the n-th partial sum of the Fourier expansion of F.

The following lemma (see Lemma 2.3 in [7]) provides an estimate of l_m.

Lemma 3 *Let $F \in \mathscr{H}(\overline{\mathbb{B}})$ and $\mu \in \mathbf{Reg}$. Then,*

$$l_m \leqslant \frac{1}{R_0 \cdots R_{m-1}} < 1, \qquad m \in \mathbb{N}.$$

Define the monic orthogonal polynomial of degree n,

$$\Phi_n(z) := \frac{\varphi_n(z)}{\kappa_n}.$$

It is well-known (see e.g., formulas (1.2) and (1.5) in [4]) that the polynomials Φ_n satisfy the following three term recurrence formula:

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \tag{13}$$

where $\Phi_n^*(z) = z^n\overline{\Phi_n(1/\bar{z})}$ is the so-called n-th reversed polynomial, and the following relation

$$1 - \left(\frac{\kappa_n}{\kappa_{n+1}}\right)^2 = |\Phi_{n+1}(0)|^2. \tag{14}$$

Using (9) and (14), it is not difficult to check that if $\mu \in \hat{\mathbf{S}}$, then

$$\lim_{n\to\infty} \frac{\kappa_{n+1}}{\kappa_n} = 1. \tag{15}$$

The following relations play an important role in the proof of the main theorem.

Lemma 4 *Let $\mu \in \hat{\mathbf{S}}$ and $F \in \mathscr{H}(\overline{\mathbb{B}})$. For each $j \in \mathbb{N}_0$, we denote by j_0 the number of poles of F in $\mathbb{B}_{R_j}$ counting their order. Let Q_j be the monic polynomial of degree j which has a zero at each pole of F in $\mathbb{B}_{R_j}$ and zeros at 0 of order $j - j_0$. Then, we have for all $k, j \in \mathbb{N}_0$,*

$$\langle z^{k+1}Q_jF, \varphi_{n+1}\rangle = \langle z^kQ_jF, \varphi_n\rangle + \delta_{n,j},$$

where

$$\limsup_{n\to\infty} |\delta_{n,j}|^{1/n} < \frac{1}{R_j}.$$

Proof of Lemma 4: It follows from the recurrence formula (13) that

$$\begin{aligned}
\frac{1}{\kappa_{n+1}}&\langle z^{k+1}Q_jF, \varphi_{n+1}\rangle \\
&= \langle z^{k+1}Q_jF, z\Phi_n + \Phi_{n+1}(0)\Phi_n^*\rangle \\
&= \langle z^{k+1}Q_jF, z\Phi_n\rangle + \langle z^{k+1}Q_jF, \Phi_{n+1}(0)\Phi_n^*\rangle \\
&= \frac{1}{\kappa_n}\langle z^kQ_jF, \varphi_n\rangle + \langle z^{k+1}Q_jF, \Phi_{n+1}(0)\Phi_n^*\rangle.
\end{aligned}$$

Then, from the above equality,

$$\begin{aligned}
|\langle z^{k+1}&Q_jF, \varphi_{n+1}\rangle - \langle z^kQ_jF, \varphi_n\rangle| \\
&\leqslant \left|\langle z^{k+1}Q_jF, \varphi_{n+1}\rangle - \frac{\kappa_n}{\kappa_{n+1}}\langle z^{k+1}Q_jF, \varphi_{n+1}\rangle\right| \\
&\quad + \left|\frac{\kappa_n}{\kappa_{n+1}}\langle z^{k+1}Q_jF, \varphi_{n+1}\rangle - \langle z^kQ_jF, \varphi_n\rangle\right| \\
&\leqslant \left|1 - \frac{\kappa_n}{\kappa_{n+1}}\right| |\langle z^{k+1}Q_jF, \varphi_{n+1}\rangle| \\
&\quad + |\kappa_n\langle z^{k+1}Q_jF, \Phi_{n+1}(0)\Phi_n^*\rangle|.
\end{aligned} \tag{16}$$

By (4), (9), (10), (14), and (15), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\left(\left|1 - \frac{\kappa_n}{\kappa_{n+1}}\right| |\langle z^{k+1}Q_jF, \varphi_{n+1}\rangle|\right)^{1/n} \\
&\leqslant \limsup_{n\to\infty}\left(\frac{|\Phi_{n+1}(0)|^2}{1 + \kappa_n\kappa_{n+1}^{-1}}\right)^{1/n} \limsup_{n\to\infty} |\langle z^{k+1}Q_jF, \varphi_{n+1}\rangle|^{1/n} \\
&< \frac{1}{R_0(z^{k+1}Q_jF)} = \frac{1}{R_j(F)}.
\end{aligned} \tag{17}$$

The expression in (16) can be rewritten as

$$\begin{aligned}
&|\kappa_n\overline{\Phi_{n+1}(0)}\langle z^{k+1}Q_jF, \Phi_n^*\rangle| \\
&\qquad = |\kappa_n\overline{\Phi_{n+1}(0)}\langle z^{k+1}(Q_jF - S_{n-k-1}), \Phi_n^*\rangle|,
\end{aligned} \tag{18}$$

where S_{n-k-1} denotes the $(n-k-1)$-th partial sum of the Fourier expansion of Q_jF. Notice that

$$\langle z^{k+1}S_{n-k-1}, \Phi_n^*\rangle = 0$$

because $z^{k+1}S_{n-k-1}$ is a polynomial of degree at most $n \geqslant 1$ with a zero of multiplicity $\geqslant 1$ at $z = 0$ and Φ_n^* is orthogonal to all such polynomials. Therefore, using (4), (9), (11), and the Holder inequality, it follows from (18) that

$$\begin{aligned}
\limsup_{n\to\infty}&|\kappa_n\overline{\Phi_{n+1}(0)}\langle zQ_jF, \Phi_n^*\rangle|^{1/n} \\
&< \limsup_{n\to\infty} \|Q_jF - S_{n-k-1}\|_2^{1/n}\|\kappa_n\Phi_n^*\|_2^{1/n} \\
&= \limsup_{n\to\infty} \|Q_jF - S_{n-k-1}\|_2^{1/n}\|\kappa_n\Phi_n\|_2^{1/n} \\
&= \limsup_{n\to\infty} \|Q_jF - S_{n-k-1}\|_2^{1/n}\|\varphi_n\|_2^{1/n} \\
&= \limsup_{n\to\infty} \|Q_jF - S_{n-k-1}\|_2^{1/n} \\
&\leqslant \frac{1}{R_0(Q_jF)} = \frac{1}{R_j(F)}.
\end{aligned} \tag{19}$$

By (17) and (19), it follows from (16) that

$$\limsup_{n\to\infty} |\langle z^{k+1}Q_jF, \varphi_{n+1}\rangle - \langle z^kQ_jF, \varphi_n\rangle|^{1/n} < \frac{1}{R_j(F)},$$

which proves the lemma. □

We will need the following simple linear algebra result in the proof of the main result.

Lemma 5 *Let*

$$A = [a_{i,j}]_{1\leqslant i,j\leqslant m} \qquad \text{and} \qquad B = [b_{i,j}]_{1\leqslant i,j\leqslant m}$$

be matrices. If all rows except the kth row of B are zero rows, then

$$\det(A + B) = \det(A) + \det(C),$$

where C is a matrix obtained from the matrix A by replacing its k-th row by the vector

$$\begin{bmatrix} b_{k,1} & b_{k,2} & \dots & b_{k,m} \end{bmatrix}.$$

PROOF OF Theorem 3

Proof of Theorem 3: The proof of this theorem is carried out by induction on $m \in \mathbb{N}_0$. By definition and Lemma 1, we recall that

$$l_0 = 1, \qquad l_1 = (R_0)^{-1}. \tag{20}$$

Fix $m \geqslant 1$ and suppose that (7) holds for all indices up to $m-1$. Let us prove that it is also true for m.

From Lemma 3, we have that $l_j \leqslant 1$ for all $j \in \mathbb{N}$. If $R_m = \infty$, according to Lemma 3, we have that $l_{m+1} = 0$. Hence, $R_m = l_m/l_{m+1}$ as needed (recall that by convention $0/0 = \infty$). Therefore, we can assume that $R_m < \infty$. Consequently, $R_j < \infty$ for $j = 0, 1, \dots, m$. By the hypothesis of induction, we have that $R_j = l_j/l_{j+1}, j = 0, 1, \dots, m-1$; therefore, $l_j > 0, j = 0, 1, \dots, m$. Multiplying these equalities, we obtain that

$$R_0 \cdots R_{m-1} = \frac{1}{l_m} < \infty. \tag{21}$$

Our main goal is to show that if $R_m < \infty$, then

$$l_{m+1} = \frac{1}{R_0 R_1 \cdots R_m}. \tag{22}$$

Suppose (22) has been proved. Combining (21) and (22), we have

$$R_m = \frac{l_m}{l_{m+1}},$$

which means that (7) holds for m.

Now, let us prove (22). By Lemma 3, to show (22), we only show that

$$l_{m+1} \geqslant \frac{1}{R_0 R_1 \dots R_m}.$$

Now, we will use the polynomials Q_j defined in Lemma 4. From Lemma 4 and Lemma 5, the fact that $Q_j \in \operatorname{span}\{1, z, \dots, z^j\}$ and $z^j \in \operatorname{span}\{Q_0, Q_1, \dots, Q_j\}$, for all $0 \leqslant j \leqslant m$, the column operation property for determinants, and the distributive law for determinants, we obtain

$$\begin{aligned}
\Delta_{n,m+1} &= \det\Big(\big[\langle z^{m+i-j}F, \varphi_n\rangle\big]_{1\leqslant i,j\leqslant m+1}\Big) \\
&= \det\Big(\big[\langle z^{i-1}Q_{m-j+1}F, \varphi_n\rangle\big]_{1\leqslant i,j\leqslant m+1}\Big) \qquad (23)\\
&= \det\Big(\big[\langle Q_{m-j+1}F, \varphi_{n-i+1}\rangle\big]_{1\leqslant i,j\leqslant m+1}\Big) + \beta_n \qquad (24)\\
&= \beta_n + \Big((-1)^{m(m+1)/2} \times \\
&\qquad \det\Big(\big[\langle Q_{m-j+1}F, \varphi_{n-m+i-1}\rangle\big]_{1\leqslant i,j\leqslant m+1}\Big)\Big) \\
&= \beta_n + \Big((-1)^{m(m+1)/2} \times \\
&\qquad \det\Big(\big[\langle z^{m-j+1}F, \varphi_{n-m+i-1}\rangle\big]_{1\leqslant i,j\leqslant m+1}\Big)\Big) \\
&= \beta_n + (-1)^{m(m+1)/2}\tilde{\Delta}_{n-m,m+1}, \qquad (25)
\end{aligned}$$

where β_n denotes the sum of the remaining $2^m - 1$ determinants and

$$b := \limsup_{n\to\infty} |\beta_n|^{1/n} < \frac{1}{R_0 R_1 \cdots R_m}. \tag{26}$$

Note that in order to get (24) from (23), we have to firstly use Lemma 4 and then expand the resulting determinant using Lemma 5 inductively from the second row to the last row. Using (5) and arguing as in (21), we can show that under the condition $R_m < \infty$,

$$\tilde{l}_{m+1} = \frac{1}{R_0 R_1 \cdots R_m}. \tag{27}$$

By (25), we have

$$\begin{aligned}
\tilde{l}_{m+1} &= \limsup_{n\to\infty} |\tilde{\Delta}_{n-m,m+1}|^{1/n} \\
&\leqslant \max\{\limsup_{n\to\infty} |\Delta_{n,m+1}|^{1/n}, \limsup_{n\to\infty} |\beta_n|^{1/n}\} \\
&= \max\{l_{m+1}, b\}. \qquad (28)
\end{aligned}$$

If $\max\{l_{m+1}, b\} = b$, then by (26), (27), and (28),

$$\frac{1}{R_0 R_1 \cdots R_m} = \tilde{l}_{m+1} \leqslant b < \frac{1}{R_0 R_1 \cdots R_m},$$

which is impossible. Therefore, $\max\{l_{m+1}, b\} = l_{m+1}$ and again, by (27) and (28),

$$\frac{1}{R_0 R_1 \cdots R_m} = \tilde{l}_{m+1} \leqslant l_{m+1}.$$

This completes the proof. □

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