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On the recurrence relations and applications of harmonic and derangement numbers

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ABSTRACT: We use elementary methods to establish three key recurrence relations: one for derangement numbers, a second for harmonic numbers, and a third for degenerate harmonic numbers. Our results not only contribute to the understanding of the underlying structure of these numbers but also highlight the effectiveness of elementary techniques in discovering new mathematical properties. We also discuss the recurrence relation of the numbers and investigate the properties of the numbers. The findings have potential applications in various fields where these numbers appear, including combinatorics, probability, and computer science.

KEYWORDS: derangement numbers, harmonic numbers, degenerate harmonic numbers

MSC2020: 11B83

INTRODUCTION

In this paper, we show by using elementary method the following recurrence relations for the derangement numbers D_n, the harmonic numbers H_n, and the degenerate harmonic numbers $H_{n,\lambda}$: for any integers $m, n \geqslant 0$, we have

$$\frac{n!}{D_{m+n}} = \sum_{l=0}^{n}\sum_{k=0}^{m}\binom{k+n-l-1}{n-l}\binom{m}{k}(-1)^{m-k}\frac{k!}{l!}D_l,$$

$$\binom{m+n}{n}H_{n+m} = \sum_{k=0}^{n}H_k\binom{m+n-k-1}{n-k} + H_m\binom{m+n}{n},$$

$$\binom{m+n}{n}H_{n+m,\lambda} = \sum_{k=0}^{n}H_{k,\lambda}\binom{m+n-k-1}{n-k} + H_{m,\lambda}\binom{m+n-\lambda}{n}.$$

The exploration of degenerate versions of special polynomials and numbers (see [1–9] and the references therein) began with Carlitz's foundational work on degenerate Bernoulli and Euler numbers. This field has since expanded to include not only polynomials and numbers but also transcendental functions like the gamma function. In addition, the development of the λ-umbral calculus has provided a more robust and effective framework for analyzing degenerate Sheffer polynomials compared to the classical umbral calculus.

A derangement is a permutation of a set of elements where no element remains in its original position. The number of derangements for a set of n elements is known as the n-th derangement number, denoted as D_n. A classic and intuitive example of a derangement is the hat-check problem: Imagine n guests at a party each check their hat, and the hats are returned randomly. The number of ways that no guest receives their own hat is the derangement number, D_n.

Harmonic numbers H_n have a rich history and are found throughout various fields of mathematics, computer science, probability, physics, and engineering. Their widespread appearance underscores their fundamental importance in many different areas of study. The degenerate harmonic numbers $H_{n,\lambda}$ are a degenerate version of the harmonic numbers.

We have structured this paper into three sections. The first section serves as an introduction to the core concepts used throughout the paper. We begin by recalling derangement and harmonic numbers, followed by a review of degenerate exponentials and logarithms. This section also introduces the notions of degenerate harmonic numbers and hyperharmonic numbers. The next section is the main contributions of this work, where we establish several important recurrence relations. Specifically, Theorem 1 presents a recurrence for derangement numbers, Theorem 2 for harmonic numbers, and Theorem 3 for degenerate harmonic numbers. In the following section, we discuss the recurrence relation of the numbers $D_n, H_n, H_{n,\lambda}$ and $H_{n,\lambda}^{(r)}$, and investigate the properties of the numbers from its graphs. Finally, the conclusion provides a summary of our results and concludes the paper. A list of general references can be found in [10–13]. In the rest of this section, we recall the facts that are needed throughout this paper.

The n-th derangement number D_n is given by (see,

([7, 10, 14])

$$D_n = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}, \tag{1}$$

where n is a nonnegative integer. From (1), we have (see, [7, 10, 14])

$$\frac{1}{1-t}\,\mathrm{e}^{-t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}. \tag{2}$$

The harmonic numbers are defined by (see [2, 6, 9, 12, 15–18])

$$H_0 = 0, \quad H_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}, \quad (n \geqslant 1). \tag{3}$$

From (3), we have (see [2, 4, 9])

$$\frac{1}{1-t}\log\left(\frac{1}{1-t}\right) = \sum_{n=1}^{\infty} H_n t^n. \tag{4}$$

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponentials are defined by (see [2–5, 7, 9])

$$\mathrm{e}_\lambda^x(t) = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \quad \mathrm{e}_\lambda = \mathrm{e}_\lambda^1(t),$$

where

$$(x)_{0,\lambda} = 1, \ (x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda), \ (n \geqslant 1).$$

The degenerate logarithm $\log_\lambda(t)$ is defined as the compositional inverse of $\mathrm{e}_\lambda(t)$ and given by

$$\log_\lambda(t) = \frac{1}{\lambda}(t^\lambda - 1). \tag{5}$$

Note here that $\lim_{\lambda\to 0}\log_\lambda(t) = \log(t)$. We use the following relation of the degenerate logarithms:

$$\begin{aligned}\log_\lambda(xy) &= \log_\lambda(x) + x^\lambda \log_\lambda(y)\\ &= \log_\lambda(y) + y^\lambda \log_\lambda(x).\end{aligned} \tag{6}$$

We note from (5) that (see [2, 4, 9])

$$\log_\lambda(1+t) = \sum_{n=1}^{\infty}\lambda^{n-1}(1)_{n,1/\lambda}\frac{t^n}{n!} = \sum_{n=1}^{\infty}\binom{\lambda-1}{n-1}\frac{t^n}{n}.$$

Kim-Kim defined the degenerate harmonic numbers given by (see [2, 4, 9])

$$\begin{aligned}H_{0,\lambda} = 0, \ H_{n,\lambda} &= \frac{1}{\lambda}\sum_{k=1}^{n}\binom{\lambda}{k}(-1)^{k-1}\\ &= \sum_{k=1}^{n}\binom{\lambda-1}{k-1}\frac{(-1)^{k-1}}{k}, \quad (n \geqslant 1).\end{aligned} \tag{7}$$

Note that

$$\lim_{\lambda\to 0} H_{n,\lambda} = H_n, \quad (n \geqslant 0).$$

From (7), we derive the generating function of degenerate harmonic numbers (see [2, 4, 9]):

$$\frac{1}{1-t}\log_{-\lambda}\left(\frac{1}{1-t}\right) = \sum_{n=1}^{\infty} H_{n,\lambda}\, t^n, \quad H_{0,\lambda} = 0. \tag{8}$$

In 1996, Conway and Guy introduced the hyperharmonic numbers, $H_n^{(r)}$, $(n, r \geqslant 0)$, which are defined by (see [11])

$$\begin{aligned}&H_0^{(r)} = 0, \quad (r \geqslant 0), \quad H_n^{(0)} = \frac{1}{n}, \quad (n \geqslant 1),\\ &H_n^{(r)} = \sum_{k=1}^{n} H_k^{(r-1)}, \quad (n, r \geqslant 1).\end{aligned} \tag{9}$$

Note that $H_n^{(1)} = H_n$ are the harmonic numbers. From (9), we have (see [11])

$$H_n^{(m+1)} = \binom{n+m}{m}\big(H_{n+m} - H_m\big), \ (m \geqslant 0), \tag{10}$$

and

$$\frac{1}{(1-t)^r}\log\left(\frac{1}{1-t}\right) = \sum_{n=1}^{\infty} H_n^{(r)} t^n. \tag{11}$$

Recently, Kim-Kim defined the degenerate hyperharmonic numbers, $H_{n,\lambda}^{(r)}$, $(n, r \geqslant 0)$, which are given by (see [2, 4, 9])

$$\begin{aligned}H_{0,\lambda}^{(r)} = 0, \quad H_{n,\lambda}^{(0)} &= \frac{1}{n!}\lambda^{n-1}(-1)^{n-1}(1)_{n,\frac{1}{\lambda}}\\ &= \frac{1}{\lambda}\binom{\lambda}{n}(-1)^{n-1}, \quad (n \geqslant 1),\\ H_{n,\lambda}^{(r)} &= \sum_{k=1}^{n} H_{k,\lambda}^{(r-1)}, \quad (n, r \geqslant 1).\end{aligned} \tag{12}$$

Observe that $H_{n,\lambda}^{(1)} = H_{n,\lambda}$ are the degenerate harmonic numbers. From (12), we note that (see [5])

$$\begin{aligned}&(-1)^m\binom{\lambda-1}{m}H_{n,\lambda}^{(m+1)}\\ &\quad = \binom{n+m}{m}(H_{n+m,\lambda} - H_{m,\lambda}), \quad (m \geqslant 0),\end{aligned} \tag{13}$$

and (see [2])

$$\frac{1}{(1-t)^r}\log_{-\lambda}\left(\frac{1}{1-t}\right) = \sum_{n=1}^{\infty} H_{n,\lambda}^{(r)} t^n, \quad (r \geqslant 0).$$

Note that (see (11))

$$\lim_{\lambda\to 0} H_{n,\lambda}^{(r)} = H_n^{(r)}, \quad (n \geqslant 0).$$

RECURRENCE RELATIONS FOR HARMONIC AND DERANGEMENT NUMBERS

In the same spirit as [3], we derive three recurrence relations for the derangement numbers D_n, the harmonic numbers H_n, and the degenerate harmonic numbers $H_{n,\lambda}$. From (2), we note that

$$\begin{aligned}
&\sum_{m,n=0}^{\infty} D_{n+m}\frac{x^n}{n!}\frac{y^m}{m!} = \sum_{m=0}^{\infty}\frac{d^m}{dx^m}\sum_{n=0}^{\infty}D_n\frac{x^n}{n!}\frac{y^m}{m!}\\
&= \sum_{n=0}^{\infty}\frac{D_n}{n!}\sum_{m=0}^{n}\binom{n}{m}x^{n-m}y^m = \sum_{n=0}^{\infty}\frac{D_n}{n!}(x+y)^n\\
&= \frac{1}{1-x-y}\,\mathrm{e}^{-(x+y)} = \frac{\mathrm{e}^{-x}}{1-x}\,\frac{1}{1-\frac{y}{1-x}}\,\mathrm{e}^{-y}\\
&= \frac{\mathrm{e}^{-x}}{1-x}\sum_{k=0}^{\infty}\left(\frac{1}{1-x}\right)^k y^k\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}y^m\\
&= \frac{1}{1-x}\,\mathrm{e}^{-x}\sum_{k=0}^{\infty}\left(\frac{1}{1-x}\right)^k y^k k!\sum_{m=k}^{\infty}\frac{(-1)^{m-k}m!}{(m-k)!k!}\,\frac{y^{m-k}}{m!}\\
&= \frac{1}{1-x}\,\mathrm{e}^{-x}\sum_{k=0}^{\infty}k!\left(\frac{1}{1-x}\right)^k\sum_{m=k}^{\infty}(-1)^{m-k}\binom{m}{k}\frac{y^m}{m!}\\
&= \sum_{m=0}^{\infty}\frac{y^m}{m!}\sum_{k=0}^{m}k!\left(\frac{1}{1-x}\right)^k(-1)^{m-k}\binom{m}{k}\frac{1}{1-x}\,\mathrm{e}^{-x}\\
&= \sum_{m=0}^{\infty}\frac{y^m}{m!}\sum_{k=0}^{m}k!\sum_{j=0}^{\infty}\binom{k+j-1}{j}x^j(-1)^{m-k}\binom{m}{k}\sum_{l=0}^{\infty}D_l\frac{x^l}{l!}\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{y^m}{m!}\frac{x^n}{n!}\\
&\qquad\times n!\sum_{l=0}^{n}\sum_{k=0}^{m}\binom{k+n-l-1}{n-l}\binom{m}{k}(-1)^{m-k}\frac{k!}{l!}D_l. \qquad (14)
\end{aligned}$$

Therefore, by comparing the coefficients on both sides of (14), we obtain the following theorem.

Theorem 1 *For any integers $m, n \geqslant 0$, we have*

$$\frac{1}{n!}D_{m+n} = \sum_{l=0}^{n}\sum_{k=0}^{m}\binom{k+n-l-1}{n-l}\binom{m}{k}(-1)^{m-k}\frac{k!}{l!}D_l.$$

Remark 1 Theorem 1 provides a new recurrence relation expressing the higher-order derangement numbers D_{m+n} in terms of lower-order derangement numbers D_l. While probabilistic and degenerate derangement polynomials have been investigated in [7, 14], the identity in Theorem 1 represents a novel combinatorial convolution identity for derangements. In the classical setting where $m = 0$, Theorem 1 trivially reduces to $D_n/n! = \sum_{l=0}^{n}(1/l!)D_l \cdot 0^{n-l}$, yielding the known base identity $D_n = n!\sum_{l=0}^{n}\frac{(-1)^l}{l!}$ (see [10]).

Remark 2 The identity in Theorem 1 expresses the derangement number D_{m+n} (the number of permutations on $m+n$ elements with no fixed points) as a double sum involving binomial coefficients, alternating signs, and lower-order derangement numbers D_l. This formula and its underlying structure connect directly to several domains across combinatorics, probability, and computer science:

(1.) *Combinatorics*: *Partitioned Derangements and Generating Functions*

• Bipartite Permutations and Restricted Fixed Points: Consider a set of $m+n$ elements split into two distinguished subsets A (size m) and B (size n). The identity computes the total number of global derangements D_{m+n} by classifying how A and B map into each other:

• k elements from A map into B, and l elements from B map into A.

• The inner factor $\binom{m}{k}(-1)^{m-k}k!D_l$ arises from applying inclusion-exclusion on A while leaving l fixed elements/derangement structures on B.

• Exponential Generating Functions (EGF) and Operational Calculus: The EGF for derangements is $D(t) = \sum_{n=0}^{\infty}D_n\frac{t^n}{n!} = \frac{\mathrm{e}^{-t}}{1-t}$. Identity expansions of D_{m+n} correspond to shifting operations $\mathrm{e}^{a\frac{d}{dt}}D(t)$ or expanding product operators in umbral calculus. Identities of this form allow for the decomposition of high-order combinatorial numbers (D_{m+n}) into localized convolution sums, useful for deriving new identities for related sequences like subfactorials, generalized derangements, and r-derangements.

(2.) *Probability Theory*: *Joint Matching Problems and Asymptotic Analysis*

• De Montmort's Matching Problem on Bipartite Systems: In classical probability, $D_k/k!$ represents the exact probability $P(E_k)$ that a uniform random permutation on k elements has zero fixed points. Normalizing both sides by $(m+n)!$ turns this identity into a formula for conditional probabilities:

$$\begin{aligned}
&\mathbb{P}(\text{No fixed points in } S_m \cup S_n)\\
&\qquad = \sum_{l,k} w(m,n,k,l)\,\mathbb{P}(\text{Derangement on } l \text{ items})
\end{aligned}$$

This structure enables exact calculation of joint coincidences when mixing two separate pools of items (e.g., matching deck problems where cards are drawn from two distinct sub-collections).

• Poisson Approximation and Limit Distributions: As $l \to \infty$, $D_l/l! \to \mathrm{e}^{-1}$. The binomial term $\binom{k+n-l-1}{n-l}$ acts as a discrete weight or kernel. Expanding D_{m+n} in terms of D_l allows analysts to perform asymptotic expansions to bound the tail probabilities of random permutations having specific cycle structures across disjoint sets.

(3.) *Computer Science*: *Algorithm Analysis, Security, and Testing*

• Analysis of Random Involutions and Shuffling Algorithms: In algorithm design (e.g., verifying de-

rangement generation algorithms like Sattolo's algorithm), one often evaluates sub-permutations on memory blocks of sizes m and n. The formula provides an analytical exact count for the number of valid global states accessible when two memory regions undergo interleaved swaps.

• Automated Software Testing and Benchmarking (Fault Injection): When testing hardware or software systems with m inputs and n configurations where no component is allowed to remain in its default state (a total derangement configuration), calculating test suite coverage across sub-modules of size m and n requires formulas that decompose D_{m+n} into smaller operational components D_l.

From (4), we have

$$\begin{aligned}\sum_{m,n=0}^{\infty}\binom{m+n}{m}H_{n+m}x^n y^m &= \sum_{m=0}^{\infty}\frac{d^m}{dx^m}\sum_{n=0}^{\infty}H_n x^n\frac{y^m}{m!}\\ &= \sum_{n=0}^{\infty}H_n\frac{d^m}{dx^m}\sum_{m=0}^{\infty}x^n\frac{y^m}{m!} = \sum_{n=0}^{\infty}H_n\sum_{m=0}^{n}\binom{n}{m}x^{n-m}y^m\\ &= \sum_{n=0}^{\infty}H_n(x+y)^n = \frac{1}{1-x-y}\log\left(\frac{1}{1-x-y}\right)\\ &= \frac{1}{1-x}\frac{1}{1-\frac{y}{1-x}}\log\left(\frac{1}{1-x}\frac{1}{1-\frac{y}{1-x}}\right) \quad (15)\end{aligned}$$

$$\begin{aligned}&= \left(\frac{1}{1-x}\log\left(\frac{1}{1-x}\right)\right)\left(\frac{1}{1-\frac{y}{1-x}}\right)\\ &\quad+\frac{1}{1-x}\frac{1}{1-\frac{y}{1-x}}\log\left(\frac{1}{1-\frac{y}{1-x}}\right)\\ &= \sum_{k=0}^{\infty}H_k x^k\sum_{m=0}^{\infty}\left(\frac{y}{1-x}\right)^m+\sum_{m=0}^{\infty}H_m y^m\left(\frac{1}{1-x}\right)^{m+1}\\ &= \sum_{k=0}^{\infty}H_k x^k\sum_{m=0}^{\infty}y^m\sum_{j=0}^{\infty}\binom{m+j-1}{j}x^j\\ &\quad+\sum_{m=0}^{\infty}H_m y^m\sum_{n=0}^{\infty}\binom{n+m}{n}x^n\\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}x^n y^m\left(\sum_{k=0}^{n}H_k\binom{m+n-k-1}{n-k}+H_m\binom{n+m}{n}\right).\end{aligned}$$

Therefore, by comparing the coefficients on both sides of (15), we obtain the following theorem.

Theorem 2 *For any integers $m,n\geqslant 0$, we have*

$$\binom{m+n}{m}H_{n+m} = \sum_{k=0}^{n}H_k\binom{m+n-k-1}{n-k}+H_m\binom{n+m}{n}.$$

By (10) and Theorem 2, we get

$$H_n^{(m+1)} = \binom{n+m}{m}(H_{n+m}-H_m) = \sum_{k=0}^{n}H_k\binom{m+n-k-1}{n-k}.$$

Remark 3 Theorem 2 establishes a fundamental convolution-type recurrence relation for classical harmonic numbers H_{m+n}. Although classical harmonic recurrences are discussed in Benjamin et al [15] and Conway–Guy [11], the identity in Theorem 2 explicitly isolates the contribution of H_{n+m} in terms of H_k $(0\leqslant k\leqslant n)$ and H_m. This identity serves as the non-degenerate $(\lambda\to 0)$ baseline for the degenerate harmonic identities in Theorem 3 (see also [18, 19]).

Now, we obtain that

$$\begin{aligned}\sum_{m,n=0}^{\infty}H_{n+m,\lambda}\binom{n+m}{m}x^n y^m &= \sum_{m=0}^{\infty}\frac{d^m}{dx^m}\sum_{n=0}^{\infty}H_{n,\lambda}x^n\frac{y^m}{m!}\\ &= \sum_{n=0}^{\infty}H_{n,\lambda}\sum_{m=0}^{\infty}\frac{d^m}{dx^m}x^n\frac{y^m}{m!} = \sum_{n=0}^{\infty}H_{n,\lambda}\sum_{m=0}^{n}\binom{n}{m}x^{n-m}y^m\\ &= \sum_{n=0}^{\infty}H_{n,\lambda}(x+y)^n = \frac{1}{1-x-y}\log_{-\lambda}\left(\frac{1}{1-x-y}\right). \quad (16)\end{aligned}$$

On the other hand, by (6) and (8), we get

$$\begin{aligned}&\frac{1}{1-x-y}\log_{-\lambda}\left(\frac{1}{1-x-y}\right)\\ &\quad= \frac{1}{1-x}\frac{1}{1-\frac{y}{1-x}}\log_{-\lambda}\left(\frac{1}{1-x}\frac{1}{1-\frac{y}{1-x}}\right)\\ &\quad= \frac{1}{1-x}\frac{1}{1-\frac{y}{1-x}}\left(\log_{-\lambda}\left(\frac{1}{1-x}\right)\right.\\ &\qquad\left.+\left(\frac{1}{1-x}\right)^{-\lambda}\log_{-\lambda}\left(\frac{1}{1-\frac{y}{1-x}}\right)\right)\\ &\quad= \frac{1}{1-x}\log_{-\lambda}\left(\frac{1}{1-x}\right)\frac{1}{1-\frac{y}{1-x}}\\ &\qquad+\left(\frac{1}{1-x}\right)^{1-\lambda}\frac{1}{1-\frac{y}{1-x}}\log_{-\lambda}\left(\frac{1}{1-\frac{y}{1-x}}\right)\\ &\quad= \sum_{l=0}^{\infty}H_{l,\lambda}x^l\sum_{m=0}^{\infty}y^m\left(\frac{1}{1-x}\right)^m\\ &\qquad+\left(\frac{1}{1-x}\right)^{1-\lambda}\sum_{m=0}^{\infty}H_{m,\lambda}\left(\frac{y}{1-x}\right)^m\\ &\quad= \sum_{l=0}^{\infty}H_{l,\lambda}x^l\sum_{m=0}^{\infty}y^m\sum_{k=0}^{\infty}\binom{m+k-1}{k}x^k\\ &\qquad+\sum_{m=0}^{\infty}H_{m,\lambda}y^m\left(\frac{1}{1-x}\right)^{m+1-\lambda}\\ &\quad= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}y^m x^n\left(\sum_{l=0}^{n}H_{l,\lambda}\binom{m+n-l-1}{n-l}\right.\\ &\qquad\left.+H_{m,\lambda}\binom{m+n-\lambda}{n}\right). \quad (17)\end{aligned}$$

Therefore, by (16) and (17), we obtain the following theorem.

Theorem 3 *For any integers $m,n\geqslant 0$, we have*

$$\binom{n+m}{n}H_{n+m,\lambda} = \sum_{l=0}^{n}H_{l,\lambda}\binom{m+n-l-1}{n-l}+H_{m,\lambda}\binom{m+n-\lambda}{n}.$$

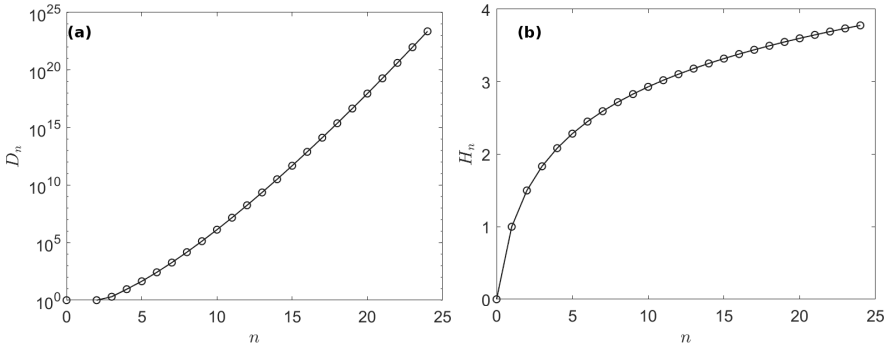

Fig. 1 The graph of the derangement number D_n and the harmonic number H_n, $n = 0, 1, \cdots, 24$.

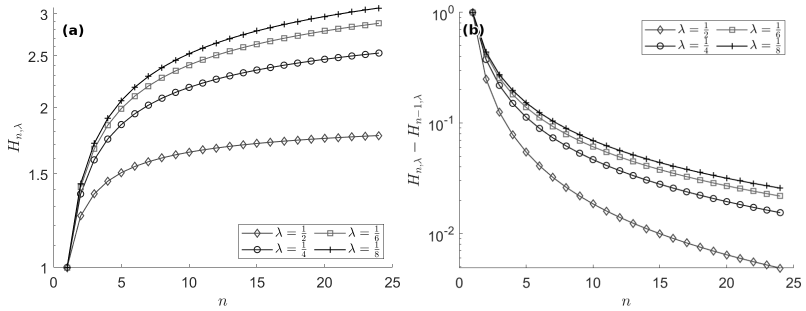

Fig. 2 The graph of the degenerate harmonic number $H_{n,\lambda}$ and $H_{n,\lambda} - H_{n-1,\lambda}$, $n = 1, \cdots, 24$ for four different $\lambda = \frac{1}{m}$, $m = 2, 4, 6, 8$.

Thus, by (13) and Theorem 3, we get

$$H_{n,\lambda}^{(m+1)} = \frac{(-1)^m}{\binom{\lambda-1}{m}} \Bigg[\sum_{l=0}^{n} H_{l,\lambda} \binom{m+n-l-1}{n-l} + \binom{m+n-\lambda}{n} H_{m,\lambda} - \binom{n+m}{n} H_{m,\lambda} \Bigg].$$

Remark 4 Theorem 3 is a direct λ-degenerate generalization of Theorem 2. When $\lambda \to 0$, the degenerate harmonic number $H_{n,\lambda}$ reduces to the classical harmonic number H_n, and $\binom{m+n-\lambda}{n}$ simplifies to $\binom{m+n}{n}$, thereby recovering the identity in Theorem 2 exactly. While degenerate harmonic numbers have been studied in recent works such as Kim-Kim [2, 4, 6], Kurt [8], and Wang et al [9], the identity in Theorem 3 is new and refines previously known recurrences by incorporating the parameter λ explicitly into the binomial weighting factor.

THE RECURRENCE RELATIONS AND THE SHAPES OF THE NUMBERS $D_n, H_n, H_{n,\lambda}$ AND $H_{n,\lambda}^{(r)}$

In this section, we discuss the recurrence relation of the numbers $D_n, H_n, H_{n,\lambda}$ and $H_{n,\lambda}^{(r)}$, and investigate the properties of the numbers from its graphs. The derangement number D_n satisfies the following recurrence relation

$$D_n = \begin{cases} 1 & \text{if } n = 0, \\ nD_{n-1} + (-1)^n & \text{if } n > 0. \end{cases}$$

The harmonic number H_n also satisfies the following recurrence relation

$$H_n = \begin{cases} 0 & \text{if } n = 0, \\ H_{n-1} + \frac{1}{n} & \text{if } n > 0. \end{cases}$$

The degenerate harmonic number $H_{n,\lambda}$ also satisfies the following recurrence relation

$$H_{n,\lambda} = \begin{cases} 0 & \text{if } n = 0, \\ H_{n-1,\lambda} + \frac{(-1)^{n-1}(\lambda)_{n,1}}{n!\lambda} & \text{if } n > 0. \end{cases}$$

Fig. 1 shows a graph of the derangement number D_n and the harmonic number H_n for $n = 0, 1, \cdots, 24$. The y-axis for the derangement number D_n is represented on a logarithmic scale. Since $D_1 = 0$, the value of D_1 is not shown in the figure.

Fig. 2 shows a graph of the degenerate harmonic number $H_{n,\lambda}$ and the difference $H_{n,\lambda} - H_{n-1,\lambda}$, $\lambda = \frac{1}{m}$, $m = 2, 4, 6, 8$ for $n = 1, \cdots, 24$. The y-axis is represented on a logarithmic scale. Fig. 3 shows that the surface of the polynomials $H_{n,\lambda}^{(r)}$ by varying $r = 1, 2, 3, 4, 5, 6$ and $\lambda \in [0, 1]$.

CONCLUSION

In this paper, we have successfully derived recurrence relations for the derangement numbers D_n, the harmonic numbers H_n, and the degenerate harmonic numbers $H_{n,\lambda}$ using elementary methods. The recurrence relations, as proven in Theorem 1, Theorem 2, and Theorem 3, provide a new and structured way to compute and analyze these numbers. The simplicity of

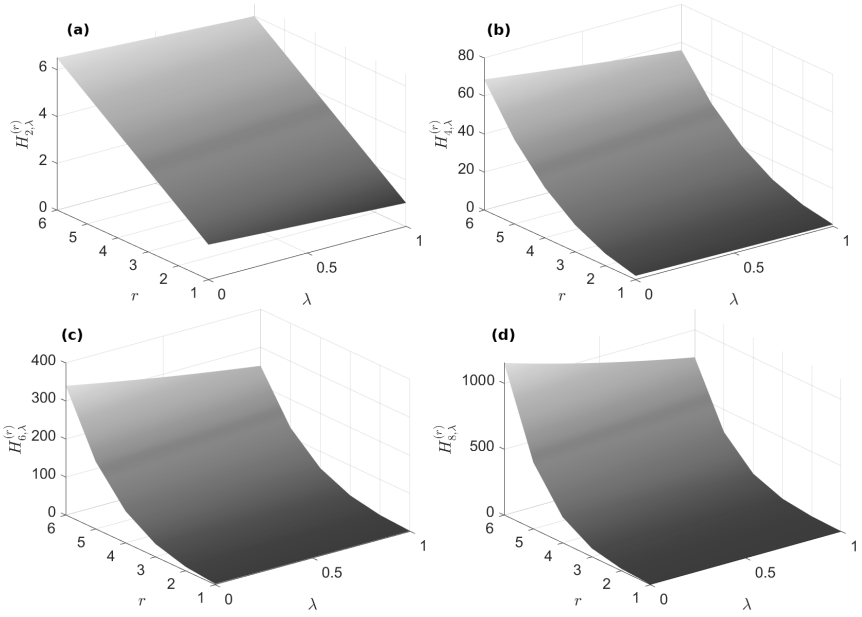

Fig. 3 The graph of the polynomials $H_{n,\lambda}^{(r)}$, $n = 2, 4, 6, 8$.

our approach suggests that similar methods could be used to discover new recurrences of other special numbers and polynomials. The recurrence relation and the properties of the numbers are discussed. This research not only enriches the theory of number sequences but also serves as a testament to the power of elementary techniques in uncovering profound mathematical truths.

It is one of our future projects to continue to explore special numbers and polynomials by employing various methods.

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