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Stability analysis of a strong order-1 stochastic Runge-Kutta method for solving stochastic differential equations

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ABSTRACT: In this paper, we study a strong order-1 stochastic Runge-Kutta method for general stochastic differential equations (SDEs). In particular, for general nonlinear SDEs, we prove that the method is mean square stable or almost surely exponentially stable, if the drift term and the diffusion term of the equation satisfy certain conditions. Numerical results are provided to confirm the theoretical analysis.

KEYWORDS: stochastic Runge-Kutta method, mean-square stability, almost sure exponential stability, stochastic differential equations (SDEs)

MSC2020: 60H10 60H35

INTRODUCTION

Stochastic differential equations (SDEs), which account for stochastic perturbations, are widely employed to model practical problems. Such equations have been applied to various fields, including physics and economics [1, 2].Unfortunately, direct analytical solutions are unavailable for most problems. Consequently, the investigation of numerical solutions to SDEs carries great significance [3, 4]. Broadly speaking, numerical methods for SDEs fall into two main categories: explicit and implicit schemes. This study primarily focuses on explicit schemes.

The Runge-Kutta method is widely utilized for solving ordinary differential equations. Owing to its compact form, it can be systematically extended to higher orders. Numerous researchers have attempted to adopt specific types of Runge-Kutta methods to solve SDEs, and have discussed the stability and convergence of the corresponding numerical schemes. Rüemelin [4] proposed the stochastic Runge-Kutta method for solving SDEs.

Burrage and Tian [5, 6] constructed a class of stochastic Runge-Kutta methods with a strong convergence order of 1.5 for Stratonovich SDEs. In addition, Burrage extended the Butcher root tree theory of Runge-Kutta methods for deterministic ordinary differential equations to Stratonovich SDEs and presented the order conditions for the corresponding stochastic Runge-Kutta methods. Rössler [7] conducted a unified study on Itô-type and Stratonovich-type SDEs and proposed a general stochastic Runge-Kutta method with a more flexible form. Tian [8] defined five stochastic Runge-Kutta methods and discussed two-stage diagonally implicit stochastic Runge-Kutta methods with strong order 1 for strong solutions of Stratonovich SDEs. Newton [9] developed asymptotically efficient Runge-Kutta methods and constructed strong order-1 stochastic Runge-Kutta methods. In recent years, Xin, Ma and Ding [10] studied the parallel waveform relaxation stochastic Runge-Kutta method for SDEs. Yang, Li and Ding [11] constructed a family of arbitrary high-order quadratic invariants and energy-conserving parametric stochastic partitioned Runge-Kutta methods (SPRK).

Exponential stability and mean-square stability are important issues in the study of analytical and numerical solutions of SDEs. Regarding the stability of analytical solutions, interested readers are referred to [12–16]. In [17], Wu and Zhang reviewed the theoretical results on the stability of stochastic continuous systems and discussed the relationships among these stability concepts. Moreover, Wu and Mao [18] discussed the almost sure exponential stability of the Euler method for stochastic delay differential equations (SDDEs). Higham [19] studied the almost sure exponential stability of the stochastic θ-method for linear SDEs and the backward Euler method for nonlinear SDEs. Wang [20] considered the mean stability of the Milstein method for SDDEs. Burrage and Butcher [21] established the stability criteria of implicit Runge-Kutta methods for $k_{l,0}$-type (where the subscript $l = 0$ denotes the zero-index stage coefficient) initial value problems in the N-dimensional Euclidean space $\mathbb{R}^N$, and proposed important concepts such as BN stability and algebraic stability. Koto [22] considered the p-stability of implicit Runge-Kutta methods and discussed their p-stability regions.

NOTIONS AND SCHEMES

Let $(\Omega, \mathscr{F}_t, P)$ be a complete probability space equipped with a filtration $\{\mathscr{F}_t : t \in \mathbb{R}_+\}$ that is increasing and right-continuous, where each $\mathscr{F}_t$ $(t \geqslant 0)$

contains all P-null sets.

Consider the following Itô SDEs:

$$dx(t) = f(t,x(t))dt + g(t,x(t))dW(t), \quad t \geqslant 0, \tag{1}$$

subject to the initial condition

$$x(0) = \xi. \tag{2}$$

Here, f and g denote the drift and diffusion coefficients, respectively. The mappings $f : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ and $g : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ are Borel-measurable, and $W = (w_1,\ldots,w_n)$ stands for an n-dimensional standard Wiener process.

To guarantee the existence and uniqueness of solutions to Eq. (1), we assume that f and g are sufficiently smooth and satisfy the following hypotheses.

Condition 1 ([2]) (Global Lipschitz condition) There exists a constant $K_1 > 0$ such that

$$|f(t,x)-f(t,y)|^2 \vee |g(t,x)-g(t,y)|^2 \leqslant K_1\left(|x-y|^2\right), \quad \forall t \geqslant 0,\ x,y \in \mathbb{R}^n.$$

Condition 2 ([2]) (Linear growth condition) There exists a constant $K_2 > 0$ such that

$$|f(t,x)|^2 \vee |g(t,x)|^2 \leqslant K_2\left(1+|x|^2\right), \quad \forall t \geqslant 0,\ x \in \mathbb{R}^n.$$

For the numerical discretization of Eq. (1), define the time step size $h = T_{\text{end}}/N$ and the equidistant grid points $t_n = nh$ $(n = 0,1,\ldots,N)$. Let $\Delta W_n = W(t_{n+1}) - W(t_n)$ denote the Wiener increment and y_n be a numerical approximation to $x(t_n)$. The explicit stochastic Runge-Kutta method proposed in Wang [23] is given by

$$Y_i = y_n + h\sum_{j=1}^{i-1} a_{ij} f(Y_j) + \sum_{j=1}^{i-1}(b_{ij}^{(1)}\theta_1 + b_{ij}^{(2)}\theta_2)g(Y_j), \quad i = 1,\ldots,s \tag{3}$$

$$y_{n+1} = y_n + h\sum_{j=1}^{s}\alpha_j f(Y_j) + \sum_{j=1}^{s}(\gamma_j^{(1)}\theta_1 + \gamma_j^{(2)}\theta_2)g(Y_j) \tag{4}$$

which can be expressed in the matrix-vector form as

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_s \end{pmatrix} = \begin{pmatrix} y_n \\ \vdots \\ y_n \end{pmatrix} + h\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{ss} \end{pmatrix}\begin{pmatrix} f(Y_1) \\ \vdots \\ f(Y_s) \end{pmatrix} + \begin{pmatrix} \sum_{l=1}^{P} b_{11}^{(l)}\theta_l & \sum_{l=1}^{P} b_{12}^{(l)}\theta_l & \cdots & \sum_{l=1}^{P} b_{1s}^{(l)}\theta_l \\ \sum_{l=1}^{P} b_{21}^{(l)}\theta_l & \sum_{l=1}^{P} b_{22}^{(l)}\theta_l & \cdots & \sum_{l=1}^{P} b_{2s}^{(l)}\theta_l \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{l=1}^{P} b_{s1}^{(l)}\theta_l & \sum_{l=1}^{P} b_{s2}^{(l)}\theta_l & \cdots & \sum_{l=1}^{P} b_{ss}^{(l)}\theta_l \end{pmatrix}\begin{pmatrix} g(Y_1) \\ \vdots \\ g(Y_s) \end{pmatrix},$$

$$y_{n+1} = y_n + h\begin{pmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_s\end{pmatrix}\begin{pmatrix} f(Y_1) \\ \vdots \\ f(Y_s) \end{pmatrix} + \begin{pmatrix} \sum_{l=1}^{P}\gamma_1^{(l)}\theta_l & \sum_{l=1}^{P}\gamma_2^{(l)}\theta_l & \cdots & \sum_{l=1}^{P}\gamma_s^{(l)}\theta_l \end{pmatrix}\begin{pmatrix} g(Y_1) \\ \vdots \\ g(Y_s) \end{pmatrix}.$$

In the present work, we consider the strong order-1 stochastic Runge-Kutta method for solving Eq. (1). Setting $\theta_1 = \Delta W_n$ and $\theta_2 = \sqrt{h}$, the above scheme can be represented by the following Butcher tableau

0	0	0	0	0	0	
0	0	a	0	$-a$	0	
1	0	$1-\frac{1}{2a}$	$\frac{1}{2a}$	$-\frac{1}{2a}$	$\frac{1}{2a}$	

where $a \neq 0$ is a free parameter.

Accordingly, the strong order-1 stochastic Runge-Kutta scheme is written as

$$Y_1 = y_n, \tag{5}$$

$$Y_2 = y_n + a\left(\Delta W_n - \sqrt{h}\right)g(y_n), \tag{6}$$

$$y_{n+1} = y_n + hf(y_n) + \left(\Delta W_n - \frac{\Delta W_n + \sqrt{h}}{2a}\right)g(y_n) + \frac{\Delta W_n + \sqrt{h}}{2a}g\left(y_n + a\left(\Delta W_n - \sqrt{h}\right)g(y_n)\right). \tag{7}$$

STABILITY OF THE STOCHASTIC RUNGE-KUTTA METHOD

In this section, we assume that the functions f and g (i.e., coefficients of Eq. (1)) are continuous. To investigate the stability of numerical solutions, we first present the following definitions.

Definition 1 ([14]) If there exists a constant $h_0 > 0$ such that, for any step size h with $0 < h < h_0$, the numerical solution produced by the scheme for Eq. (1) satisfies

$$\lim_{n\to\infty} E|y_n|^2 = 0,$$

then the numerical solution $\{y_n\}$ is said to be mean-square stable.

Definition 2 ([18]) If there exist constants $\gamma > 0$ and $p > 0$ such that, for any initial value with $|\xi| < \infty$, the numerical solution $y(t_n,\xi)$ of Eq. (1) satisfies

$$\overline{\lim_{t_n\to\infty}}\frac{\ln E|y(t_n,\xi)|^p}{t_n} \leqslant -\gamma \quad (a.s.),$$

then the numerical solution $\{y(t_n,\xi)\}$ is said to be p-th-moment (or almost-surely) exponentially stable, where γ denotes the exponential decay rate.

Since ΔW_n denotes the Wiener increment, we have

$$\begin{aligned} &E(\Delta W_n) = E(\Delta W_n^3) = 0; \\ &E(\Delta W_n^2) = h; \;\; E(\Delta W_n^4) = 3h^2. \end{aligned} \tag{8}$$

The main theoretical results of this paper are stated below, where we restrict our attention to one-dimensional systems.

Theorem 1 *Under Conditions 1 and 2, for any $x \in \mathbb{R}$, suppose that f and g satisfy the following conditions*
(1.) there exist positive constants $\lambda_1, \lambda_2, \lambda_3$ such that

$$xf(t,x) \leqslant -\lambda_1 x^2 \tag{9}$$

$$f(t,x)^2 \leqslant \lambda_2 x^2 \tag{10}$$

$$g(t,x)^2 \leqslant \lambda_3 x^2 \tag{11}$$

(2.) $\rho = 3\big[2\lambda_1 - \big(1 - \frac{1}{a} + \frac{3}{2a^2}\big)\lambda_3\big] \geqslant 2\sqrt{6\big(\lambda_2 + \lambda_3^2\big)}$,
(3.) $\rho + \sqrt{\Delta} \leqslant 6\big(\lambda_2 + \lambda_3^2\big)$, *where* $\Delta = \rho^2 - 24\big(\lambda_2 + \lambda_3^2\big)$, *then the numerical solution $\{y_n\}$ generated by Eq. (7) is mean-square stable, and the critical step-size bound is*

$$h_0 = \frac{\rho + \sqrt{\Delta}}{6\big(\lambda_2 + \lambda_3^2\big)}$$

Proof: Squaring both sides of Eq. (7) yields

$$\begin{aligned} y_{n+1}^2 = &\bigg[y_n + hf(y_n) + \bigg(\Delta W_n - \frac{\Delta W_n + \sqrt{h}}{2a}\bigg)g(y_n)\bigg]^2 \\ &+ \frac{\Delta W_n + \sqrt{h}}{a} g\bigg(y_n + a\big(\Delta W_n - \sqrt{h}\big)g(y_n)\bigg) \\ &\times \bigg[y_n + hf(y_n) + \bigg(\Delta W_n - \frac{\Delta W_n + \sqrt{h}}{2a}\bigg)g(y_n)\bigg] \\ &+ \frac{(\Delta W_n + \sqrt{h})^2}{4a^2} g^2\bigg(y_n + a\big(\Delta W_n - \sqrt{h}\big)g(y_n)\bigg) \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{12}$$

where

$$\begin{aligned} I_1 &= [y_n + hf(y_n) + (\Delta W_n - \frac{\Delta W_n + \sqrt{h}}{2a})g(y_n)]^2, \\ I_2 &= \frac{1}{a}(\Delta W_n + \sqrt{h})g(y_n + a(\Delta W_n - \sqrt{h})g(y_n)) \\ &\quad \times [y_n + hf(y_n) + (\Delta W_n - \frac{\Delta W_n + \sqrt{h}}{2a})g(y_n)], \\ I_3 &= \frac{1}{4a^2}(\Delta W_n + \sqrt{h})^2 g^2(y_n + a(\Delta W_n - \sqrt{h})g(y_n)). \end{aligned}$$

Taking expectations on both sides of Eq. (12), we obtain

$$Ey_{n+1}^2 = EI_1 + EI_2 + EI_3. \tag{13}$$

We first bound the term I_1 by applying the inequality $(a+b)^2 \leqslant 2a^2 + 2b^2$

$$\begin{aligned} I_1 &= \bigg[y_n + hf(y_n) + \big(\Delta W_n - \tfrac{\Delta W_n + \sqrt{h}}{2a}\big)g(y_n)\bigg]^2 \\ &\leqslant 2\big(y_n + hf(y_n)\big)^2 + 2\bigg[(1 - \tfrac{1}{2a})\Delta W_n - \tfrac{\sqrt{h}}{2a}\bigg]^2 g^2(y_n) \\ &\leqslant 2\big(y_n^2 + 2hy_n f(y_n) + h^2 f^2(y_n)\big) + 2\bigg[(1 - \tfrac{1}{2a})^2 \Delta W_n^2 \\ &\quad - \tfrac{\sqrt{h}}{a}(1 - \tfrac{1}{2a})\Delta W_n + \tfrac{h}{4a^2}\bigg]g^2(y_n) \\ &\leqslant 2y_n^2 - 4h\lambda_1 y_n^2 + 2h^2\lambda_2 y_n^2 + 2\bigg[(1 - \tfrac{1}{2a})^2 \Delta W_n^2 \\ &\quad - \tfrac{\sqrt{h}}{a}(1 - \tfrac{1}{2a})\Delta W_n + \tfrac{h}{4a^2}\bigg]\lambda_3 y_n^2 \\ &= \bigg[2(1 - \tfrac{1}{2a})^2 \lambda_3 \Delta W_n^2 - \tfrac{2\sqrt{h}}{a}(1 - \tfrac{1}{2a})\lambda_3 \Delta W_n \\ &\qquad + 2\lambda_2 h^2 + (\tfrac{\lambda_3}{2a^2} - 4\lambda_1)h + 2\bigg]y_n^2. \end{aligned} \tag{14}$$

For the second term, using the inequality $a^2 + b^2 \geqslant 2ab$, we get

$$\begin{aligned} I_2 &= \tfrac{1}{a}(\Delta W_n + \sqrt{h})g(y_n + a(\Delta W_n - \sqrt{h})g(y_n)) \\ &\quad \times \bigg[y_n + hf(y_n) + \big(\Delta W_n - \tfrac{\Delta W_n + \sqrt{h}}{2a}\big)g(y_n)\bigg] \\ &\leqslant \tfrac{1}{2}\bigg[\tfrac{1}{a}(\Delta W_n + \sqrt{h})g(y_n + a(\Delta W_n - \sqrt{h})g(y_n))\bigg]^2 \\ &\quad + \tfrac{1}{2}\bigg[y_n + hf(y_n) + \big(\Delta W_n - \tfrac{\Delta W_n + \sqrt{h}}{2a}\big)g(y_n)\bigg]^2. \end{aligned} \tag{15}$$

From Eq. (11), it follows that

$$\begin{aligned} &\tfrac{1}{2}\bigg[\tfrac{1}{a}(\Delta W_n + \sqrt{h})g(y_n + a(\Delta W_n - \sqrt{h})g(y_n))\bigg]^2 \\ &\leqslant \tfrac{1}{2a^2}(\Delta W_n + \sqrt{h})^2 \lambda_3 (y_n + a(\Delta W_n - \sqrt{h})g(y_n))^2 \\ &\leqslant \tfrac{\lambda_3}{2a^2}(\Delta W_n + \sqrt{h})^2 \bigg[2y_n^2 + 2a^2\big(\Delta W_n - \sqrt{h}\big)^2 g^2(y_n)\bigg] \\ &\leqslant \tfrac{\lambda_3}{2a^2}(\Delta W_n + \sqrt{h})^2 \bigg[2y_n^2 + 2a^2(\Delta W_n - \sqrt{h})^2 \lambda_3 y_n^2\bigg] \\ &= \bigg[\lambda_3^2 \Delta W_n^4 + (\tfrac{\lambda_3}{a^2} - 2\lambda_3^2 h)\Delta W_n^2 \\ &\qquad + \tfrac{2\lambda_3\sqrt{h}}{a^2}\Delta W_n + \tfrac{\lambda_3 h}{a^2} + \lambda_3^2 h^2\bigg]y_n^2. \end{aligned} \tag{16}$$

Substituting Eq. (14) and Eq. (16) into Eq. (15) yields

$$\begin{aligned} I_2 &\leqslant \bigg[\lambda_3^2 \Delta W_n^4 + (\tfrac{\lambda_3}{a^2} - 2\lambda_3^2 h)\Delta W_n^2 + \tfrac{2\lambda_3\sqrt{h}}{a^2}\Delta W_n + \tfrac{\lambda_3 h}{a^2} + \lambda_3^2 h^2\bigg]y_n^2 \\ &\quad + \bigg[(1 - \tfrac{1}{2a})^2 \lambda_3 \Delta W_n^2 - \tfrac{\sqrt{h}}{a}(1 - \tfrac{1}{2a})\lambda_3 \Delta W_n \\ &\qquad + \lambda_2 h^2 + (\tfrac{\lambda_3}{4a^2} - 2\lambda_1)h + 1\bigg]y_n^2 \\ &= \bigg[\lambda_3^2 \Delta W_n^4 + (-2\lambda_3^2 h + (1 - \tfrac{1}{a} + \tfrac{5}{4a^2})\lambda_3)\Delta W_n^2 \\ &\qquad + (\tfrac{5}{2a^2} - \tfrac{1}{a})\lambda_3\sqrt{h}\Delta W_n + (\tfrac{5\lambda_3}{4a^2} - 2\lambda_1)h \\ &\qquad + (\lambda_3^2 + \lambda_2)h^2 + 1\bigg]y_n^2. \end{aligned} \tag{17}$$

For the third term, we derive

$$\begin{aligned}I_3 &= \tfrac{1}{4a^2}(\Delta W_n+\sqrt{h})^2 g^2(y_n+a(\Delta W_n-\sqrt{h})g(y_n))\\ &\leqslant \tfrac{1}{4a^2}(\Delta W_n+\sqrt{h})^2\lambda_3(y_n+a(\Delta W_n-\sqrt{h})g^2(y_n))^2\\ &\leqslant \tfrac{\lambda_3}{4a^2}(\Delta W_n+\sqrt{h})^2(2y_n^2+2a^2(\Delta W_n-\sqrt{h})^2 g^2(y_n))\\ &\leqslant \tfrac{\lambda_3}{2a^2}(\Delta W_n+\sqrt{h})^2 y_n^2+\tfrac{\lambda_3}{2}(\Delta W_n-\sqrt{h})^2\lambda_3 y_n^2\\ &= \Big[\tfrac{\lambda_3^2}{2}\Delta W_n^4+(\tfrac{\lambda_3}{2a^2}-\lambda_3^2 h)\Delta W_n^2+\tfrac{\lambda_3\sqrt{h}}{a^2}\Delta W_n\\ &\qquad +\tfrac{\lambda_3 h}{2a^2}+\tfrac{\lambda_3^2h^2}{2}\Big]y_n^2. \quad (18)\end{aligned}$$

Taking expectations of I_1, I_2, I_3 and using Eq. (8),

$$EI_1 \leqslant \Big[2\lambda_2 h^2+(-4\lambda_1+(2-\tfrac{2}{a}+\tfrac{1}{a^2})\lambda_3)h+2\Big]Ey_n^2 \quad (19)$$

$$\begin{aligned}EI_2 \leqslant \Big[&(2\lambda_3^2+\lambda_2)h^2+(-2\lambda_1\\ &+(1-\tfrac{1}{a}+\tfrac{5}{2a^2})\lambda_3)h+1\Big]Ey_n^2 \quad (20)\end{aligned}$$

$$EI_3 \leqslant (\lambda_3^2 h^2+\tfrac{\lambda_3 h}{a^2})Ey_n^2. \quad (21)$$

Substituting Eqs. (19)–(21) into Eq. (13) gives

$$\begin{aligned}Ey_{n+1}^2 &\leqslant \Big[2\lambda_2 h^2+(-4\lambda_1+(2-\tfrac{2}{a}+\tfrac{1}{a^2})\lambda_3)h+2\Big]Ey_n^2\\ &\quad+\Big[(2\lambda_3^2+\lambda_2)h^2+(-2\lambda_1+(1-\tfrac{1}{a}+\tfrac{5}{2a^2})\lambda_3)h+1\Big]Ey_n^2\\ &\quad+(\lambda_3^2h^2+\tfrac{\lambda_3 h}{a^2})Ey_n^2\\ &=\Big[(3\lambda_3^2+3\lambda_2)h^2-[6\lambda_1-(3-\tfrac{3}{a}+\tfrac{9}{2a^2})\lambda_3]h+3\Big]Ey_n^2\\ &=\Big[3(\lambda_3^2+\lambda_2)h^2-\rho h+3\Big]Ey_n^2. \quad (22)\end{aligned}$$

Define $\Delta=\rho^2-24(\lambda_2+\lambda_3^2)$. By the hypotheses of the theorem, we have $\Delta \geqslant 0$. Let

$$h_0=\frac{\rho+\sqrt{\Delta}}{6\left(\lambda_2+\lambda_3^2\right)}.$$

Clearly, for $h<h_0$, we have

$$3(\lambda_3^2+\lambda_2)h^2-\rho h+3<1,$$

which implies

$$Ey_{n+1}^2 \leqslant CEy_n^2,$$

where C is a positive constant, which implies that

$$\lim_{n\to\infty} E|y_n|^2=0.$$

This completes the proof. □

Theorem 2 *Under Conditions 1 and 2, for any $x\in\mathbb{R}$, assume that f and g satisfy the following conditions:*
(i) *conditions (9)–(11) hold,*
(ii) $\tilde{\rho}=2\lambda_1-\left(1-\frac{1}{a}+\frac{3}{2a^2}\right)\lambda_3>0$,
(iii) $3\left(\lambda_3^2+\lambda_2-\tilde{\rho}\right)+2\leqslant 0$,
then the numerical solution $\{y_n\}$ generated by Eq. (7) is almost surely exponentially stable.

Proof: From the proof of Theorem 1, it follows that

$$Ey_{n+1}^2 \leqslant \Big[3\left(\lambda_3^2+\lambda_2\right)h^2-3\tilde{\rho}h+3\Big]Ey_n^2. \quad (23)$$

For any constant $C>1$ and $h\in(0,1)$, we have

$$\begin{aligned}&C^{(n+1)h}Ey_{n+1}^2-C^{nh}Ey_n^2\\ &\quad\leqslant C^{nh}\left\{C^h\left[3\left(\lambda_3^2+\lambda_2-\tilde{\rho}\right)h+3\right]-1\right\}Ey_n^2, \quad (24)\end{aligned}$$

which implies

$$\begin{aligned}C^{nh}Ey_n^2 &\leqslant Ey_0^2\\ &\quad+\left\{C^h\left[3\left(\lambda_3^2+\lambda_2-\tilde{\rho}\right)h+3\right]-1\right\}\sum_{i=0}^{n-1}C^{ih}Ey_i^2\\ &=Ey_0^2+\psi(C)\sum_{i=0}^{n-1}C^{ih}Ey_i^2, \quad (25)\end{aligned}$$

where

$$\psi(C)=C^h\left[3\left(\lambda_3^2+\lambda_2-\tilde{\rho}\right)h+3\right]-1. \quad (26)$$

Obviously, $\psi'(C)>0$ for all $C>1$. Furthermore, from the hypotheses of Theorem 2, we obtain

$$\begin{aligned}\psi(1)&=3\left(\lambda_3^2+\lambda_2-\tilde{\rho}\right)h+2\\ &\leqslant 3\left(\lambda_3^2+\lambda_2-\tilde{\rho}\right)+2\leqslant 0. \quad (27)\end{aligned}$$

Since $\psi(C)\to\infty$ as $C\to\infty$, there exists a unique constant $C_h>1$ satisfying $\psi(C_h)=0$. Setting $C_h=\mathrm{e}^{\gamma_h}$ and applying Eq. (27), we obtain the inequality $\mathrm{e}^{\gamma_h nh}Ey_n^2\leqslant Ey_0^2$, which gives

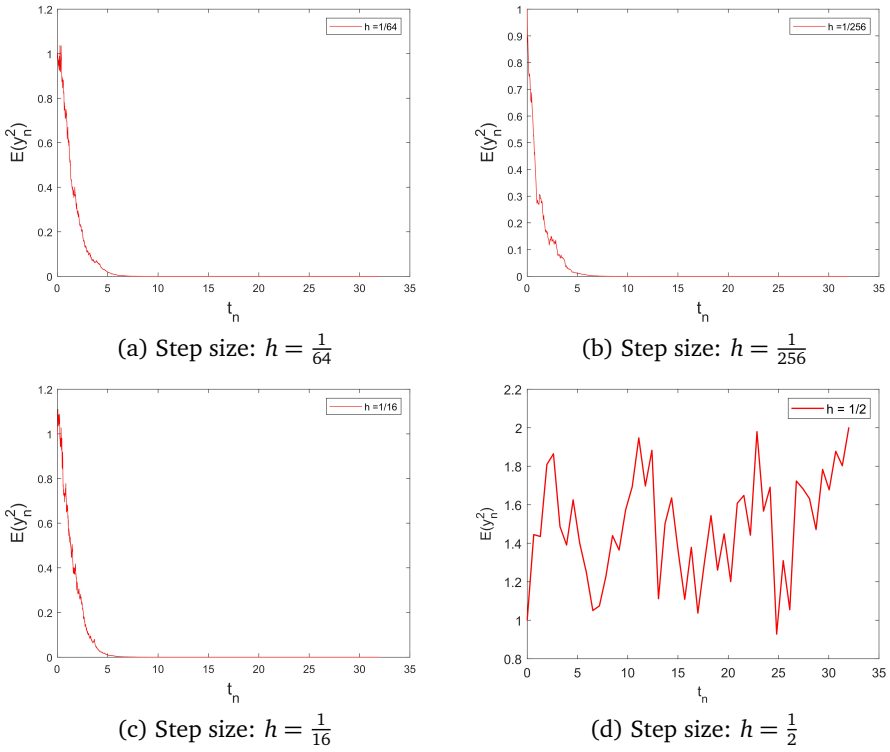
$$\mathrm{e}^{\gamma_h nh}Ey_n^2\leqslant C_0Ey_0^2. \quad (28)$$

where $C_0>0$ is a constant. □

When $a=1/2$ in Eq. (7), the scheme reduces to Newton's "FRKI" method [9], from which we immediately derive the following two corollaries.

Corollary 1 *Under Conditions 1 and 2, for any $x\in\mathbb{R}$, suppose that f and g satisfy the following conditions:*
(i) *conditions (9)–(11) hold,*
(ii) $\rho=3(2\lambda_1-5\lambda_3)\geqslant 2\sqrt{6\left(\lambda_2+\lambda_3^2\right)}$,
(iii) $\rho+\sqrt{\Delta}\leqslant 6\left(\lambda_2+\lambda_3^2\right)$,
then the numerical solution $\{y_n\}$ generated by the "FRKI" method is mean-square stable, with the critical step-size bound h_0.

Corollary 2 *Under Conditions 1 and 2, for any $x\in\mathbb{R}$, suppose that f and g satisfy the following conditions:*
(i) *conditions (9)–(11) hold,*
(ii) $\tilde{\rho}=2\lambda_1-5\lambda_3>0$,
(iii) $3\left(\lambda_3^2+\lambda_2-\tilde{\rho}\right)+2\leqslant 0$,
then the numerical solution $\{y_n\}$ generated by Eq. (7) is almost surely exponentially stable.



(a) Step size: $h = \frac{1}{64}$ (b) Step size: $h = \frac{1}{256}$

(c) Step size: $h = \frac{1}{16}$ (d) Step size: $h = \frac{1}{2}$

Fig. 1 Mean-square stability of the Runge-Kutta method.

Table 1 Coefficients of Eq. (29).

Case	b	c	d	ρ	γ_h
case 1	$\frac{99}{100}$	$\frac{1}{100}$	$\frac{1}{10}$	$\frac{191}{100}$	1.42×10^{-15}
case 2	$\frac{23}{20}$	$\frac{1}{20}$	$\frac{1}{8}$	$\frac{106}{50}$	1.78×10^{-15}
case 3	$\frac{57}{50}$	$\frac{1}{25}$	$\frac{-1}{9}$	$\frac{213}{100}$	1.17×10^{-15}
case 4	$\frac{123}{100}$	$\frac{3}{100}$	$\frac{-1}{7}$	$\frac{23}{10}$	1.11×10^{-15}

NUMERICAL RESULTS

In this section, numerical examples are provided to verify the theoretical conclusions derived in the previous section, where we set $a = 1/2$ in Eq. (7).

Consider the following nonlinear SDEs

$$\begin{cases} dx(t) &= (-bx(t)+c\sin x(t))\,\mathrm{d}t \\ &\quad +d\sin x(t)\,\mathrm{d}W(t), \qquad t \geqslant 0 \\ x(t) &= 0, \qquad\qquad\qquad t \in [-1,0], \end{cases} \tag{29}$$

where $b > 0, c > 0, d > 0$.

We first perform numerical experiments to verify the mean-square-stability result in Theorem 1. We set $b = \frac{5\sqrt{18}+18}{20}, c = \frac{5\sqrt{18}-18}{20}$ and $d = \frac{1}{\sqrt{10}}$. It can be verified that Eq. (29) satisfies all assumptions of Theorem 1 with the parameters

$$\lambda_1 = \tfrac{9}{5}, \lambda_2 = \tfrac{15}{4}, \lambda_3 = \tfrac{1}{10}, \rho = \tfrac{93}{10}, h_0 = \tfrac{186+2\sqrt{69}}{429}. \tag{30}$$

Consequently, the stochastic Runge-Kutta scheme (7) is mean-square stable for any step size $h > 0$ satisfying

$$h < h_0 = \tfrac{186+2\sqrt{69}}{429} \approx 0.473.$$

We adopt the step sizes

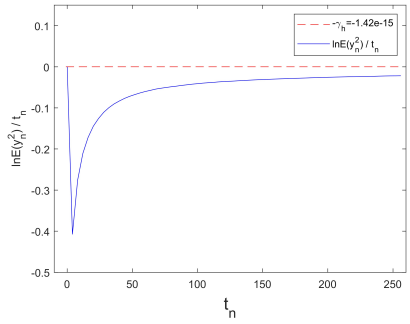
$$h = 2^{-8},\ 2^{-6},\ 2^{-4},\ 2^{-1}$$

to compute four sets of numerical solutions for Eq. (29), as shown in Fig. 1.

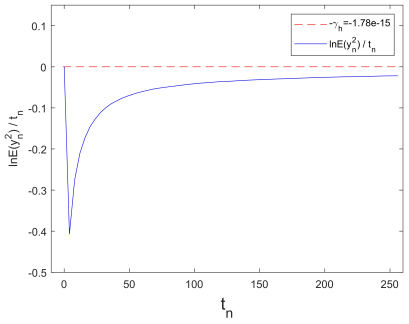
To visualize our theoretical results, we generate 500 discrete Brownian paths on the time interval $[0, 35]$ to investigate how the step-size h affects the mean-square stability of the stochastic Runge-Kutta scheme.

In Fig. 1, subplots (a–c) show that the numerical solution is mean-square stable, whereas subplot (d) corresponds to $h = \frac{1}{2} > h_0$. One can observe severe oscillations for large t, which indicates that stability is no longer preserved. In other words, the stochastic Runge-Kutta scheme is mean-square stable provided $h \in (0, h_0)$. This observation agrees well with the theoretical analysis in the previous section.

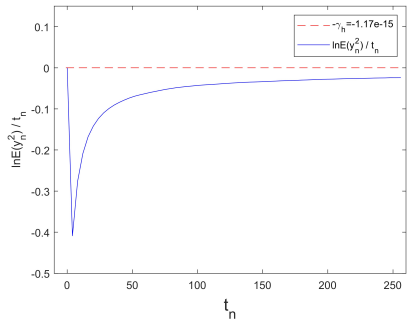
Next, we provide numerical simulations to validate the almost-sure exponential stability. Four groups of parameters are selected for computation. Solving $\psi(C) = 0$ via the bisection method yields γ_h, as listed in Table 1.



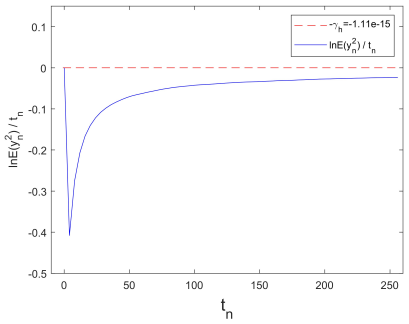
(a) case 1: $b = \frac{99}{100}, c = \frac{1}{100}, d = \frac{1}{10}, \rho = \frac{191}{100}$



(b) case 2: $b = \frac{23}{20}, c = \frac{1}{20}, d = \frac{1}{8}, \rho = \frac{106}{50}$



(c) case 3: $b = \frac{57}{50}, c = \frac{1}{25}, d = \frac{-1}{9}, \rho = \frac{213}{100}$



(d) case 4: $b = \frac{123}{100}, c = \frac{3}{100}, d = \frac{-1}{7}, \rho = \frac{23}{10}$.

Fig. 2 Almost sure exponential stability of the Runge-Kutta method.

For each case, 500 discrete Brownian paths are simulated on $[0, 250]$ with step size $h = 2^{-5}$. We compute the numerical solution y_n for Eq. (29) and approximate the expectation by the sample average over all paths.

In Fig. 2, the solid-blue curves depict the numerically approximated values of $\frac{\ln E\left[y_n^2\right]}{t_n}$ at discrete instants $t_n = nh$, while the red dotted lines denote the corresponding upper bounds $-\gamma_h$. One can see that all solid-blue curves remain below the red dotted lines, confirming that the numerical solutions are almost-surely exponentially stable. This result further supports our theoretical conclusions in the previous section.

CONCLUSION

Stochastic Runge-Kutta methods offer key advantages, including high accuracy and a derivative-free structure. However, constructing high-order schemes remains challenging. In this paper, a stochastic Runge-Kutta method of strong order 1 is proposed. Furthermore, for general nonlinear SDEs under appropriate conditions, it is proved that the numerical solution preserves both mean-square and almost sure exponential stability whenever the exact solution is stable in these senses. Numerical experiments validate the theoretical analysis.

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