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General Popoviciu-type functional equations and their stability

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ABSTRACT: Let n be a positive integer and let $(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n\backslash\{0\}$. For real linear spaces X and Y, we show that a function $f:X\to Y$ satisfies the Popoviciu-type functional equation

$$\sum_{k=1}^{n}\lambda_k\left(\sum_{1\leqslant i_1<\cdots<i_k\leqslant n} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right)\right)=0$$

if and only if f is a general polynomial function of degree at most $n-1$, where the actual highest possible degree of f can be determined in terms of $\lambda_1,\dots,\lambda_n$. The Hyers-Ulam stability of the Popoviciu-type functional equations is also investigated.

KEYWORDS: Popoviciu-type functional equation, general polynomial function, stability

MSC2020: 39B52, 39B82

INTRODUCTION

Popoviciu [1] proved that if $I\subset\mathbb{R}$ is a nonempty interval, then a continuous function $f:I\to\mathbb{R}$ is convex if and only if

$$3f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z)\geqslant 2\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right] \quad (1)$$

for all $x,y,z\in I$, or generally for given positive integers n and k, with $2\leqslant k\leqslant n-1$,

$$n\binom{n-2}{k-2}f\left(\frac{x_1+\cdots+x_n}{n}\right)+\binom{n-2}{k-1}\sum_{i=1}^{n}f(x_i)\geqslant k\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right) \quad (2)$$

for all $x_1,\dots,x_n\in I$. Several authors have studied functional equation in connection with these Popoviciu inequalities. Trif [2, 3] solved the functional equations obtained by replacing the greater or equal signs in (1) and (2) by the equal signs and proved their stability (cf. [4]). The modified Trif functional equation (2) was also studied by Park [5] and Lee and Shin [6]. A general functional equation of the form

$$bf\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z)=a\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right], \quad (3)$$

was studied by Smajdor [7]. A case of (3) when $a=4$ and $b=9$ was investigated by Lee [8] and Lee and Chung [9].

Let n be an integer and let $\Lambda_n=(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n\backslash\{0\}$. We will generalize the Popoviciu-type functional equation to the *Λ_n-Popoviciu-type functional equation*

$$\sum_{k=1}^{n}\lambda_k\left(\sum_{1\leqslant i_1<\cdots<i_k\leqslant n} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right)\right)=0. \quad (4)$$

The functional equation analogue of (1) and the functional equation (3) can be regarded as Λ_n-Popoviciu-type functional equations with $\Lambda_3=(1,-2,3)$ and $\Lambda_3=(1,-a,b)$, respectively. While the functional equation analogue of (2) is more complicated but can still be regarded as a Λ_n-Popoviciu-type functional equation with

$$\lambda_1=\binom{n-2}{k-1},\lambda_k=-k,\lambda_n=n\binom{n-2}{k-2}$$

and all other λ_i's are zero.

In this paper, we will show that, for real linear spaces X and Y, a function $f:X\to Y$ satisfies the general Λ_n-Popoviciu-type functional equation (4) if and only if it is a general polynomial function of degree at most $n-1$, where the actual highest possible degree of f can be determined in terms of Λ_n. Finally, we will also establish the Hyers-Ulam stability of (4) when Y is a Banach space.

NOTATIONS AND PRELIMINARY RESULTS

Throughout the paper, we will let X and Y be real linear spaces, and let B be a Banach space. In order to simplify the notion of (4), we will define the following notations.

Definition 1 For a function $f : X \to Y$, we denote the sum of the function values at the arithmetic means of $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ taken k elements at a time by $\mathscr{M}_f^k[\mathbf{x}]$, i.e.,

$$\mathscr{M}_f^k[\mathbf{x}] = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right).$$

With the notion of $\mathscr{M}_f^k[\mathbf{x}]$, we may formally define a Λ_n-Popoviciu-type functional equation (4) as follows.

Definition 2 For $\Lambda_n = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \backslash \{0\}$, the functional equation

$$\sum_{k=1}^{n} \lambda_k \mathscr{M}_f^k[\mathbf{x}] = 0 \qquad \text{for every } \mathbf{x} = (x_1, \ldots, x_n)$$

is said to be the Λ_n-*Popoviciu-type functional equation*.

Definition 3 For $\Lambda_n = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \backslash \{0\}$, if f satisfies the Λ_n-Popoviciu-type functional equation, then f is called a Λ_n-*Popoviciu function*.

Trif [2] showed that in the class of functions on real linear spaces the $(1, -2, 3)$-Popoviciu function is a Jensen function, $A(x) + B$, while Lee [8] showed that the $(1, -4, 9)$-Popoviciu function is $Q(x) + A(x) + B$, where Q is a quadratic function, A is an additive function and B is a constant. Smajdor [7] discovered that the $(1, -a, b)$-Popoviciu function is $Q(x) + A(x) + B$ when $b = 3(a-1)$. Trif [3] proved that all Λ_n-Popoviciu functions satisfying the functional equation analogue of (2) are Jensen functions.

To study Λ_n-Popoviciu functions and the stability of Λ_n-Popoviciu-type functional equation, we will make use of general polynomial functions, which will be briefly summarized as follows.

Definition 4 Let k be a positive integer. A function $A_k : X^k \to Y$ is k-additive when it is additive in each of its argument, i.e.,

$$\begin{aligned} &A_k(x_1, \ldots, x_{i-1}, x_i + y_i, x_{i+1}, \ldots, x_k) \\ &\quad = A_k(x_1, \ldots, x_k) + A_k(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_k) \end{aligned}$$

for every $x_1, \ldots, x_n, y_i \in X$ and for every $i = 1, \ldots, k$.

In addition, the *diagonalization* of A_k is the function $A^k : X \to Y$ defined by $A^k(x) = A_k(x, \ldots, x)$ for every $x \in X$.

The following two propositions illustrate basic properties drawn from the k-additivity of A_k.

Proposition 1 *If $A^k : X \to Y$ is the diagonalization of a k-additive function $A_k : X^k \to Y$, then $A^k(rx) = r^k A(x)$ for every $x \in X$ and for every rational number r.*

Proposition 2 *If $A_k : X^k \to Y$ is a k-additive function, then*

$$A^k(x_1 + \cdots + x_k) = \sum_{\mathbf{w} \in \{x_1, \ldots, x_k\}^k} A_k(\mathbf{w}).$$

We may say that general polynomial functions generalize ordinary polynomial functions, and we will define a general polynomial functions of a given degree as follows.

Definition 5 A function $f : X \to Y$ is called a *general polynomial function* of degree m if, for every $x \in X$,

$$f(x) = A^0 + A^1(x) + \cdots + A^m(x),$$

where A^0 is a constant and, for $k = 1, \ldots, m$, A^k is the diagonalization of a symmetric, k-additive function, where $A^m \neq 0$.

Remark 1 An ordinary polynomial function of degree m always is a general polynomial function of degree m, but the converse is not necessarily true.

The difference operators are proven useful in the study of general polynomial functions. We give some important definitions and results as follows.

Definition 6 For a function $f : X \to Y$. We denote, for every $x, h \in X$,

$$\Delta_h f(x) = f(x + h) - f(x),$$

and denote, for every $x, h_1, \ldots, h_m \in X$,

$$\begin{aligned} &\Delta_{h_1, \ldots, h_m} f(x) = \\ &\quad \sum_{\varepsilon_1, \ldots, \varepsilon_m = 0}^{1} (-1)^{m - (\varepsilon_1 + \cdots + \varepsilon_m)} f(x + \varepsilon_1 h_1 + \cdots + \varepsilon_m h_m). \end{aligned}$$

In particular, when $h_1 = \cdots = h_m = h$,

$$\Delta_h^m f(x) = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} f(x + ih).$$

Theorem 1 *Let m be a positive integer. A function $f : X \to Y$ satisfies*

$$\Delta_{h_1, \ldots, h_m} f(x) = 0$$

for all $x, h_1, \ldots, h_m \in X$ if and only if f is a general polynomial function of degree at most $m - 1$.

The following remarkable theorem is due to Djoković [10].

Theorem 2 *Let m be an integer. A function $f : X \to Y$ satisfies*

$$\Delta_{h_1,\ldots,h_m} f(x) = 0$$

for every $x, h_1, \ldots, h_m \in X$ if and only if

$$\Delta_h^m f(x) = 0$$

for every $x, h \in X$.

We finally give an important stability theorem of a functional equation characterizing polynomial functions proven by Hyers [11].

Theorem 3 *Let S be a convex cone in a rational vector space, and let B be a Banach space. Let $f : S \to B$ be a function satisfying the inequality*

$$\left\|\Delta_{y_1,\ldots,y_n} f(x)\right\| \leqslant \delta, \qquad x, y_1, \ldots, y_n \in S,$$

for some $\delta \geqslant 0$. Then there exists a polynomial γ of degree at most $n-1$ such that

$$\|f(x) - \gamma(x)\| \leqslant \delta, \qquad x \in S.$$

Moreover γ is given by the following formula when $\gamma_0(x) = f(0)$,

$$\gamma_{n-1}(x) = \lim_{s\to+\infty} \frac{1}{(n-1)!s^{n-1}} \Delta_{sx}^{n-1} f(0)$$

$$\gamma_k(x) = \lim_{s\to+\infty} \frac{1}{k!s^k} \left\{ \Delta_{sx}^k f(0) - \sum_{j=k+1}^{n-1} \Delta_{sx}^k \gamma_j(0) \right\},$$

for $1 \leqslant k \leqslant n-2$.

GENERAL POLYNOMIAL SOLUTION

This section provides some rudimental results towards the Λ_n-Popoviciu-type functional equation as well as its Hyers-Ulam stability.

Lemma 1 *Let $\varepsilon \geqslant 0$ and let $\beta, \beta_0, \ldots, \beta_m \in \mathbb{R}$ such that $\beta_m \neq 0$. If a function $f : X \to Y$ satisfies the inequality*

$$\left\| \beta f(x) + \sum_{k=0}^{m} \beta_k f\left(\frac{kx+y}{k+1}\right) \right\| \leqslant \varepsilon, \tag{5}$$

for every $x, y \in X$, then

$$\left\|\Delta_{h_1,\ldots,h_{m+1}} f(x)\right\| \leqslant \frac{2^{m+1}\varepsilon}{|\beta_m|}$$

for every $x \in X$.

Proof: Let $x, h_1, \ldots, h_{m+1} \in X$.
For $\mathbf{e} = (\varepsilon_1, \ldots, \varepsilon_{m+1}) \in \{0,1\}^{m+1}$, we let

$$X_{\mathbf{e}} = x + (m+1)\left(\sum_{i=1}^{m-1} \frac{\varepsilon_i h_i}{m-i} + \frac{\varepsilon_{m+1} h_{m+1}}{m} \right),$$

$$Y_{\mathbf{e}} = x - (m+1)\left(\sum_{i=1}^{m-1} \frac{i\varepsilon_i h_i}{m-i} - \varepsilon_m h_m \right).$$

Observe that $X_{\mathbf{e}}$ is independent of ε_m and $Y_{\mathbf{e}}$ is independent of ε_{m+1}. Moreover, for each integer k with $1 \leqslant k \leqslant m$,

$$kX_{\mathbf{e}} + Y_{\mathbf{e}} = (k+1)x + (m+1)\left(\sum_{i=1}^{m-1} \frac{(k-i)\varepsilon_i h_i}{m-i} + \varepsilon_m h_m + \frac{k\varepsilon_{m+1} h_{m+1}}{m} \right).$$

Therefore, $kX_{\mathbf{e}} + Y_{\mathbf{e}}$ is independent of ε_k when $1 \leqslant k < m$, and when $k = m$,

$$mX_{\mathbf{e}} + Y_{\mathbf{e}} = (m+1)\left(x + \sum_{i=1}^{m+1} \varepsilon_i h_i \right).$$

Since $X_{\mathbf{e}}$ is independent of ε_m, the sum of $(-1)^{\varepsilon_m}$ over $\varepsilon_m \in \{0,1\}$ will produce zero. Hence,

$$\sum_{\varepsilon_1,\ldots,\varepsilon_{m+1}=0}^{1} (-1)^{\varepsilon_1+\cdots+\varepsilon_{m+1}} f\left(X_{\mathbf{e}}\right) = 0. \tag{6}$$

Similarly, for each integer k with $0 \leqslant k < m$,

$$\sum_{\varepsilon_1,\ldots,\varepsilon_{m+1}=0}^{1} (-1)^{\varepsilon_1+\cdots+\varepsilon_{m+1}} f\left(\frac{kX_{\mathbf{e}} + Y_{\mathbf{e}}}{k+1}\right) = 0. \tag{7}$$

In addition, when $k = m$,

$$\sum_{\varepsilon_1,\ldots,\varepsilon_{m+1}=0}^{1} (-1)^{\varepsilon_1+\cdots+\varepsilon_{m+1}} f\left(\frac{mX_{\mathbf{e}} + Y_{\mathbf{e}}}{m+1}\right) = (-1)^{m+1} \Delta_{h_1,\ldots,h_{m+1}} f(x). \tag{8}$$

Since f satisfies (5), we will have

$$\left\| \sum_{\varepsilon_1,\ldots,\varepsilon_{m+1}=0}^{1} (-1)^{\varepsilon_1+\cdots+\varepsilon_{m+1}} \left(\beta f(X_{\mathbf{e}}) + \sum_{k=0}^{m} \beta_k f\left(\frac{kX_{\mathbf{e}} + Y_{\mathbf{e}}}{k+1}\right) \right) \right\| \leqslant 2^{m+1}\varepsilon.$$

Using (6) and swapping the order of the summation, the above inequality becomes

$$\left\| \sum_{k=0}^{m} \beta_k \sum_{\varepsilon_1,\ldots,\varepsilon_{m+1}=0}^{1} (-1)^{\varepsilon_1+\cdots+\varepsilon_{m+1}} f\left(\frac{kX_{\mathbf{e}} + Y_{\mathbf{e}}}{k+1}\right) \right\| \leqslant 2^{m+1}\varepsilon.$$

Taking (7) and (8) into account, only the term with $k = m$ in the summation survives, and the above inequality reduces to

$$\left\|\Delta_{h_1,\ldots,h_{m+1}} f(x)\right\| \leqslant \frac{2^{m+1}\varepsilon}{|\beta_m|}.$$

□

As an application of Lemma 1, we prove the following result on a Λ_n-Popoviciu function.

Lemma 2 *Let $\Lambda_n=(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n\backslash\{0\}$ and let m be the largest integer for which $\lambda_m\neq 0$. Let $\varepsilon\geqslant 0$. If a function $f:X\to Y$ satisfies the inequality*

$$\left\|\sum_{k=1}^{m}\lambda_k\mathscr{M}_f^k[\mathbf{x}]\right\|\leqslant\varepsilon, \tag{9}$$

for every $\mathbf{x}=(x_1,\dots,x_n)\in X^n$, then

$$\left\|\Delta_{h_1,\dots,h_m}f(x)\right\|\leqslant\frac{2^m\varepsilon}{\binom{n-1}{m-1}|\lambda_m|}.$$

Proof: We set $x_1=\ldots=x_{n-1}=x$ and $x_n=y$ in $\mathbf{x}=(x_1,\dots,x_n)$. For $k=1,\dots,m$, we have

$$\mathscr{M}_f^k[\mathbf{x}]=\binom{n-1}{k}f(x)+\binom{n-1}{k-1}f\left(\frac{(k-1)x+y}{k}\right).$$

Now (9) becomes

$$\left\|\beta f(x)+\sum_{k=1}^{m}\beta_k f\left(\frac{(k-1)x+y}{k}\right)\right\|\leqslant\varepsilon, \tag{10}$$

where

$$\beta=\sum_{k=1}^{m}\binom{n-1}{k}\lambda_k\ \text{ and }\ \beta_k=\binom{n-1}{k-1}\lambda_k,\ 1\leqslant k\leqslant m$$

with $\beta_m\neq 0$. The desired result simply follows from Lemma 1. □

Proposition 3 *Let $\Lambda_n=(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n\backslash\{0\}$ and let m be the largest integer for which $\lambda_m\neq 0$. If a function $f:X\to Y$ is a Λ_n-Popoviciu function, then f is a general polynomial function of degree at most $m-1$.*

Proof: Setting $\varepsilon=0$ in Lemma 2, we have

$$\Delta_{h_1,\dots,h_m}f(x)=0.$$

By Theorem 1, f is a general polynomial function of degree at most $m-1$. □

Remark 2 The actual highest degree of the general polynomial in Proposition 3 may be less than $m-1$, and the precise value can be determined using a classification in the next two sections.

CLASSIFICATION OF Λ_n

In this section, we will give a degree classification of $\Lambda_n\in\mathbb{R}^n\backslash\{0\}$, which will be of use in the study of Λ_n-Popoviciu functions in the next section. Note that when $\Lambda_n=0$, the Λ_n-Popoviciu function equation simply reduces to an identity; so, we may simply assume $\Lambda_n\neq 0$.

Definition 7 Let $\Lambda_n=(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n\backslash\{0\}$. Define

$$\sigma_l^k(\Lambda_n)=\sum_{i=l}^{n}\binom{n-l}{i-l}\frac{\lambda_i}{i^k}$$

for $k=1,\dots,n$ and $l=1,\dots,k$.

Definition 8 For each $k=1,\dots,n$ and for $\Lambda_n\in\mathbb{R}^n$. Define

$$\boldsymbol{\sigma}^k(\Lambda_n)=\left(\sigma_1^k(\Lambda_n),\dots,\sigma_k^k(\Lambda_n)\right).$$

In addition, if $\sigma_l^k(\Lambda_n)=0$ for every $l=1,\dots,k$, then we say that $\boldsymbol{\sigma}^k(\Lambda_n)=0$; otherwise, we say that $\boldsymbol{\sigma}^k(\Lambda_n)\neq 0$.

Remark 3 Let $\Lambda_n=(\lambda_1,\lambda_2,\dots,\lambda_n)$. Note that the identity $n\binom{n-1}{k-1}=k\binom{n}{k}$ gives

$$\sum_{k=1}^{n}\binom{n}{k}\lambda_k=n\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{\lambda_k}{k}=n\sigma_1^1(\Lambda_n).$$

The following lemma gives some relationship between $\sigma_l^k(\Lambda_n)$ for some different k and l.

Lemma 3 *Let $\Lambda_n\in\mathbb{R}^n\backslash\{0\}$. For integers k and l with $1\leqslant l<k\leqslant n$,*

$$\sigma_l^{k-1}(\Lambda_n)=l\sigma_l^k(\Lambda_n)+(n-l)\sigma_{l+1}^k(\Lambda_n).$$

Proof: Let $\Lambda_n=(\lambda_1,\dots,\lambda_n)$.

$$\begin{aligned}
&l\sigma_l^k(\Lambda_n)+(n-l)\sigma_{l+1}^k(\Lambda_n)\\
&\quad=l\sum_{i=l}^{n}\tbinom{n-l}{i-l}\tfrac{\lambda_i}{i^k}+(n-l)\sum_{i=l+1}^{n}\tbinom{n-l-1}{i-l-1}\tfrac{\lambda_i}{i^k}\\
&\quad=\tfrac{\lambda_l}{l^{k-1}}+\sum_{i=l+1}^{n}\left(l\tbinom{n-l}{i-l}+(n-l)\tbinom{n-l-1}{i-l-1}\right)\tfrac{\lambda_i}{i^k}\\
&\quad=\tfrac{\lambda_l}{l^{k-1}}+\sum_{i=l+1}^{n}\left(l\tbinom{n-l}{i-l}+(i-l)\tbinom{n-l}{i-l}\right)\tfrac{\lambda_i}{i^k}\\
&\quad=\sum_{i=l}^{n}\tbinom{n-l}{i-l}\tfrac{\lambda_i}{i^{k-1}}=\sigma_l^{k-1}(\Lambda_n).
\end{aligned}$$

□

The following proposition proves that $\boldsymbol{\sigma}^k(\Lambda_n)$ cannot all vanish.

Proposition 4 *For $\Lambda_n=(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n\backslash\{0\}$, if $\lambda_m\neq 0, m\in\{1,\dots,n\}$, then $\boldsymbol{\sigma}^m(\Lambda_n)\neq 0$.*

Proof: Observe that

$$\sigma_m^m(\Lambda_n)=\sum_{i=m}^{m}\binom{n-m}{i-m}\frac{\lambda_i}{i^m}=\frac{\lambda_m}{m^m}\neq 0.$$

Therefore, $\boldsymbol{\sigma}^m(\Lambda_n)\neq 0$. □

Proposition 4 permits us to classify Λ_n by a degree $d(\Lambda_n)$ which is basically a measure of the smallest degree at which $\boldsymbol{\sigma}^k(\Lambda_n)$ becomes nontrivial, thereby determining the maximum polynomial degree that can survive the Λ_n-Popoviciu functional equation.

Definition 9 The *degree* of $\Lambda_n\in\mathbb{R}^n\backslash\{0\}$ is defined by

$$d(\Lambda_n)=\min\left\{k\in\{0,\dots,n-1\}:\boldsymbol{\sigma}^{k+1}(\Lambda_n)\neq 0\right\}.$$

Remark 4 We can see that $0 \leqslant d(\Lambda_n) < m$, where m is the largest integer for which $\lambda_m \neq 0$

The following proposition gives simpler alternatives for the evaluation of the degree of Λ_n.

Proposition 5 *Let* $\Lambda_n = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \backslash \{0\}$.
(i) $\sigma_l^{d(\Lambda_n)+1}(\Lambda_n) \neq 0$ *for every* $l = 1, \ldots, d(\Lambda_n)+1$.
(ii) $d(\Lambda_n) = \min\left\{k \in \{0, \ldots, n-1\} : \sigma_1^{k+1}(\Lambda_n) \neq 0\right\}$,
(iii) $d(\Lambda_n) = \min\left\{k \in \{0, \ldots, n-1\} : \sigma_{k+1}^{k+1}(\Lambda_n) \neq 0\right\}$.

Proof: Let $p = d(\Lambda_n)$. By the definition of $d(\Lambda_n)$,

$$\boldsymbol{\sigma}^k(\Lambda_n) = 0 \quad \text{for every } k = 1, \ldots, p, \tag{11}$$

but $\boldsymbol{\sigma}^{p+1}(\Lambda_n) \neq 0$, i.e., there exists $l_0 \in \{1, \ldots, p+1\}$ such that

$$\sigma_{l_0}^{p+1}(\Lambda_n) \neq 0. \tag{12}$$

For any $l = 1, \ldots, p$, Lemma 3 gives

$$\sigma_l^p(\Lambda_n) = l\sigma_l^{p+1}(\Lambda_n) + (n-l)\sigma_{l+1}^{p+1}(\Lambda_n). \tag{13}$$

From (11), we know that $\boldsymbol{\sigma}^p(\Lambda_n) = 0$; so, $\sigma_l^p(\Lambda_n) = 0$. Also note that $1 \leqslant l \leqslant p = d(\Lambda_n) < n$. Therefore, we conclude from (13) that $\sigma_l^{p+1}(\Lambda_n)$ and $\sigma_{l+1}^{p+1}(\Lambda_n)$ are both zero, or both non-zero. Applying this fact to (12), we infer that

$$\sigma_l^{p+1}(\Lambda_n) \neq 0 \quad \text{for every } l = 1, \ldots, p+1. \tag{14}$$

From (11) and (14), we are allowed to evaluate $d(\Lambda_n)$ using any component of $\boldsymbol{\sigma}^k(\Lambda_n)$. □

Probably the most convenient way to evaluate $d(\Lambda_n)$ is to employ statement (iii) in Proposition 5. We will begin with two numerical examples on the evaluation of the degree of Λ_3 based on the $(1,-2,3)$- and $(1,-4,9)$-Popoviciu-type functional equations from literature.

Example 1 Let $\Lambda_3 = (1,-2,3)$

$$\sigma_1^1(\Lambda_3) = \tbinom{2}{0}\tfrac{1}{1^1} + \tbinom{2}{1}\tfrac{(-2)}{2^1} + \tbinom{2}{2}\tfrac{3}{3^1} = 0,$$
$$\sigma_2^2(\Lambda_3) = \tbinom{1}{0}\tfrac{1}{1^2} + \tbinom{1}{1}\tfrac{(-2)}{2^2} = \tfrac{1}{2}.$$

Therefore, $d(\Lambda_3) = 1$.

Example 2 Let $\Lambda_3 = (1,-4,9)$.

$$\sigma_1^1(\Lambda_3) = \tbinom{2}{0}\tfrac{1}{1^1} + \tbinom{2}{1}\tfrac{(-4)}{2^1} + \tbinom{2}{2}\tfrac{9}{3^1} = 0,$$
$$\sigma_2^2(\Lambda_3) = \tbinom{1}{0}\tfrac{1}{1^2} + \tbinom{1}{1}\tfrac{(-4)}{2^2} = 0,$$
$$\sigma_3^3(\Lambda_3) = \tbinom{0}{0}\tfrac{1}{1^3} = 1.$$

Therefore, $d(\Lambda_3) = 2$.

The following example clarifies the degree of $\Lambda_3 = (1,-a,b)$ taken from the functional equation considered by Smajdor [7].

Example 3 Let $\Lambda_3 = (1,-a,b)$.

$$\sigma_1^1(\Lambda_3) = \tbinom{2}{0}\tfrac{1}{1^1} + \tbinom{2}{1}\tfrac{(-a)}{2^1} + \tbinom{2}{2}\tfrac{b}{3^1} = 1 - a + \tfrac{b}{3},$$
$$\sigma_2^2(\Lambda_3) = \tbinom{1}{0}\tfrac{1}{1^2} + \tbinom{1}{1}\tfrac{(-a)}{2^2} = 1 - \tfrac{a}{4},$$
$$\sigma_3^3(\Lambda_3) = \tbinom{0}{0}\tfrac{1}{1^3} = 1.$$

If $b \neq 3(a-1)$, then $d(\Lambda_3) = 0$. If $b = 3(a-1) \neq 9$, then $d(\Lambda_3) = 1$. Therefore, $d(\Lambda_3) = 2$ if and only if $a = 4$ and $b = 9$.

The next more complicated example was taken from the functional equation analogue of (2) considered by Trif [3], and we will calculate $d(\Lambda_n)$ by the second alternative in Proposition 5.

Example 4 Let k be an integer with $2 \leqslant k < n$. For $\Lambda_n = (\lambda_1, \ldots, \lambda_n)$ with all λ_i's zero except

$$\lambda_1 = \tbinom{n-2}{k-1},\ \lambda_k = -k,\ \lambda_n = n\tbinom{n-2}{k-2}.$$

$$\sigma_1^1(\Lambda_n) = \sum_{i=1}^{n} \tbinom{n-1}{i-1}\tfrac{\lambda_i}{i} = \tbinom{n-2}{k-1} - \tbinom{n-1}{k-1} + \tbinom{n-2}{k-2} = 0,$$
$$\sigma_2^2(\Lambda_n) = \sum_{i=2}^{n} \tbinom{n-2}{i-2}\tfrac{\lambda_i}{i^2} = -\tbinom{n-2}{k-2}\tfrac{1}{k} + \tbinom{n-2}{k-2}\tfrac{1}{n} \neq 0.$$

Therefore, $d(\Lambda_n) = 1$.

The following example illustrates a typical Λ_n for an arbitrary positive integer n.

Example 5 Let $\Lambda_n = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_i = (-1)^i i^m$ $(i = 1, \ldots, n)$ for an integer m.

$$\sigma_k^k(\Lambda_n) = \sum_{i=k}^{n} \tbinom{n-k}{i-k}\tfrac{(-1)^i i^m}{i^k}.$$

Using the $(n-k)$-order differences of a polynomial of degree $m-k$, $\sigma_k^k(\Lambda_n) = 0$ if and only if $0 \leqslant m < n$ and $k < n$. Therefore, $d(\Lambda_n) = m$ if $1 \leqslant m < n$ and $d(\Lambda_n) = 0$ otherwise.

The following lemma and the following proposition will provide some facts on $\sigma_l^k(\Lambda_n)$ which will be fruitful in the study of Λ_n-Popoviciu functions in the next section.

Lemma 4 *Let* $\Lambda_n \in \mathbb{R}^n \backslash \{0\}$. *If* $\boldsymbol{\sigma}^{k_0}(\Lambda_n) = 0$ *for some integer* k_0, *then* $\boldsymbol{\sigma}^k(\Lambda_n) = 0$ *for every positive integer* $k \leqslant k_0$.

Proof: It suffices to prove that $\boldsymbol{\sigma}^k(\Lambda_n) = 0$ implies $\boldsymbol{\sigma}^{k-1}(\Lambda_n) = 0$ for $k = 2, \ldots, n$, and what we need simply follows from Lemma 3. □

Remark 5 $\boldsymbol{\sigma}^k(\Lambda_n) = 0$ when $k \leqslant d(\Lambda_n)$ and $\boldsymbol{\sigma}^k(\Lambda_n) \neq 0$ when $k > d(\Lambda_n)$.

Proposition 6 *Let $\Lambda_n \in \mathbb{R}^n\backslash\{0\}$. $\boldsymbol{\sigma}^k(\Lambda_n) \neq 0$ when $k > d(\Lambda_n)$; i.e, $\sigma_l^k(\Lambda_n) \neq 0$ for some $l \in \{1,\dots,k\}$.*

Proof: Suppose there exists $k_0 > d(\Lambda_n)$ such that $\boldsymbol{\sigma}^{k_0}(\Lambda_n) = 0$, then by Lemma 4, $\boldsymbol{\sigma}^k(\Lambda_n) = 0$ for every $k = 1,\dots,k_0$. Since $1 \leqslant d(\Lambda_n)+1 \leqslant k_0$, we have $\boldsymbol{\sigma}^k(\Lambda_n) = 0$ for $k = d(\Lambda_n)+1$, which contradicts the definition of $d(\Lambda_n)$. □

Λ_n-POPOVICIU FUNCTIONS

We are now ready to determine the general form of Λ_n-Popoviciu functions based on the classification of Λ_n discussed in the previous section. We first give one more definition and prove one more auxiliary lemma regarding an expansion of multi-additive functions. Please recall that the diagonalization $A^p : X \to Y$ of a p-additive function $A_p : X^n \to Y$ is defined by

$$A^p(x) = A_p(x,\dots,x).$$

Definition 10 Let $\mathbf{x} = \{x_1,\dots,x_n\}$ be a set of n different objects. For each p-permutation $\mathbf{w} = (w_1,\dots,w_p) \in \mathbf{x}^p$, we define

$$|\mathbf{w}| = |\{w_1,\dots,w_p\}|,$$

i.e., $|\mathbf{w}|$ is the number of *distinct* elements in the set $\{w_1,\dots,w_p\}$.

Example 6 Let $\mathbf{x} = \{x_1,x_2,x_3,x_4\}$.
(i) $|(x_1,x_1,x_1)| = 1$.
(ii) $|(x_1,x_2,x_2,x_1)| = 2$.
(iii) $|(x_3,x_1,x_2,x_4)| = 4$.

Lemma 5 *Let p be a positive integer. Let $A_p : X^p \to Y$ be a p-additive function and $A^p : X \to Y$ be its diagonalization. For $\Lambda_n = (\lambda_1,\dots,\lambda_n)$ and $\mathbf{x} \in X^p$,*

$$\sum_{k=1}^{n} \lambda_k \mathscr{M}_{A^p}^k[\mathbf{x}] = \sum_{\mathbf{w}\in\mathbf{x}^p} \sigma_{|\mathbf{w}|}^p(\Lambda_n) A_p(\mathbf{w}).$$

Proof: Using Proposition 1, we have

$$\sum_{k=1}^{n} \lambda_k \mathscr{M}_{A^p}^k[\mathbf{x}] = \sum_{k=1}^{n} \frac{\lambda_k}{k^p} \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} A^p(x_{i_1}+\cdots+x_{i_k}).$$

By Proposition 2,

$$A^p(x_{i_1}+\cdots+x_{i_k}) = \sum_{\mathbf{w}\in\{x_{i_1},\dots,x_{i_k}\}^p} A_p(\mathbf{w}),$$

once it is summed over $1 \leqslant i_1 < \cdots < i_k \leqslant n$, we will get $A_p(\mathbf{w})$ up to some multiplicity for all $\mathbf{w} \in \mathbf{x}^p$. For w with $|\mathbf{w}|$ distinct elements, the multiplicity of $A_p(w)$ is the number of ways in choosing $k-|\mathbf{w}|$ more distinct elements to form k-subsets of $\mathbf{x}$. Therefore,

$$\begin{aligned}\sum_{k=1}^{n} \lambda_k \mathscr{M}_{A^p}^k[\mathbf{x}] &= \sum_{k=1}^{n} \frac{\lambda_k}{k^p} \sum_{\mathbf{w}\in\mathbf{x}^p} \binom{n-|\mathbf{w}|}{k-|\mathbf{w}|} A_p(\mathbf{w}) \\ &= \sum_{\mathbf{w}\in\mathbf{x}^p} \left(\sum_{k=1}^{n} \binom{n-|\mathbf{w}|}{k-|\mathbf{w}|} \frac{\lambda_k}{k^p}\right) A_p(\mathbf{w}) \\ &= \sum_{\mathbf{w}\in\mathbf{x}^p} \sigma_{|\mathbf{w}|}^p(\Lambda_n) A_p(\mathbf{w}).\end{aligned}$$

The following main theorem proves that a Λ_n-Popoviciu function is basically a general polynomial function of degree at most $d(\Lambda_n)$.

Theorem 4 *A function $f : X \to Y$ is a Λ_n-Popoviciu function if and only if*
(i) *f is a general polynomial function of degree at most $d(\Lambda_n)$ if $d(\Lambda_n) > 0$,*
(ii) *f is the zero function if $d(\Lambda_n) = 0$*

Proof: Let $\Lambda_n = (\lambda_1,\lambda_2,\dots,\lambda_n)$.

(ii) We will first treat the case when $d(\Lambda_n) = 0$, i.e., $\sigma_1^1(\Lambda_n) \neq 0$. Letting $x_1 = \cdots = x_n = x$ in $\mathbf{x} = (x_1,\dots,x_n)$ and using Remark 3, the Λ_n-Popoviciu-type functional equation reduces to

$$n\sigma_1^1(\Lambda_n) f(x) = 0.$$

Therefore, the Λ_n-Popoviciu function is the zero function when $d(\Lambda_n) = 0$.

(i) Now we may assume that $d(\Lambda_n) > 0$.

($\Rightarrow$) If a function $f : X \to Y$ is a Λ_n-Popoviciu function, then, by Proposition 3, f is a general polynomial function of degree at most $n-1$. It should be remarked that Proposition 3 actually proves that the degree of f is at most $m-1$, where m is the largest integer for which $\lambda_m \neq 0$, but leaving the degree to be at most $n-1$ produces no effect to the proof and the statement of the theorem. Therefore, we can express f as

$$f(x) = A^0 + A^1(x) + \cdots + A^{n-1}(x).$$

As a result, the Λ_n-Popoviciu-type functional equation becomes

$$\sum_{k=1}^{n} \lambda_k \left(\sum_{p=0}^{n-1} \mathscr{M}_{A^p}^k[\mathbf{x}]\right) = 0.$$

Replacing $\mathbf{x}$ with $r\mathbf{x} = (rx_1,\dots,rx_n)$, where r is an arbitrary rational numbers, and noting that $A^p(rx) = r^p A^p(x)$, the above equation becomes

$$\sum_{p=0}^{n-1} r^p \left(\sum_{k=1}^{n} \lambda_k \mathscr{M}_{A^p}^k[\mathbf{x}]\right) = 0.$$

Since the above equation holds for any rational number r, we conclude that

$$\sum_{k=1}^{n} \lambda_k \mathscr{M}_{A^p}^k[\mathbf{x}] = 0$$

for every $p = 0,\dots,n-1$. From Lemma 5,

$$\sum_{\mathbf{w}\in\mathbf{x}^p} \sigma_{|\mathbf{w}|}^p(\Lambda_n) A_p(\mathbf{w}) = 0$$

for every $p = 0,\dots,n-1$.

Consider a fixed $p > d(\Lambda_n)$. Replacing $\mathbf{w} = (w_1, \ldots, w_p)$ with $(r_1 w_1, \ldots, r_p w_p)$, where $r_1, \ldots, r_p$ are arbitrary rational numbers, $\sum_{\mathbf{w}\in\mathbf{x}^p} \sigma^p_{|\mathbf{w}|}(\Lambda_n) A_p(\mathbf{w})$ becomes a polynomial in the variables $r_1, \ldots, r_p$. From Proposition 6, $\sigma^p(\Lambda_n) \neq 0$; that is $\sigma^p_{|\mathbf{w}|}(\Lambda_n) \neq 0$ for some $\mathbf{w} \in \mathbf{x}^p$. By virtue of the identity theorem, we conclude that $A_p(\mathbf{w}) = 0$ for some $\mathbf{w} \in \mathbf{x}^p$, which consequently yields $A^p(x) = 0$ by setting $\mathbf{w} = (x, \ldots, x)$. Therefore, f is a general polynomial function of degree at most $d(\Lambda_n)$.

($\Leftarrow$) Suppose f is a general polynomial function of degree at most $p = d(\Lambda_n) > 0$, i.e.

$$f(x) = A^0 + A^1(x) + \cdots + A^p(x).$$

Using Lemma 5,

$$\sum_{k=1}^{n} \lambda_k \mathscr{M}_f^k[\mathbf{x}] = \sum_{k=1}^{n} \binom{n}{k} \lambda_k A^0 + \sum_{q=1}^{p} \sum_{\mathbf{w}\in\mathbf{x}^q} \sigma^q_{|\mathbf{w}|}(\Lambda_n) A_q(\mathbf{w}).$$

By the definition of $d(\Lambda_n)$, we know that $\boldsymbol{\sigma}^q(\Lambda_n) = 0$ for every $q = 1, \ldots, p$. Moreover, Remark 3 tells us that $\sum_{k=1}^{n} \binom{n}{k} \lambda_k = n\sigma^1_1(\Lambda_n) = 0$. Therefore, f is indeed a Λ_n-Popoviciu function. □

Example 7 From examples after Proposition 5, we have the following conclusions.

(i) For $\Lambda_3 = (1, -2, 3)$, $\Lambda_3 = (1, -a, b)$ when $b = 3(a-1) \neq 9$, and $\Lambda_n = (\lambda_1, \ldots, \lambda_n)$ with

$$\lambda_1 = \binom{n-2}{k-1},\ \lambda_k = -k,\ \lambda_n = n\binom{n-2}{k-2}$$

and all other λ_i's are zero, where $2 \leqslant k < n$, the Λ_n-Popoviciu functions are general polynomial functions of degree at most 1.

(ii) For $\Lambda_3 = (1, -4, 9)$, the Λ_n-Popoviciu functions are general polynomial functions of degree at most 2.

(iii) For $\Lambda_n = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_i = (-1)^i i^m$ $(i = 1, \ldots, n)$ for an integer m, the Λ_n-Popoviciu functions are general polynomial functions of degree at most m if $1 \leqslant m < n$ and the zero function otherwise.

STABILITY OF Λ_n-POPOVICIU-TYPE FUNCTIONAL EQUATION

We will give a conclusion on the Hyers-Ulam stability of the Λ_n-Popoviciu-type functional equation in the following theorem.

Theorem 5 *Let $\Lambda_n = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n \backslash \{0\}$ and let m be the largest integer for which $\lambda_m \neq 0$. Let X be a real linear space and let Y be a real Banach space. If a function $f : X \to Y$ satisfies*

$$\left\| \sum_{k=1}^{m} \lambda_k \mathscr{M}_f^k[\mathbf{x}] \right\| \leqslant \varepsilon \tag{15}$$

for every $\mathbf{x} \in X^n$, then there exists a Λ_n-Popoviciu function $P : X \to Y$ such that

$$\|f(x) - P(x)\| \leqslant \frac{\varepsilon}{K} \tag{16}$$

for every $x \in X$, where $K = \frac{1}{2^{m+1}}\binom{n-1}{m-1}|\lambda_m|$ if $d(\Lambda_n) > 0$ and $K = \left|\sum_{k=1}^{m} \binom{n}{k}\lambda_k\right|$ if $d(\Lambda_n) = 0$.

Proof: First consider the case when $d(\Lambda_n) = 0$. Settings $\mathbf{x} = (x, \ldots, x)$ in (15) gives

$$\left\| \sum_{k=1}^{m} \binom{n}{k} \lambda_k f(x) \right\| \leqslant \varepsilon \quad \text{for every } x \in X.$$

By Theorem 4, the Λ_n-Popoviciu function $d(\Lambda_n) = 0$ is the zero function, and (16) simply follows.

Now we assume that $d(\Lambda_n) > 0$. From Lemma 2, we have that

$$\left\| \Delta_{h_1,\ldots,h_m} f(x) \right\| \leqslant \frac{\varepsilon}{K}$$

for every $x, h_1, \ldots, h_m \in X$. Since X can also be viewed as a rational space and also serves as a convex cone under addition and nonnegative rational scaling, Theorem 3 tells us that there exists a general polynomial function $P : X \to Y$ of degree at most $m-1$ such that

$$\|f(x) - P(x)\| \leqslant \frac{\varepsilon}{K} \tag{17}$$

and P is unique up to a constant. So, we let

$$P(x) = A^0 + A^1(x) + \cdots + A^{m-1}(x).$$

Let r be an arbitrary rational number.

$$\sum_{k=1}^{m} \lambda_k \mathscr{M}_P^k[r\mathbf{x}] = \sum_{k=1}^{m} \lambda_k \binom{n}{k} A^0 + \sum_{p=1}^{m-1} r^p \sum_{k=1}^{m} \lambda_k \mathscr{M}_{A^p}^k[\mathbf{x}] \tag{18}$$

On the other hand,

$$\left\| \sum_{k=1}^{m} \lambda_k \mathscr{M}_P^k[r\mathbf{x}] \right\| \leqslant \left\| \sum_{k=1}^{m} \lambda_k \mathscr{M}_f^k[r\mathbf{x}] \right\| + \left\| \sum_{k=1}^{m} \lambda_k \mathscr{M}_{f-p}^k[r\mathbf{x}] \right\| \leqslant \varepsilon + \sum_{k=1}^{m} |\lambda_k| \binom{n}{k} \frac{\varepsilon}{K}.$$

The left side of (18) is bounded, therefore $\sum_{k=1}^{m} \lambda_k \mathscr{M}_{A^p}^k[\mathbf{x}] = 0$ when $p \geqslant 1$. We are left with

$$\sum_{k=1}^{m} \lambda_k \mathscr{M}_P^k[r\mathbf{x}] = \sum_{k=1}^{m} \lambda_k \binom{n}{k} A^0 = n\sigma^1_1(\Lambda_n) A^0 = 0,$$

where Remark 3 and the fact that $\sigma^1_1(\Lambda_n) = 0$ when $d(\Lambda_n) \neq 0$ have been used. Hence, P is a Λ_n-Popoviciu function. □

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