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Maturity effects and payoff risk of futures-based moment swaps under a time-dependent mean-reverting model

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ABSTRACT: Moment swaps written on commodity futures provide direct exposure to realized powers of futures returns. This paper studies these contracts under a trending Ornstein–Uhlenbeck commodity model with a time-dependent mean-reversion level. The spot price follows the original trending Ornstein–Uhlenbeck specification, and the price of a fixed-maturity futures contract is derived from the associated risk-neutral conditional expectation. The moment-swap maturity and the futures delivery date are treated separately. The resulting futures dynamics have deterministic time-dependent volatility, and log-futures returns over nonoverlapping observation intervals are independent Gaussian random variables with explicit means and variances. This result shows that the return law is independent of the deterministic trend and the initial spot level, while it depends on the mean-reversion speed, volatility, observation dates, and futures delivery date. Closed-form fair delivery prices are obtained for futures-based moment swaps of arbitrary integer order, with explicit formulas for the second-, third-, and fourth-order contracts. Exact risk-neutral payoff variances and cross-order covariances are also derived, and the effects of the futures delivery date and sampling frequency are examined. An explicit two-delivery-date identification result is obtained in the continuous-sampling limit, and a nonlinear least-squares procedure based on the exact finite-grid prices is proposed for several delivery dates. Numerical illustrations and Monte Carlo verification based on the exact return law support the analytical formulas and describe the main maturity and sampling effects.

KEYWORDS: trending Ornstein–Uhlenbeck process, commodity futures, moment swaps, payoff covariance, maturity effects

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INTRODUCTION

Commodity markets support the trading and risk management of energy products, metals, and agricultural goods through spot, futures, and other derivative contracts. Commodity prices are affected by storage costs, inventories, seasonality, and convenience yields, and these features distinguish them from standard financial assets. Stochastic models for commodity prices therefore often include mean reversion and other market-specific effects to support the valuation of futures and options [1, 2]. The models of Gibson and Schwartz [3] and Schwartz [1] established important approaches for commodity derivative pricing and showed the role of mean-reverting dynamics in commodity markets. Empirical studies have also provided evidence that many commodity prices move around long-run levels rather than follow unrestricted permanent trends [4, 5].

The Ornstein–Uhlenbeck process is widely adopted to describe mean reversion because it has a Gaussian transition law and allows many quantities to be evaluated in closed form. Its classical form has a constant long-run mean, while commodity markets may exhibit deterministic trends caused by seasonal patterns, changes in production, and long-term economic conditions. Trending Ornstein–Uhlenbeck models address this limitation by allowing the mean-reversion level to vary with time [6–8]. Mongkolsin et al [9] extended this direction to time-fractional Cauchy problems associated with Ornstein–Uhlenbeck, Cox–Ingersoll–Ross, and Jacobi processes with time-dependent parameters and obtained analytical conditional moments. Closed-form and recursive conditional-moment formulas have also been developed for several time-inhomogeneous, nonlinear, and regime-switching diffusion models [10–12]. Related moment calculations based on transition densities have been obtained for the extended Cox–Ingersoll–Ross process with time-varying dimension [13]. For the time-inhomogeneous 3/2 model, Meesa et al [14] derived an infinite-series transition density and explicit conditional moments by relating the model to processes associated with squared Bessel processes. In a fractional setting, Sutthimat and Ketrat [15] obtained explicit conditional moments for time-changed Ornstein–Uhlenbeck, Cox–Ingersoll–Ross, and Jacobi diffusions within fractional Cauchy

problems. Conditional-expectation methods have also been developed for derivative valuation under more general stochastic models. In particular, Kharrat and Alruwaili [16] introduced a localization-function method for computing conditional expectations associated with American-option pricing when the asset price is generated by a multidimensional J-process. For the trending Ornstein–Uhlenbeck process, Thamrongrat et al [6] derived conditional moments through recursive ordinary differential equations. These results provide effective analytical tools for derivative valuation and for the study of time-dependent mean-reverting processes.

Moment swaps extend variance swaps by taking realized powers of returns as their underlying quantities. They allow market participants to obtain exposure to second- and higher-order return risks without trading a large collection of options. Schoutens [17] introduced moment swaps as extensions of variance swaps, and Rompolis and Tzavalis [18] studied their role in pricing and hedging contingent claims. In commodity markets, Weraprasertsakun and Rujivan [19] obtained a closed-form variance-swap price under a one-factor commodity model. Chumpong et al [20, 21] developed analytical formulas for discretely sampled higher-order and general moment swaps, while Duangpan et al [22] studied generalized swaps under stochastic convenience yields. More recently, Bunchak et al [7] considered contracts based on exact one-step innovations of a trending Ornstein–Uhlenbeck process, and Ruengsinsub et al [8] derived simplified formulas for moment swaps in commodity markets. These studies show that the Gaussian structure of mean-reverting models can lead to tractable prices for several realized-moment contracts. Related work has also developed Gaussian frameworks for discretely monitored volatility derivatives. Lim et al [23] considered variance- and volatility-linked derivatives under generalized mixed fractional Brownian dynamics, where realized variance is represented as a quadratic form in correlated Gaussian returns.

Earlier futures-based moment-swap results are especially relevant to the present study. Chumpong et al [21] considered moment swaps with a futures price as the underlying under an extended Black–Scholes model with deterministic time-dependent parameters, where the futures expiration coincides with the swap horizon. Doffou [24] also studied higher-order moment swaps driven by futures prices under different pricing models and with a different empirical objective, without introducing a separate futures delivery date beyond the contract horizon. In the deterministic-volatility setting of Chumpong et al [21], the fair delivery price is obtained from moments of Gaussian futures log returns. The same normal-moment calculation appears in the present paper after the futures dynamics are derived from the commodity model. Here, however, the volatility function $\sigma\, e^{-\kappa(\mathcal{T}-t)}$ follows from the mean-reverting structure, and the futures delivery date $\mathcal{T}$ is allowed to differ from the swap maturity T. This distinction is essential for the analysis of the futures delivery date and the identification of κ and σ from prices observed at different delivery dates. The payoff-covariance formulas and the sampling asymptotics are additional results for the resulting futures-return law. The innovation-based contracts of Bunchak et al [7] and the log-price formulation of Ruengsinsub et al [8] do not introduce this separate fixed futures delivery date.

The present paper considers a different question. The underlying variables are the raw log returns of fixed-maturity commodity futures contracts, rather than one-step forecast errors of the spot process. A futures price must therefore be derived from the risk-neutral conditional expectation implied by the spot-price model. This distinction introduces a separate delivery date for the futures contract and makes it possible to study how moment-swap prices change across the futures maturity structure. Existing conditional-moment formulas alone do not describe these maturity effects, the sampling-frequency behavior of raw futures moments, or the dependence between payoffs of different moment orders. These questions are important because contracts with the same swap maturity and different futures delivery dates need not have the same risk exposure.

This paper keeps the original trending Ornstein–Uhlenbeck spot-price model and derives the corresponding fixed-maturity futures price in closed form. The resulting futures dynamics have deterministic time-dependent volatility, and log-futures returns over nonoverlapping observation intervals are independent Gaussian random variables. Their distributions do not depend on the deterministic trend or the initial spot level, but they depend on the mean-reversion speed, volatility, observation dates, and futures delivery date. This return law leads to a closed-form fair delivery price for a futures-based moment swap of any integer order. Explicit formulas are then obtained for the second-, third-, and fourth-order contracts. For an equally spaced observation grid, the time sums are reduced to geometric power sums, which display the dependence on the futures delivery date directly. The analysis is further extended to exact risk-neutral payoff variances and cross-order covariances, together with the behavior of the fair delivery prices as the sampling frequency increases. These results also support the identification of the mean-reversion and volatility parameters from moment-swap prices observed at several futures delivery dates.

TIME-DEPENDENT MEAN-REVERTING MODEL

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{Q})$ be a filtered probability space satisfying the usual conditions, where $\mathbb{Q}$ denotes the risk-neutral measure and W_t is a standard Brownian motion. Let $\mu \in C^1([0,\infty))$ be a deterministic func-

tion. The commodity spot price S_t is assumed to satisfy the time-dependent mean-reverting stochastic differential equation

$$dS_t = \kappa\left(\mu(t) + \frac{\mu'(t)}{\kappa} - \ln S_t\right)S_t\,dt + \sigma S_t\,dW_t, \quad (1)$$

where $S_0 > 0$, $\kappa > 0$ is the mean-reversion speed, and $\sigma > 0$ is the volatility parameter. The term $\mu'(t)/\kappa$ adjusts the drift for changes in the deterministic mean level and preserves the trending Ornstein–Uhlenbeck structure of the log-price process.

Let $Y_t = \ln S_t$. Applying Itô's formula to (1) gives

$$\begin{aligned} dY_t &= \big(\kappa(\alpha(t) - Y_t) + \alpha'(t)\big)\,dt + \sigma\,dW_t, \\ \alpha(t) &= \mu(t) - \frac{\sigma^2}{2\kappa}. \end{aligned} \quad (2)$$

Thus, Y_t follows a trending Ornstein–Uhlenbeck process with the deterministic mean-reversion level $\alpha(t)$. The time dependence can be separated from the random component by defining

$$Z_t = Y_t - \alpha(t).$$

It follows directly from (2) that

$$dZ_t = -\kappa Z_t\,dt + \sigma\,dW_t. \quad (3)$$

Therefore, the deterministic trend changes the level of the log-price process but does not change the Ornstein–Uhlenbeck dynamics of its centered component. The following lemma records the exact transition law applied throughout the paper.

Lemma 1 *Suppose that $0 \leqslant s < t$ and $Y_s = y$. Then the solution of (2) satisfies*

$$Y_t = \alpha(t) + \mathrm{e}^{-\kappa(t-s)}\big(Y_s - \alpha(s)\big) + \sigma\int_s^t \mathrm{e}^{-\kappa(t-u)}\,dW_u. \quad (4)$$

Consequently, the conditional distribution of Y_t given $Y_s = y$ is Gaussian with conditional mean and variance

$$\begin{aligned} m_{s,t}(y) &= \alpha(t) + \mathrm{e}^{-\kappa(t-s)}\big(y - \alpha(s)\big), \\ v_{s,t} &= \frac{\sigma^2}{2\kappa}\big(1 - \mathrm{e}^{-2\kappa(t-s)}\big). \end{aligned} \quad (5)$$

Proof: Define $Z_t = Y_t - \alpha(t)$. Since α is deterministic and continuously differentiable, subtracting $d\alpha(t) = \alpha'(t)\,dt$ from (2) gives

$$dZ_t = -\kappa Z_t\,dt + \sigma\,dW_t.$$

The integrating factor for this linear stochastic differential equation is $\mathrm{e}^{\kappa t}$. Because $\mathrm{e}^{\kappa t}$ has finite variation, its quadratic covariation with Z_t is zero. The stochastic product rule therefore gives

$$\begin{aligned} d\left(\mathrm{e}^{\kappa t}Z_t\right) &= \kappa\,\mathrm{e}^{\kappa t}Z_t\,dt + \mathrm{e}^{\kappa t}dZ_t \\ &= \kappa\,\mathrm{e}^{\kappa t}Z_t\,dt + \mathrm{e}^{\kappa t}\left(-\kappa Z_t\,dt + \sigma\,dW_t\right) \\ &= \sigma\,\mathrm{e}^{\kappa t}\,dW_t. \end{aligned}$$

Integrating from s to t and multiplying by $\mathrm{e}^{-\kappa t}$ yield

$$Z_t = \mathrm{e}^{-\kappa(t-s)}Z_s + \sigma\int_s^t \mathrm{e}^{-\kappa(t-u)}\,dW_u.$$

Substituting $Z_t = Y_t - \alpha(t)$ and $Z_s = Y_s - \alpha(s)$ proves (4).

It remains to identify the conditional law. The integrand $u \mapsto \sigma\,\mathrm{e}^{-\kappa(t-u)}$ is deterministic, so the stochastic integral is Gaussian. It depends only on the Brownian increments after time s and is therefore independent of $\mathscr{F}_s$. Its conditional mean is zero, and Itô's isometry gives

$$\begin{aligned} \mathrm{Var}\left(\sigma\int_s^t \mathrm{e}^{-\kappa(t-u)}\,dW_u\right) &= \sigma^2\int_s^t \mathrm{e}^{-2\kappa(t-u)}\,du \\ &= \sigma^2\int_0^{t-s} \mathrm{e}^{-2\kappa z}\,dz \\ &= \frac{\sigma^2}{2\kappa}\big(1 - \mathrm{e}^{-2\kappa(t-s)}\big). \end{aligned}$$

Conditional on $Y_s = y$, the first two terms on the right-hand side of (4) are deterministic. Adding the centered Gaussian stochastic integral therefore gives the conditional mean and variance in (5). □

The state variable can be related to the usual storage-based interpretation without adding an independent stochastic equation for the convenience yield. Suppose that the risk-neutral spot drift is written in the standard net-carrying-cost form $r(t) + c(t) - \delta_t$, where $r(t)$ is the deterministic short rate, $c(t)$ is a deterministic storage-cost rate, and δ_t is the convenience yield. The drift in (1) can be rewritten as

$$\kappa\left(\mu(t) + \frac{\mu'(t)}{\kappa} - Y_t\right) = \kappa\mu(t) + \mu'(t) - \kappa Y_t.$$

Consequently, the specification

$$\delta_t = \kappa Y_t \quad (6)$$

is consistent with the standard net-carrying-cost form precisely when the deterministic carrying-cost component satisfies

$$r(t) + c(t) = \kappa\mu(t) + \mu'(t). \quad (7)$$

Indeed, substitution of (6) and (7) into $r(t) + c(t) - \delta_t$ reproduces the drift coefficient in (1). Equations (6)–(7) are used only to give a consistency interpretation of the one-factor spot model in the usual net-carrying-cost form. In particular, δ_t is determined by the log spot price through (6) and is not introduced as an independent stochastic state variable. The subsequent futures and moment-swap results require only the risk-neutral dynamics in (2) and the exact transition law in Lemma 1. No separate model for $r(t)$, $c(t)$, or δ_t is required.

CONDITIONAL AND MIXED MOMENTS

Conditional moments of the trending Ornstein–Uhlenbeck process have been studied by several analytical methods. In particular, Thamrongrat et al [6] derived closed-form conditional moments by reducing the associated partial differential equation to recursive ordinary differential equations and solving the transformed coefficient system through linear difference equations in Laplace space. Conditional mixed moments have also been studied for other time-dependent diffusion models. Duangpan et al [25] obtained analytical formulas for conditional moments and conditional mixed moments of a generalized stochastic correlation process by using a partial differential equation and the tower property. Explicit solutions of general second-order linear recurrences with variable coefficients were developed by Laohakosol et al [26]. Sutthimat et al [27] subsequently gave a unified extension to generalized second-order linear recurrences in which the recurrence data may depend on an additional variable. These recurrence results provide relevant discrete-mathematical tools whenever transformed coefficient systems arising in conditional-moment calculations reduce to second-order linear recurrences. More recent studies examined the same class of time-dependent mean-reverting processes for pricing moment swaps and constructing innovation-based variance-type and volatility-type contracts[7, 8]. The present section therefore treats the conditional and mixed moments as supporting results. They provide a self-contained probabilistic basis for the subsequent analysis, while the main new results of this paper concern fixed-maturity futures returns, maturity effects, payoff risk, and sampling behavior.

Because the process is time-inhomogeneous, a conditional moment generally depends on both calendar dates and not only on their difference. For integers $n \geqslant 0$ and dates $0 \leqslant s < t$, define

$$U_{s,t}^{(n)}(y) = E^{\mathbb{Q}}\left[Y_t^n \mid Y_s = y\right]. \tag{8}$$

For a fixed terminal date T and time to maturity $0 \leqslant \tau \leqslant T$, set

$$U_T^{(n)}(\tau, y) = U_{T-\tau,T}^{(n)}(y) = E^{\mathbb{Q}}\left[Y_T^n \mid Y_{T-\tau} = y\right]. \tag{9}$$

By convention, $U_{s,t}^{(0)}(y) = U_T^{(0)}(\tau, y) = 1$. The two-date notation in (8) applies whenever the initial date is not determined by a fixed terminal date and a time-to-maturity variable. The exact transition law in Lemma 1 shows that every conditional moment is finite and is a polynomial in the current state. The following theorem gives the recursive differential system for its coefficients.

Theorem 1 *Let $n \in \mathbb{N}$ and fix $T > 0$. Then the conditional moment in (9) has the polynomial form*

$$U_T^{(n)}(\tau, y) = \sum_{k=0}^{n} A_k(\tau) y^k. \tag{10}$$

The coefficient functions satisfy

$$\frac{dA_k(\tau)}{d\tau} - (k+1)\beta_T(\tau)A_{k+1}(\tau) + \kappa k A_k(\tau) - \frac{\sigma^2}{2}(k+2)(k+1)A_{k+2}(\tau) = 0, \tag{11}$$

for $k = 0, 1, \ldots, n$, where

$$\beta_T(\tau) = \kappa\alpha(T-\tau) + \alpha'(T-\tau).$$

The initial conditions are

$$A_n(0) = 1, \quad A_k(0) = 0, \quad k = 0, 1, \ldots, n-1, \tag{12}$$

with the convention

$$A_{n+1}(\tau) = A_{n+2}(\tau) = 0. \tag{13}$$

Proof: Apply Lemma 1 with $s = T - \tau$ and $t = T$. Conditional on $Y_{T-\tau} = y$,

$$Y_T = a_T(\tau) + \mathrm{e}^{-\kappa\tau} y + \sqrt{D(\tau)}Z,$$

where $a_T(\tau) = \alpha(T) - \mathrm{e}^{-\kappa\tau}\alpha(T-\tau)$ and Z is standard normal. Hence, $U_T^{(n)}(\tau, y)$ is a polynomial in y of degree n.

To obtain the coefficient equations, define

$$u(s, y) = E^{\mathbb{Q}}\left[Y_T^n \mid Y_s = y\right].$$

The backward Kolmogorov equation for (2) is

$$\frac{\partial u}{\partial s} + \left(\kappa\alpha(s) + \alpha'(s) - \kappa y\right)\frac{\partial u}{\partial y} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial y^2} = 0,$$

with $u(T, y) = y^n$. Since $U_T^{(n)}(\tau, y) = u(T-\tau, y)$, this becomes

$$\frac{\partial U_T^{(n)}}{\partial \tau} - (\beta_T(\tau) - \kappa y)\frac{\partial U_T^{(n)}}{\partial y} - \frac{\sigma^2}{2}\frac{\partial^2 U_T^{(n)}}{\partial y^2} = 0. \tag{14}$$

Substituting (10) into (14) and equating coefficients of equal powers of y gives (11). Finally, $U_T^{(n)}(0, y) = y^n$ gives (12), while (13) is the convention used at the upper indices. □

The recursive system in Theorem 1 is convenient when the coefficient functions themselves are required. For the present model, however, the exact Gaussian transition law gives a shorter and more direct expression for the conditional moments. Define

$$M_T(\tau, y) = \alpha(T) + \mathrm{e}^{-\kappa\tau}\left(y - \alpha(T-\tau)\right)$$

and

$$D(\tau) = \frac{\sigma^2}{2\kappa}\left(1 - \mathrm{e}^{-2\kappa\tau}\right).$$

The next theorem gives the closed-form formula for every integer order.

Theorem 2 *Let $n \in \mathbb{N}$ and $0 \leqslant \tau \leqslant T$. Then*

$$U_T^{(n)}(\tau, y) = \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2r)!r!2^r} M_T^{n-2r}(\tau, y) D^r(\tau). \quad (15)$$

Proof: Apply Lemma 1 with $s = T - \tau$ and $t = T$. Conditional on $Y_{T-\tau} = y$, the terminal value has the form

$$Y_T = M_T(\tau, y) + \sigma \int_{T-\tau}^{T} \mathrm{e}^{-\kappa(T-u)}\, dW_u.$$

The stochastic integral is independent of $\mathscr{F}_{T-\tau}$, has mean zero, and has variance

$$\sigma^2 \int_{T-\tau}^{T} \mathrm{e}^{-2\kappa(T-u)}\, du = D(\tau).$$

Consequently, under the conditional law, there is a standard normal random variable Z such that

$$Y_T = M_T(\tau, y) + \sqrt{D(\tau)} Z.$$

All moments are finite because the right-hand side is Gaussian. Raising it to the integer power n and applying the binomial theorem give

$$\begin{aligned} U_T^{(n)}(\tau, y) &= E\left[\left(M_T(\tau, y) + \sqrt{D(\tau)} Z\right)^n\right] \\ &= \sum_{j=0}^{n} \binom{n}{j} M_T^{n-j}(\tau, y) D^{j/2}(\tau) E[Z^j]. \end{aligned}$$

The symmetry of the standard normal law gives $E[Z^j] = 0$ when j is odd. For an even index $j = 2r$,

$$E[Z^{2r}] = (2r-1)!! = \frac{(2r)!}{r!2^r}.$$

Therefore, only $r = 0, 1, \ldots, \lfloor n/2 \rfloor$ remain. For each such r,

$$\binom{n}{2r} \frac{(2r)!}{r!2^r} = \frac{n!}{(n-2r)!r!2^r}.$$

Substitution of this coefficient yields (15). At $\tau = 0$,

$$D(0) = 0, \quad M_T(0, y) = y.$$

Therefore, the formula reduces to $U_T^{(n)}(0, y) = y^n$, as required. □

For example, the first four conditional moments are

$$\begin{aligned} U_T^{(1)}(\tau, y) &= M_T(\tau, y), \\ U_T^{(2)}(\tau, y) &= M_T^2(\tau, y) + D(\tau), \\ U_T^{(3)}(\tau, y) &= M_T^3(\tau, y) + 3M_T(\tau, y) D(\tau), \\ U_T^{(4)}(\tau, y) &= M_T^4(\tau, y) + 6M_T^2(\tau, y) D(\tau) + 3D^2(\tau). \end{aligned}$$

These formulas show that the time-dependent trend enters the conditional moments only through the conditional mean, whereas the conditional variance depends only on κ, σ, and the elapsed time.

Products of log-prices at two observation dates arise when powers of spot log-price increments are expanded. Although the futures-return analysis developed later will admit a simpler independent-increment form, a two-time mixed-moment formula is recorded here for completeness and for comparison with earlier spot-based formulations. Let $0 \leqslant t_0 < T_1 < T_2$, let $p, q \in \mathbb{N}_0$, and define

$$\tau_1 = T_1 - t_0, \quad \tau_2 = T_2 - T_1.$$

The mixed conditional moment is

$$U_{t_0,T_1,T_2}^{(p,q)}(y) = E^{\mathbb{Q}}\left[Y_{T_1}^p Y_{T_2}^q \mid Y_{t_0} = y\right]. \quad (16)$$

Theorem 3 *For $p, q \in \mathbb{N}_0$, the mixed conditional moment in (16) satisfies*

$$\begin{aligned} U_{t_0,T_1,T_2}^{(p,q)}(y) = \sum_{r=0}^{\lfloor q/2 \rfloor} \sum_{j=0}^{q-2r} &\frac{q!}{(q-2r)!r!2^r} \binom{q-2r}{j} \\ &\times G^{q-2r-j}(T_1, T_2) Q^j(\tau_2) D^r(\tau_2) U_{t_0,T_1}^{(p+j)}(y), \quad (17) \end{aligned}$$

where $Q(\tau_2) = \mathrm{e}^{-\kappa\tau_2}$ and

$$G(T_1, T_2) = \alpha(T_2) - \alpha(T_1) Q(\tau_2).$$

Proof: By Lemma 1, conditionally on $\mathscr{F}_{T_1}$,

$$Y_{T_2} = G(T_1, T_2) + Q(\tau_2) Y_{T_1} + \sqrt{D(\tau_2)} Z_2,$$

where Z_2 is standard normal and independent of $\mathscr{F}_{T_1}$. Therefore, Theorem 2 gives

$$\begin{aligned} E^{\mathbb{Q}}\left[Y_{T_2}^q \mid \mathscr{F}_{T_1}\right] = \sum_{r=0}^{\lfloor q/2 \rfloor} &\frac{q!}{(q-2r)!r!2^r} \\ &\times \left(G(T_1, T_2) + Q(\tau_2) Y_{T_1}\right)^{q-2r} D^r(\tau_2). \end{aligned}$$

Multiplying by $Y_{T_1}^p$, applying the tower property, and expanding the remaining power by the binomial theorem give

$$E^{\mathbb{Q}}\left[Y_{T_1}^{p+j} \mid Y_{t_0} = y\right] = U_{t_0,T_1}^{(p+j)}(y),$$

for $j = 0, 1, \ldots, q-2r$. Substitution yields (17). □

The formula in Theorem 3 includes the one-time conditional moments as special cases, with their calendar dates retained explicitly. If $q = 0$, then $Y_{T_2}^q = 1$ and

$$U_{t_0,T_1,T_2}^{(p,0)}(y) = U_{t_0,T_1}^{(p)}(y).$$

If $p = 0$, the tower property gives

$$\begin{aligned} U_{t_0,T_1,T_2}^{(0,q)}(y) &= E^{\mathbb{Q}}\left[E^{\mathbb{Q}}\left[Y_{T_2}^q \mid \mathscr{F}_{T_1}\right] \middle| Y_{t_0} = y\right] \\ &= E^{\mathbb{Q}}\left[Y_{T_2}^q \mid Y_{t_0} = y\right] = U_{t_0,T_2}^{(q)}(y). \end{aligned}$$

The explicit dates are essential here because the deterministic mean-reversion level is time dependent. In general, a conditional moment cannot be indexed only by the elapsed time. Repeated application of the tower property extends the same argument to products at three or more observation dates. The subsequent sections do not require this extension because fixed-maturity futures returns have a simpler form, but the results above clarify the relation between the underlying log-price process and earlier spot-based moment-swap formulas.

FIXED-MATURITY FUTURES PRICES AND RETURNS

The previous section described the conditional moments of the spot log-price process. The futures price is now derived directly from the same risk-neutral model. This step is needed because a futures contract is determined by its delivery date, which may be later than the final observation date of a moment swap. Previous studies applied the trending Ornstein–Uhlenbeck model to conditional moments, innovation-based contracts, and moment swaps written from log-price changes [6–8]. In contrast, the present section starts from the model-implied price of a fixed-maturity futures contract and derives the distribution of its log returns.

Let $\mathscr{T} > 0$ be the delivery date of a futures contract. Under the risk-neutral measure and deterministic interest rates, the futures price at time $t \in [0, \mathscr{T}]$ is

$$F(t,\mathscr{T}) = E^{\mathbb{Q}}[S_{\mathscr{T}} \mid \mathscr{F}_t]. \tag{18}$$

The symbol $\mathscr{T}$ denotes the futures delivery date, while T later denotes the final observation date of a moment swap. The following theorem gives the futures price implied by the model in (1)–(2).

Theorem 4 *Suppose that the spot price follows* (1)*, and let* $0 \leqslant t \leqslant \mathscr{T}$*. Then the futures price in* (18) *is*

$$F(t,\mathscr{T}) = \exp\left\{\alpha(\mathscr{T}) + \mathrm{e}^{-\kappa(\mathscr{T}-t)}\big(Y_t - \alpha(t)\big) + \tfrac{1}{2}D(\mathscr{T}-t)\right\}, \tag{19}$$

where

$$D(u) = \frac{\sigma^2}{2\kappa}\left(1 - \mathrm{e}^{-2\kappa u}\right), \quad u \geqslant 0.$$

Proof: By Lemma 1, the conditional on $\mathscr{F}_t$, the terminal log-price can be written as

$$Y_{\mathscr{T}} = m(t,\mathscr{T}) + \sqrt{D(\mathscr{T}-t)}Z,$$

where Z is the standard normal independent of $\mathscr{F}_t$,

$$m(t,\mathscr{T}) = \alpha(\mathscr{T}) + \mathrm{e}^{-\kappa(\mathscr{T}-t)}\left(Y_t - \alpha(t)\right).$$

Since $S_{\mathscr{T}} = \mathrm{e}^{Y_{\mathscr{T}}}$, (18) becomes

$$F(t,\mathscr{T}) = \mathrm{e}^{m(t,\mathscr{T})} E\left[\mathrm{e}^{\sqrt{D(\mathscr{T}-t)}Z}\right].$$

The moment-generating function of a standard normal random variable satisfies

$$E[\mathrm{e}^{aZ}] = \mathrm{e}^{a^2/2}.$$

Taking $a = \sqrt{D(\mathscr{T}-t)}$ gives

$$F(t,\mathscr{T}) = \exp\left\{m(t,\mathscr{T}) + \tfrac{1}{2}D(\mathscr{T}-t)\right\}.$$

Substituting the displayed expression for $m(t,\mathscr{T})$ proves (19). □

The formula in Theorem 4 separates the deterministic level $\alpha(\mathscr{T})$ from the stochastic exposure to the current log-price. It also shows explicitly how the mean-reversion speed changes the response of the futures price to the current state. The next theorem gives the risk-neutral dynamics of the futures price.

Theorem 5 *For a fixed delivery date* $\mathscr{T}$*, the futures price in* (19) *satisfies*

$$\frac{dF(t,\mathscr{T})}{F(t,\mathscr{T})} = \sigma e^{-\kappa(\mathscr{T}-t)}\, dW_t, \tag{20}$$

for $0 \leqslant t \leqslant \mathscr{T}$*. Equivalently, its logarithm satisfies*

$$d\ln F(t,\mathscr{T}) = -\tfrac{1}{2}\sigma^2 \mathrm{e}^{-2\kappa(\mathscr{T}-t)}dt + \sigma \mathrm{e}^{-\kappa(\mathscr{T}-t)}dW_t. \tag{21}$$

Proof: Let $b(t,\mathscr{T}) = \mathrm{e}^{-\kappa(\mathscr{T}-t)}$ and recall that $Z_t = Y_t - \alpha(t)$ satisfies (3). Formula (19) can be written as

$$\ln F(t,\mathscr{T}) = \alpha(\mathscr{T}) + b(t,\mathscr{T})Z_t + \tfrac{1}{2}D(\mathscr{T}-t).$$

The derivative of b with respect to t is κb. Since b is deterministic, the product rule and (3) give

$$\begin{aligned} d\left(b(t,\mathscr{T})Z_t\right) &= \kappa b(t,\mathscr{T})Z_t\, dt + b(t,\mathscr{T})dZ_t \\ &= \kappa b Z_t\, dt + b\left(-\kappa Z_t\, dt + \sigma\, dW_t\right) \\ &= \sigma\, b(t,\mathscr{T})\, dW_t. \end{aligned}$$

For $u \geqslant 0$, differentiate $D(u) = \sigma^2(1-\mathrm{e}^{-2\kappa u})/(2\kappa)$ gives $D'(u) = \sigma^2\,\mathrm{e}^{-2\kappa u}$. Since $u = \mathscr{T} - t$ has derivative -1,

$$dD(\mathscr{T}-t) = -\sigma^2\,\mathrm{e}^{-2\kappa(\mathscr{T}-t)}\,dt = -\sigma^2 b^2(t,\mathscr{T})\,dt.$$

Combining the last two equalities yields

$$d\ln F(t,\mathscr{T}) = -\tfrac{1}{2}\sigma^2 b^2(t,\mathscr{T})\,dt + \sigma\, b(t,\mathscr{T})\,dW_t,$$

which is (21). Finally, Itô's formula for $F = \mathrm{e}^{\ln F}$ gives

$$\frac{dF}{F} = d\ln F + \frac{1}{2}d\langle \ln F\rangle.$$

Since $d\langle \ln F\rangle = \sigma^2 b^2(t,\mathscr{T})\,dt$, the two drift terms cancel and (20) follows. □

The diffusion coefficient in (20) is deterministic. It decreases as the time to delivery increases and approaches σ as t approaches $\mathscr{T}$. This maturity dependence will enter all moment-swap prices derived

below. More importantly, the deterministic trend $\alpha(t)$ does not appear in the futures return dynamics, although it remains present in the level of the futures price. The same return dynamics are obtained from the corresponding constant-mean model with the same values of κ and σ. Thus, the time-dependent trend changes the futures price level through (19), but it does not change the law of futures log returns. This cancellation is a structural property of the present one-factor specification. Let

$$0 = t_0 < t_1 < \ldots < t_N = T \leqslant \mathscr{T}$$

be the observation dates of a moment swap written on the futures contract with delivery date $\mathscr{T}$. Define the log return over $[t_{i-1}, t_i]$ by

$$X_i(\mathscr{T}) = \ln \frac{F(t_i, \mathscr{T})}{F(t_{i-1}, \mathscr{T})}, \quad i = 1, 2, \ldots, N. \tag{22}$$

The following theorem gives its exact finite-grid law.

Theorem 6 *For $i = 1, 2, \ldots, N$, define*

$$\begin{aligned} v_i(\mathscr{T}) &= \int_{t_{i-1}}^{t_i} \sigma^2 \, \mathrm{e}^{-2\kappa(\mathscr{T}-s)} \, ds \\ &= \frac{\sigma^2}{2\kappa} \left[\mathrm{e}^{-2\kappa(\mathscr{T}-t_i)} - \mathrm{e}^{-2\kappa(\mathscr{T}-t_{i-1})} \right]. \end{aligned} \tag{23}$$

Then

$$X_i(\mathscr{T}) = -\tfrac{1}{2} v_i(\mathscr{T}) + \sqrt{v_i(\mathscr{T})}\, \xi_i, \tag{24}$$

where $\xi_1, \xi_2, \ldots, \xi_N$ are independent standard normal random variables. Consequently,

$$X_i(\mathscr{T}) \sim N\left(-\tfrac{1}{2} v_i(\mathscr{T}), v_i(\mathscr{T})\right), \quad i = 1, 2, \ldots, N, \tag{25}$$

and the returns $X_1(\mathscr{T}), X_2(\mathscr{T}), \ldots, X_N(\mathscr{T})$ are independent.

Proof: Integrate (21) from t_{i-1} to t_i. By (22),

$$X_i(\mathscr{T}) = -\frac{1}{2} \int_{t_{i-1}}^{t_i} \sigma^2 \, \mathrm{e}^{-2\kappa(\mathscr{T}-s)} \, ds + \int_{t_{i-1}}^{t_i} \sigma \, \mathrm{e}^{-\kappa(\mathscr{T}-s)} \, dW_s.$$

The deterministic integral is

$$\begin{aligned} \int_{t_{i-1}}^{t_i} \sigma^2 \, \mathrm{e}^{-2\kappa(\mathscr{T}-s)} \, ds &= \frac{\sigma^2}{2\kappa} \left[\mathrm{e}^{-2\kappa(\mathscr{T}-t_i)} - \mathrm{e}^{-2\kappa(\mathscr{T}-t_{i-1})} \right] \\ &= v_i(\mathscr{T}). \end{aligned}$$

The stochastic integral has mean zero. By Itô's isometry, its variance is the same deterministic integral, namely $v_i(\mathscr{T})$. Thus, the standardized random variable

$$\xi_i = \frac{1}{\sqrt{v_i(\mathscr{T})}} \int_{t_{i-1}}^{t_i} \sigma \, \mathrm{e}^{-\kappa(\mathscr{T}-s)} \, dW_s$$

is standard normal, and (24) follows.

For $i \neq j$, the stochastic integrals are taken over disjoint intervals and have deterministic integrands. They are jointly Gaussian and have covariance zero because Brownian increments over disjoint intervals are independent. Hence, the variables $\xi_1, \xi_2, \ldots, \xi_N$ are independent. Adding deterministic means does not change independence, so the returns are independent and have the distributions in (25). □

The distribution in Theorem 6 depends on κ, σ, the observation dates, and the delivery date. It does not depend on the deterministic trend $\mu(t)$ or the current spot price. Since the convenience yield in (6) is not an independent state variable, no separate initial convenience-yield value enters the return law. This conclusion follows from the futures return dynamics and does not require a change in the spot-price model. It will allow the fair delivery prices of futures-based moment swaps to be obtained directly from normal moments without the mixed moments of the spot log-price.

FUTURES-BASED MOMENT SWAPS

The earlier section shows that the log returns of a fixed-maturity futures contract are independent normal random variables with means and variances determined by the observation dates and the delivery date. This return law yields the fair delivery prices of futures-based moment swaps directly from normal moments. It also avoids the spot–futures relation adopted in earlier formulations and keeps the pricing analysis consistent with the risk-neutral futures price in (18). Let

$$0 = t_0 < t_1 < \ldots < t_N = T \leqslant \mathscr{T}$$

be the observation dates, where T is the maturity of the moment swap and $\mathscr{T}$ is the delivery date of the underlying futures contract. For an integer $m \geqslant 2$, define the annualized realized mth moment of the futures log returns by

$$M^{(m)}(T, N, \mathscr{T}) = \frac{1}{T} \sum_{i=1}^{N} X_i^m(\mathscr{T}), \tag{26}$$

where $X_i(\mathscr{T})$ is defined in (22). The quantity in (26) is a realized raw moment. In particular, the third- and fourth-order quantities are not standardized skewness and kurtosis. The common factor $1/T$ is used to express every contract as a raw moment per unit of calendar time, in line with the annualized moment-swap convention used in earlier futures-based formulations [21]. This normalization is kept fixed across moment orders so that the contracts are defined on the same observation horizon. As shown later, it yields a nonzero continuous-sampling limit for the second-order contract, whereas the higher-order raw moments converge to zero.

A futures-based moment swap with notional amount $L > 0$ pays

$$L\left(M^{(m)}(T,N,\mathscr{T}) - K_m(T,N,\mathscr{T})\right)$$

at time T, where $K_m(T,N,\mathscr{T})$ is the delivery price fixed at inception. Since the interest rate is deterministic, the zero-value condition at inception gives

$$K_m(T,N,\mathscr{T}) = E^{\mathbb{Q}}\left[M^{(m)}(T,N,\mathscr{T})\right]. \tag{27}$$

The next theorem gives the fair delivery price for every integer order.

Theorem 7 *Let $m \geqslant 2$ be an integer, and suppose that the futures log returns are observed at $0 = t_0 < t_1 < \ldots < t_N = T \leqslant \mathscr{T}$. Then the fair delivery price in (27) is*

$$K_m(T,N,\mathscr{T}) = \frac{1}{T}\sum_{i=1}^{N}\sum_{r=0}^{\lfloor m/2\rfloor}\frac{m!}{(m-2r)!r!2^r}\left(-\frac{1}{2}\right)^{m-2r} v_i^{m-r}(\mathscr{T}), \tag{28}$$

where $v_i(\mathscr{T})$ is given by (23).

Proof: The payoff is settled at the common date T. With a deterministic discount factor, the time-zero value is proportional to

$$E^{\mathbb{Q}}\left[M^{(m)}(T,N,\mathscr{T}) - K_m(T,N,\mathscr{T})\right].$$

Therefore, the zero-value condition is exactly (27). By Theorem 6, for each observation interval,

$$X_i(\mathscr{T}) = -\tfrac{1}{2}v_i(\mathscr{T}) + \sqrt{v_i(\mathscr{T})}\,\xi_i,$$

where ξ_i is standard normal. Expanding the mth power gives

$$X_i^m(\mathscr{T}) = \sum_{j=0}^{m}\binom{m}{j}\left(-\frac{v_i(\mathscr{T})}{2}\right)^{m-j} v_i^{j/2}(\mathscr{T})\xi_i^j.$$

Taking expectations removes all terms with odd j. Write the remaining index as $j = 2r$. Since

$$E[\xi_i^{2r}] = \frac{(2r)!}{r!2^r},$$

the coefficient of the rth remaining term is

$$\binom{m}{2r}\frac{(2r)!}{r!2^r} = \frac{m!}{(m-2r)!r!2^r}.$$

The power of $v_i(\mathscr{T})$ is

$$m - 2r + r = m - r,$$

and the sign and the remaining power of 2 are represented by $(-1/2)^{m-2r}$. Hence,

$$E^{\mathbb{Q}}\left[X_i^m(\mathscr{T})\right] = \sum_{r=0}^{\lfloor m/2\rfloor}\frac{m!}{(m-2r)!r!2^r}\left(-\frac{1}{2}\right)^{m-2r} v_i^{m-r}(\mathscr{T}).$$

Finally, apply the definition in (26), linearity of expectation, and (27) to sum this expression over $i = 1, 2, \ldots, N$. This gives (28). □

The normal-moment calculation in Theorem 7 is consistent with the deterministic time-dependent volatility setting used for futures-based moment swaps by Chumpong et al [21]. In the present model, however, the volatility function $\sigma\,\mathrm{e}^{-\kappa(\mathscr{T}-t)}$ is obtained from the mean-reverting commodity dynamics rather than specified directly. In addition, the futures delivery date $\mathscr{T}$ is allowed to exceed the swap maturity T. This separate delivery date is used below to study the effects of the futures delivery date and parameter identification across futures maturities. The payoff-covariance formulas and sampling asymptotics extend the analysis of the same exact return law.

The formula in Theorem 7 is valid for arbitrary observation dates. For equally spaced observations, the remaining sums can also be evaluated explicitly. Let

$$t_i = i\Delta, \quad i = 0, 1, \ldots, N, \quad \Delta = \frac{T}{N},$$

and for an integer $p \geqslant 1$ define

$$S_p(T,N,\mathscr{T}) = \sum_{i=1}^{N} v_i^p(\mathscr{T}). \tag{29}$$

The following corollary gives the closed form of these sums and removes the summation over the observation dates from the general pricing formula.

Corollary 1 *For equally spaced observation dates,*

$$S_p(T,N,\mathscr{T}) = \left(\frac{\sigma^2}{2\kappa}\right)^p \frac{\left(1-\mathrm{e}^{-2\kappa\Delta}\right)^p}{1-\mathrm{e}^{-2p\kappa\Delta}} \times \mathrm{e}^{-2p\kappa(\mathscr{T}-T)}\left(1-\mathrm{e}^{-2p\kappa T}\right), \tag{30}$$

for $p = 1, 2, \ldots$. Consequently,

$$K_m(T,N,\mathscr{T}) = \frac{1}{T}\sum_{r=0}^{\lfloor m/2\rfloor}\frac{m!}{(m-2r)!r!2^r} \times \left(-\frac{1}{2}\right)^{m-2r} S_{m-r}(T,N,\mathscr{T}). \tag{31}$$

Proof: For $t_i = i\Delta$, the two exponentials in (23) satisfy

$$\mathrm{e}^{-2\kappa(\mathscr{T}-t_{i-1})} = \mathrm{e}^{-2\kappa(\mathscr{T}-t_i)}\,\mathrm{e}^{-2\kappa\Delta}.$$

Therefore,

$$v_i(\mathscr{T}) = \frac{\sigma^2}{2\kappa}\left(1-\mathrm{e}^{-2\kappa\Delta}\right)\mathrm{e}^{-2\kappa(\mathscr{T}-i\Delta)}.$$

Raising this expression to the power p and summing gives

$$S_p(T,N,\mathscr{T}) = \left(\frac{\sigma^2}{2\kappa}\right)^p\left(1-\mathrm{e}^{-2\kappa\Delta}\right)^p\sum_{i=1}^{N}\mathrm{e}^{-2p\kappa(\mathscr{T}-i\Delta)}.$$

Evaluating the final sum using $j = N-i$; since $N\Delta = T$,

$$\begin{aligned}\sum_{i=1}^{N} \mathrm{e}^{-2p\kappa(\mathscr{T}-i\Delta)} &= \mathrm{e}^{-2p\kappa(\mathscr{T}-T)} \sum_{j=0}^{N-1} \mathrm{e}^{-2p\kappa j\Delta} \\ &= \mathrm{e}^{-2p\kappa(\mathscr{T}-T)} \frac{1-\mathrm{e}^{-2p\kappa N\Delta}}{1-\mathrm{e}^{-2p\kappa\Delta}} \\ &= \mathrm{e}^{-2p\kappa(\mathscr{T}-T)} \frac{1-\mathrm{e}^{-2p\kappa T}}{1-\mathrm{e}^{-2p\kappa\Delta}}.\end{aligned}$$

Substitution proves (30). In (28), the rth term contains $\sum_{i=1}^{N} \nu_i^{m-r}(\mathscr{T})$, which is $S_{m-r}(T,N,\mathscr{T})$ by (29). Replacing each such sum gives (31). □

The general formula immediately gives explicit expressions for the second-, third-, and fourth-order contracts. These orders are reported separately because they are the most common cases in applications.

Corollary 2 *The fair delivery price of the second-order futures-based moment swap is*

$$K_2(T,N,\mathscr{T}) = \frac{1}{T}\left(S_1(T,N,\mathscr{T}) + \frac{1}{4}S_2(T,N,\mathscr{T})\right). \quad (32)$$

Proof: Set $m = 2$ in (31). For $r = 0$, the coefficient is

$$\frac{2!}{2!0!}\left(-\frac{1}{2}\right)^2 = \frac{1}{4},$$

and the corresponding power sum is S_2. For $r = 1$, the coefficient is

$$\frac{2!}{0!1!2} = 1,$$

and the corresponding power sum is S_1. Adding the two terms and dividing by T gives (32). □

Corollary 3 *The fair delivery price of the third-order futures-based moment swap is*

$$K_3(T,N,\mathscr{T}) = -\frac{1}{T}\left(\frac{3}{2}S_2(T,N,\mathscr{T}) + \frac{1}{8}S_3(T,N,\mathscr{T})\right). \quad (33)$$

Proof: Set $m = 3$ in (31). The term $r = 0$ has coefficient

$$\frac{3!}{3!0!}\left(-\frac{1}{2}\right)^3 = -\frac{1}{8}$$

and multiplies S_3. The term $r = 1$ has coefficient

$$\frac{3!}{1!1!2}\left(-\frac{1}{2}\right) = -\frac{3}{2}$$

and multiplies S_2. Their sum gives (33). □

Corollary 4 *The fair delivery price of the fourth-order futures-based moment swap is*

$$K_4(T,N,\mathscr{T}) = \frac{1}{T}\left(3S_2(T,N,\mathscr{T}) + \frac{3}{2}S_3(T,N,\mathscr{T}) + \frac{1}{16}S_4(T,N,\mathscr{T})\right). \quad (34)$$

Proof: Set $m = 4$ in (31). The three possible values of r give

$$\begin{aligned} r = 0: &\quad \frac{4!}{4!0!}\left(-\frac{1}{2}\right)^4 S_4 = \frac{1}{16}S_4, \\ r = 1: &\quad \frac{4!}{2!1!2}\left(-\frac{1}{2}\right)^2 S_3 = \frac{3}{2}S_3, \\ r = 2: &\quad \frac{4!}{0!2!2^2}S_2 = 3S_2. \end{aligned}$$

Adding these terms and dividing by T proves (34). □

Since $S_p(T,N,\mathscr{T}) > 0$ for every $p \geqslant 1$, Corollary 2 and Corollary 4 show that the second- and fourth-order fair delivery prices are positive, whereas Corollary 3 shows that the third-order fair delivery price is negative under the present Gaussian futures model. This sign follows from the negative drift of the futures log return under the risk-neutral measure. The formulas also show that the fair delivery prices depend on κ, σ, the observation grid, and the futures delivery date, but not on the deterministic trend or the initial state. These maturity and sampling effects will be studied in the subsequent sections.

PAYOFF VARIANCES, COVARIANCES, AND SAMPLING BEHAVIOR

The fair delivery price describes the risk-neutral mean of a realized moment, but it does not describe the uncertainty of the payment at maturity. The variances of the realized moments and the covariances between realized moments of different orders are therefore examined. These quantities characterize portfolios of moment swaps and permit comparisons of contract risk across delivery dates. The independent-return law in Theorem 6 makes the required calculations explicit because products involving returns from different observation intervals factor into products of their expectations. Throughout this section, "payoff risk" refers to variance and covariance under the risk-neutral measure $\mathbb{Q}$. These quantities are risk-neutral payoff variances and covariances and are not physical-measure forecasts of realized contract risk.

For an integer $m \geqslant 0$ and $\nu > 0$, let

$$g_m(\nu) = \sum_{r=0}^{\lfloor m/2 \rfloor} \frac{m!}{(m-2r)!r!2^r}\left(-\frac{1}{2}\right)^{m-2r} \nu^{m-r}. \quad (35)$$

Thus, if $X \sim N(-\nu/2, \nu)$, then $g_m(\nu) = E[X^m]$. This notation only shortens the covariance formulas below.

Theorem 8 *Let $p, q \geqslant 2$ be integers. The covariance between the realized moments $M^{(p)}(T,N,\mathscr{T})$ and $M^{(q)}(T,N,\mathscr{T})$ is*

$$\begin{aligned}&\mathrm{Cov}\left(M^{(p)}(T,N,\mathscr{T}), M^{(q)}(T,N,\mathscr{T})\right) \\ &= \frac{1}{T^2}\sum_{i=1}^{N}\left[g_{p+q}(v_i(\mathscr{T})) - g_p(v_i(\mathscr{T}))\, g_q(v_i(\mathscr{T}))\right]. \quad (36)\end{aligned}$$

In particular,

$$\begin{aligned}&\operatorname{Var}\left(M^{(p)}(T,N,\mathscr{T})\right)\\&\qquad=\frac{1}{T^2}\sum_{i=1}^{N}\Big[g_{2p}\left(v_i(\mathscr{T})\right)-g_p^2\left(v_i(\mathscr{T})\right)\Big].\end{aligned}\tag{37}$$

Proof: For readability, suppress the common arguments $(T,N,\mathscr{T})$. From (26),

$$M^{(p)}=\frac{1}{T}\sum_{i=1}^{N}X_i^p,\quad M^{(q)}=\frac{1}{T}\sum_{j=1}^{N}X_j^q.$$

Bilinearity of covariance gives

$$\operatorname{Cov}\left(M^{(p)},M^{(q)}\right)=\frac{1}{T^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\operatorname{Cov}\left(X_i^p,X_j^q\right).$$

When $i\neq j$, Theorem 6 implies that X_i and X_j are independent. Measurable powers of independent random variables remain independent, so

$$\operatorname{Cov}\left(X_i^p,X_j^q\right)=0,\quad i\neq j.$$

For $i=j$,

$$\begin{aligned}\operatorname{Cov}\left(X_i^p,X_i^q\right)&=E\left[X_i^{p+q}\right]-E\left[X_i^p\right]E\left[X_i^q\right]\\&=g_{p+q}\left(v_i(\mathscr{T})\right)-g_p\left(v_i(\mathscr{T})\right)g_q\left(v_i(\mathscr{T})\right),\end{aligned}$$

where the final equality follows from (35) and the distribution in (25). Only the diagonal terms remain in the double sum, which proves (36). Taking $q=p$ gives (37). □

For contract-specific notionals L_m and L_q, define the time-T payoffs by

$$P_m=L_m\left(M^{(m)}-K_m\right),\quad P_q=L_q\left(M^{(q)}-K_q\right).\tag{38}$$

Because each delivery price is deterministic at inception, the corresponding payoff-risk measures are

$$\begin{aligned}\operatorname{Var}^{\mathbb{Q}}(P_m)&=L_m^2\operatorname{Var}^{\mathbb{Q}}\left(M^{(m)}\right),\\\operatorname{Cov}^{\mathbb{Q}}(P_m,P_q)&=L_mL_q\operatorname{Cov}^{\mathbb{Q}}\left(M^{(m)},M^{(q)}\right).\end{aligned}\tag{39}$$

All variance and covariance quantities in this section are evaluated under $\mathbb{Q}$. They are risk-neutral payoff variances and covariances and should not be interpreted as physical-measure forecasts of realized contract risk.

For equally spaced observations, the formulas in Theorem 8 can be written in terms of the power sums in (29). The next corollary records the complete covariance structure of the second-, third-, and fourth-order realized moments. In the displayed formulas, S_j denotes $S_j(T,N,\mathscr{T})$.

Corollary 5 *For equally spaced observation dates,*

$$\begin{aligned}\operatorname{Var}\left(M^{(2)}\right)&=\tfrac{1}{T^2}\left(S_3+2S_2\right),\\\operatorname{Cov}\left(M^{(2)},M^{(3)}\right)&=-\tfrac{3}{4T^2}\left(S_4+8S_3\right),\\\operatorname{Cov}\left(M^{(2)},M^{(4)}\right)&=\tfrac{1}{2T^2}\left(S_5+18S_4+24S_3\right),\\\operatorname{Var}\left(M^{(3)}\right)&=\tfrac{3}{16T^2}\left(3S_5+48S_4+80S_3\right),\\\operatorname{Cov}\left(M^{(3)},M^{(4)}\right)&=-\tfrac{3}{8T^2}\left(S_6+28S_5+128S_4\right),\\\operatorname{Var}\left(M^{(4)}\right)&=\tfrac{1}{4T^2}\left(S_7+42S_6+384S_5+384S_4\right).\end{aligned}\tag{40}$$

Proof: Formula (35) gives the moments required up to order eight:

$$\begin{aligned}g_2(v)&=v+\tfrac{1}{4}v^2,\\g_3(v)&=-\tfrac{3}{2}v^2-\tfrac{1}{8}v^3,\\g_4(v)&=3v^2+\tfrac{3}{2}v^3+\tfrac{1}{16}v^4,\\g_5(v)&=-\tfrac{15}{2}v^3-\tfrac{5}{4}v^4-\tfrac{1}{32}v^5,\\g_6(v)&=15v^3+\tfrac{45}{4}v^4+\tfrac{15}{16}v^5+\tfrac{1}{64}v^6,\\g_7(v)&=-\tfrac{105}{2}v^4-\tfrac{105}{8}v^5-\tfrac{21}{32}v^6-\tfrac{1}{128}v^7,\\g_8(v)&=105v^4+105v^5+\tfrac{105}{8}v^6+\tfrac{7}{16}v^7+\tfrac{1}{256}v^8.\end{aligned}$$

Substituting these polynomials into the six covariance expressions appearing in (36) gives

$$\begin{aligned}g_4(v)-g_2^2(v)&=2v^2+v^3,\\g_5(v)-g_2(v)g_3(v)&=-6v^3-\tfrac{3}{4}v^4,\\g_6(v)-g_2(v)g_4(v)&=12v^3+9v^4+\tfrac{1}{2}v^5,\\g_6(v)-g_3^2(v)&=15v^3+9v^4+\tfrac{9}{16}v^5,\\g_7(v)-g_3(v)g_4(v)&=-48v^4-\tfrac{21}{2}v^5-\tfrac{3}{8}v^6,\\g_8(v)-g_4^2(v)&=96v^4+96v^5+\tfrac{21}{2}v^6+\tfrac{1}{4}v^7.\end{aligned}$$

Equivalently, these are the factored expressions

$$\begin{aligned}g_4-g_2^2&=v^2(v+2),\\g_5-g_2g_3&=-\tfrac{3}{4}v^3(v+8),\\g_6-g_2g_4&=\tfrac{1}{2}v^3(v^2+18v+24),\\g_6-g_3^2&=\tfrac{3}{16}v^3(3v^2+48v+80),\\g_7-g_3g_4&=-\tfrac{3}{8}v^4(v^2+28v+128),\\g_8-g_4^2&=\tfrac{1}{4}v^4(v^3+42v^2+384v+384).\end{aligned}$$

Apply (36) to each expression and substitute $\sum_{i=1}^{N}v_i^j(\mathscr{T})=S_j(T,N,\mathscr{T})$. Collecting equal powers proves all six equalities in (40). □

The signs in (40) have a direct interpretation under the present model. The second- and fourth-order realized moments have positive covariance, whereas the third-order realized moment has negative covariance with both even-order realized moments. These signs follow from the negative mean of each risk-neutral log-futures return and are specific to raw powers of the returns. They should not be interpreted

as general statements about standardized skewness or kurtosis.

The futures delivery date also changes the size of every fair delivery price and every payoff variance. The dependence is especially clear because the return variance satisfies

$$v_i(\mathscr{T}) = \mathrm{e}^{-2\kappa(\mathscr{T}-T)} v_i(T), \quad \mathscr{T} \geqslant T. \tag{41}$$

The next corollary gives the resulting monotonicity of the fair delivery prices.

Corollary 6 *Fix T, N, κ, and σ. For every integer $q \geqslant 1$, the even-order fair delivery price $K_{2q}(T,N,\mathscr{T})$ is positive and strictly decreasing in $\mathscr{T}$, while the odd-order fair delivery price $K_{2q+1}(T,N,\mathscr{T})$ is negative and strictly increasing in $\mathscr{T}$. Moreover,*

$$K_m(T,N,\mathscr{T}) \to 0 \quad \text{as } \mathscr{T} \to \infty,$$

for every integer $m \geqslant 2$.

Proof: Let $L = \mathscr{T} - T \geqslant 0$. Equation (41) gives

$$S_p(T,N,\mathscr{T}) = \mathrm{e}^{-2p\kappa L} S_p(T,N,T), \quad p = 1,2,\ldots.$$

For each r, define the positive coefficient

$$a_{m,r} = \frac{m!}{(m-2r)!\,r!\,2^r}\left(\frac{1}{2}\right)^{m-2r} > 0.$$

Because $m-2r$ has the same parity as m, the sign of $(-1/2)^{m-2r}$ is $(-1)^m$. Formula (31) can therefore be written as

$$K_m(T,N,\mathscr{T}) = \frac{(-1)^m}{T}\sum_{r=0}^{\lfloor m/2\rfloor} a_{m,r} S_{m-r}(T,N,T)\,\mathrm{e}^{-2(m-r)\kappa L}.$$

Every term inside the sum is strictly positive. Hence, K_m is positive when m is even and negative when m is odd. Differentiating with respect to $\mathscr{T}$, which is the same as differentiating with respect to L, gives

$$\frac{\partial K_m}{\partial \mathscr{T}} = \frac{(-1)^{m+1}2\kappa}{T}\sum_{r=0}^{\lfloor m/2\rfloor}(m-r) a_{m,r} S_{m-r}(T,N,T)\,\mathrm{e}^{-2(m-r)\kappa L}.$$

The sum is strictly positive. Thus, the derivative is negative for even m and positive for odd m. Finally, every exponential term converges to zero as $L \to \infty$, so $K_m(T,N,\mathscr{T}) \to 0$. □

The effect of increasing the number of equally spaced observations is examined next while the swap maturity and the futures delivery date remain fixed. This analysis is important because raw realized moments of different orders do not have the same limiting behavior. Let $\Delta = T/N$. The next theorem gives the exact leading orders as Δ tends to zero.

Theorem 9 *For each integer $p \geqslant 1$, define*

$$C_p(T,\mathscr{T}) = \frac{\sigma^{2p}}{2p\kappa}\,\mathrm{e}^{-2p\kappa(\mathscr{T}-T)}\left(1-\mathrm{e}^{-2p\kappa T}\right). \tag{42}$$

As $N \to \infty$, or equivalently $\Delta = T/N \to 0$,

$$S_p(T,N,\mathscr{T}) = C_p(T,\mathscr{T})\Delta^{p-1} + O\left(\Delta^p\right). \tag{43}$$

Consequently, for every integer $q \geqslant 1$,

$$K_{2q}(T,N,\mathscr{T}) = \frac{(2q)!}{q!\,2^q\,T} C_q(T,\mathscr{T})\Delta^{q-1} + O\left(\Delta^q\right) \tag{44}$$

and

$$K_{2q+1}(T,N,\mathscr{T}) = -\frac{(2q+1)!}{q!\,2^{q+1}\,T} C_{q+1}(T,\mathscr{T})\Delta^q + O\left(\Delta^{q+1}\right). \tag{45}$$

Proof: Fix T, $\mathscr{T}$, κ, σ, and p. From (30),

$$S_p = \left(\frac{\sigma^2}{2\kappa}\right)^p \frac{\left(1-\mathrm{e}^{-2\kappa\Delta}\right)^p}{1-\mathrm{e}^{-2p\kappa\Delta}}\,\mathrm{e}^{-2p\kappa(\mathscr{T}-T)}\left(1-\mathrm{e}^{-2p\kappa T}\right).$$

Taylor expansion at $\Delta = 0$ gives

$$1-\mathrm{e}^{-2\kappa\Delta} = 2\kappa\Delta\left(1-\kappa\Delta+O(\Delta^2)\right)$$

and

$$1-\mathrm{e}^{-2p\kappa\Delta} = 2p\kappa\Delta\left(1-p\kappa\Delta+O(\Delta^2)\right).$$

Consequently,

$$\begin{aligned}\frac{\left(1-\mathrm{e}^{-2\kappa\Delta}\right)^p}{1-\mathrm{e}^{-2p\kappa\Delta}} &= \frac{(2\kappa\Delta)^p\,(1+O(\Delta))}{2p\kappa\Delta\,(1+O(\Delta))} \\ &= \frac{(2\kappa)^{p-1}}{p}\Delta^{p-1}\,(1+O(\Delta)).\end{aligned}$$

Multiplication by $(\sigma^2/(2\kappa))^p$ changes the leading constant to $\sigma^{2p}/(2p\kappa)$. The remaining factors do not depend on Δ. Therefore,

$$S_p = C_p(T,\mathscr{T})\Delta^{p-1}\,(1+O(\Delta)) = C_p(T,\mathscr{T})\Delta^{p-1} + O(\Delta^p),$$

which proves (43).

Now consider (31). For $m = 2q$, the index r ranges from 0 to q, and the power sum in the rth term is S_{2q-r}. Its order is

$$S_{2q-r} = O\left(\Delta^{2q-r-1}\right).$$

The smallest exponent occurs at $r = q$, where the term is $S_q = C_q\Delta^{q-1} + O(\Delta^q)$. Its coefficient is

$$\frac{(2q)!}{0!\,q!\,2^q}\left(-\frac{1}{2}\right)^0 = \frac{(2q)!}{q!\,2^q}.$$

The next possible power sum is $S_{q+1} = O(\Delta^q)$, so every other term is absorbed into the $O(\Delta^q)$ remainder. This proves (44).

For $m = 2q+1$, the largest index is again $r = q$, and the corresponding power sum is $S_{q+1} = C_{q+1}\Delta^q + O(\Delta^{q+1})$. Its coefficient is

$$\frac{(2q+1)!}{1!q!2^q}\left(-\frac{1}{2}\right) = -\frac{(2q+1)!}{q!2^{q+1}}.$$

Every other term contains S_{q+2} or a higher power sum and is therefore $O(\Delta^{q+1})$. This proves (45). □

The second-order fair delivery price has a nonzero continuous-sampling limit, whereas all fair delivery prices of order three or higher converge to zero under the normalization in (26). The leading terms for the first three contracts are recorded below.

Corollary 7 *As $N \to \infty$,*

$$K_2(T,N,\mathscr{T}) = \frac{1}{T}\int_0^T \sigma^2 \mathrm{e}^{-2\kappa(\mathscr{T}-s)}\,ds + O(\Delta) = \frac{\sigma^2}{2\kappa T}\mathrm{e}^{-2\kappa(\mathscr{T}-T)}\left(1-\mathrm{e}^{-2\kappa T}\right) + O(\Delta), \quad (46)$$

$$K_3(T,N,\mathscr{T}) = -\frac{3\sigma^4}{8\kappa T}\mathrm{e}^{-4\kappa(\mathscr{T}-T)}\left(1-\mathrm{e}^{-4\kappa T}\right)\Delta + O\left(\Delta^2\right), \quad (47)$$

and

$$K_4(T,N,\mathscr{T}) = \frac{3\sigma^4}{4\kappa T}\mathrm{e}^{-4\kappa(\mathscr{T}-T)}\left(1-\mathrm{e}^{-4\kappa T}\right)\Delta + O\left(\Delta^2\right). \quad (48)$$

Proof: For the second-order contract, set $q = 1$ in (44). Since $(2q)!/(q!2^q) = 1$ when $q = 1$, the result is

$$K_2(T,N,\mathscr{T}) = \frac{C_1(T,\mathscr{T})}{T} + O(\Delta).$$

The definition in (42) gives

$$C_1(T,\mathscr{T}) = \frac{\sigma^2}{2\kappa}\mathrm{e}^{-2\kappa(\mathscr{T}-T)}\left(1-\mathrm{e}^{-2\kappa T}\right).$$

On the other hand, direct integration yields

$$\int_0^T \sigma^2 \mathrm{e}^{-2\kappa(\mathscr{T}-s)}\,ds = \sigma^2\mathrm{e}^{-2\kappa\mathscr{T}}\int_0^T \mathrm{e}^{2\kappa s}\,ds = \frac{\sigma^2}{2\kappa}\mathrm{e}^{-2\kappa\mathscr{T}}\left(\mathrm{e}^{2\kappa T}-1\right) = \frac{\sigma^2}{2\kappa}\mathrm{e}^{-2\kappa(\mathscr{T}-T)}\left(1-\mathrm{e}^{-2\kappa T}\right) = C_1(T,\mathscr{T}).$$

This proves both equalities in (46).

For the third-order contract, set $q = 1$ in (45). The coefficient is

$$\frac{(2q+1)!}{q!2^{q+1}T} = \frac{3!}{1!2^2T} = \frac{3}{2T}.$$

Moreover,

$$C_2(T,\mathscr{T}) = \frac{\sigma^4}{4\kappa}\mathrm{e}^{-4\kappa(\mathscr{T}-T)}\left(1-\mathrm{e}^{-4\kappa T}\right).$$

Therefore,

$$-\frac{3}{2T}C_2(T,\mathscr{T})\Delta = -\frac{3\sigma^4}{8\kappa T}\mathrm{e}^{-4\kappa(\mathscr{T}-T)}\left(1-\mathrm{e}^{-4\kappa T}\right)\Delta,$$

which proves (47), including the $O(\Delta^2)$ remainder supplied by (45).

For the fourth-order contract, set $q = 2$ in (44). The coefficient is

$$\frac{(2q)!}{q!2^q T} = \frac{4!}{2!2^2T} = \frac{3}{T}.$$

Substitution of the same value of $C_2(T,\mathscr{T})$ gives

$$\frac{3}{T}C_2(T,\mathscr{T})\Delta = \frac{3\sigma^4}{4\kappa T}\mathrm{e}^{-4\kappa(\mathscr{T}-T)}\left(1-\mathrm{e}^{-4\kappa T}\right)\Delta,$$

which proves (48), with the $O(\Delta^2)$ remainder supplied by (44). □

The limits above distinguish the second-order contract from higher raw-moment contracts. The second-order contract approaches the annualized integrated variance of the fixed-maturity futures return over $[0,T]$, namely the integrated variance divided by T. By contrast, the third- and fourth-order contracts decrease at rate $1/N$, and higher orders decrease at the rates given in Theorem 9. These results support the numerical comparison of exact fair delivery prices with their leading high-frequency approximations.

PARAMETER IDENTIFICATION FROM MULTIPLE DELIVERY DATES

The maturity dependence of the second-order contract provides a direct route to identifying the mean-reversion speed and volatility. Define the continuous-sampling second-order price by

$$K_2^{\mathrm{cs}}(T,\mathscr{T};\kappa,\sigma) = \frac{\sigma^2}{2\kappa T}\mathrm{e}^{-2\kappa(\mathscr{T}-T)}\left(1-\mathrm{e}^{-2\kappa T}\right). \quad (49)$$

Theorem 10 *Let $T > 0$ and $T \leqslant \mathscr{T}_1 < \mathscr{T}_2$. Suppose that positive fair delivery prices k_1 and k_2 are observed and are required to satisfy*

$$k_j = K_2^{\mathrm{cs}}(T,\mathscr{T}_j;\kappa,\sigma), \quad j = 1,2,$$

for $\kappa > 0$ and $\sigma > 0$. Such a parameter pair exists if and only if $k_1 > k_2$. When it exists, it is unique and is given by

$$\kappa = \frac{\log(k_1/k_2)}{2(\mathscr{T}_2-\mathscr{T}_1)}, \quad \sigma^2 = \frac{2\kappa T k_1 e^{2\kappa(\mathscr{T}_1-T)}}{1-\mathrm{e}^{-2\kappa T}}. \quad (50)$$

Proof: Dividing the two pricing equalities eliminates every factor except the delivery-date term and gives

$$\frac{k_1}{k_2} = \mathrm{e}^{2\kappa(\mathscr{T}_2-\mathscr{T}_1)}.$$

A positive value of κ therefore exists exactly when $k_1 > k_2$, and taking logarithms gives the first formula in (50). Substitution into the equality for k_1 gives the formula for σ^2. Every factor in that formula is positive, so the resulting σ is positive and unique. If $k_1 \leqslant k_2$, the strict decrease with respect to the futures delivery date established in Corollary 6 rules out any $\kappa > 0$. □

When a finite observation grid is retained, the exact second-order formula in (32) replaces (49). For observed fair delivery prices $\widehat{K}_{2,j}$ at $J \geqslant 2$ delivery dates and positive weights w_j, a finite-grid nonlinear least-squares estimate is

$$(\widehat{\kappa}, \widehat{\sigma}) = \underset{\kappa>0,\sigma>0}{\arg\min} \sum_{j=1}^{J} w_j \left[\widehat{K}_{2,j} - K_2(T, N, \mathscr{T}_j; \kappa, \sigma)\right]^2. \quad (51)$$

Here, K_2 is evaluated from (32) and the closed-form power sums in (30). Equal weights give an unweighted fit, whereas inverse error-variance weights account for heterogeneous price precision. Positive parameter constraints may be imposed directly or through a logarithmic parameterization.

The explicit two-delivery-date formula is useful for identification, but its numerical sensitivity depends on the spacing of the delivery dates and on the relative precision of the two observed prices. To see this directly, suppose that the observed prices are perturbed to $\widetilde{k}_j = k_j(1+\varepsilon_j)$, $j = 1, 2$, with $1+\varepsilon_j > 0$. The estimate obtained from (50) then satisfies

$$\widetilde{\kappa} - \kappa = \frac{\log(1+\varepsilon_1) - \log(1+\varepsilon_2)}{2(\mathscr{T}_2 - \mathscr{T}_1)}. \quad (52)$$

For small relative perturbations,

$$\widetilde{\kappa} - \kappa \approx \frac{\varepsilon_1 - \varepsilon_2}{2(\mathscr{T}_2 - \mathscr{T}_1)},$$

and hence

$$\frac{\widetilde{\kappa} - \kappa}{\kappa} \approx \frac{\varepsilon_1 - \varepsilon_2}{\log(k_1/k_2)}.$$

Thus, closely spaced delivery dates can amplify pricing errors because the denominator $2(\mathscr{T}_2 - \mathscr{T}_1)$ is small. The relative sensitivity is also large when k_1/k_2 is close to one. In practical estimation, the delivery dates should therefore be sufficiently separated to reveal the maturity decay, while maturities so distant that the second-order prices become very small should be avoided when price precision deteriorates. When several delivery dates are available, the finite-grid fit in (51) uses more information and reduces reliance on any single price ratio.

NUMERICAL RESULTS

This section provides numerical verification and illustrations of the analytical results. The first experiment checks the moment calculations by simulating directly from the exact Gaussian futures-return law. A second Monte Carlo check starts from the trending Ornstein–Uhlenbeck state process and constructs the futures prices before computing the returns. The remaining experiments examine the effects of the futures delivery date, risk-neutral payoff variances and covariances, sampling-frequency behavior, and parameter recovery. Unless stated otherwise, time is measured in years, so κ has unit year^{-1} and σ has unit year$^{-1/2}$. The values $\kappa = 0.099$ and $\sigma = 0.129$ used in the representative numerical experiments are adopted from the numerical example of Chumpong et al [20] for comparison and illustration. They are not obtained from a new calibration in this paper.

The first Monte Carlo experiment is intended to verify the analytical moment calculations conditional on the exact return law in Theorem 6. It does not independently verify the derivation of that return law. Let $\{X_i(\mathscr{T}; \omega_\ell)\}_{i=1}^{N}$ denote the simulated futures log returns in replication ℓ, where $\ell = 1, 2, \ldots, N_{\mathrm{MC}}$, N is the number of monitoring intervals, and N_{MC} is the number of Monte Carlo replications. The simulated returns and the corresponding estimator are

$$X_i(\mathscr{T}; \omega_\ell) = -\frac{1}{2} v_i(\mathscr{T}) + \sqrt{v_i(\mathscr{T})}\, \xi_{i,\ell}, \quad \xi_{i,\ell} \overset{\text{iid}}{\sim} N(0,1),$$

and

$$\widehat{K}_m = \frac{1}{N_{\mathrm{MC}}} \sum_{\ell=1}^{N_{\mathrm{MC}}} \left(\frac{1}{T} \sum_{i=1}^{N} X_i(\mathscr{T}; \omega_\ell)^m \right), \quad m = 2, 3, 4.$$

The numerical accuracy is assessed through the percentage relative error

$$\mathrm{PRE} = \frac{\left|\widehat{K}_m - K_m\right|}{|K_m|} \times 100\%.$$

Because a relative error can be unstable when K_m is close to zero, the mean absolute error (MAE) across the parameter grid is also reported. Here, MAE is the average of $|\widehat{K}_m - K_m|$ over the parameter grid. The Monte Carlo standard error at a fixed parameter point is the sample standard deviation of the realized moment divided by $\sqrt{N_{\mathrm{MC}}}$.

For Fig. 1, $T = 1$, $N = 252$, and $\mathscr{T} = 1.5$ are fixed. The manuscript benchmark grid is $\kappa = 0.025, 0.050, \ldots, 0.200$ and $\sigma = 0.05, 0.10, \ldots, 0.30$, giving 48 parameter pairs. The error statistics in Table 1 are computed on this 48-point benchmark grid. For the plotted Fig. 1, the σ grid is refined to $0.05, 0.075, 0.10, \ldots, 0.30$, giving 88 displayed parameter pairs, while the pricing formulas, $N_{\mathrm{MC}} = 50\,000$, random seed 20260926, and common-random-number construction are unchanged. Fig. 1(a–c) compares the analytical prices with Monte Carlo estimates generated from the exact return law. The PRE for the third-order contract is larger because K_3 is close to zero over part of the grid, while its MAE remains small in absolute value.

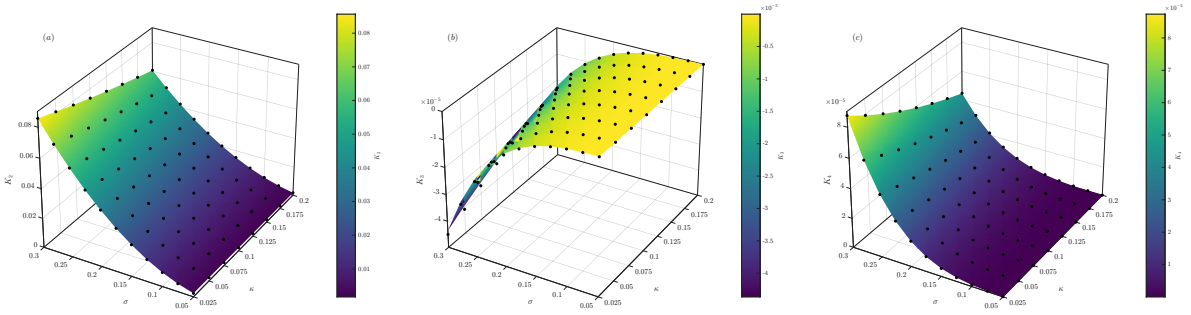

Fig. 1 Comparison between analytical prices (surfaces) and Monte Carlo estimates (markers) for futures-based moment swaps. Panels (a–c) correspond to the second-, third-, and fourth-order contracts, respectively.

Table 1 Error statistics for the Monte Carlo verification based on the exact return law.

Moment swap	Mean PRE (%)	Max PRE (%)	MAE
K_2	0.005576	0.008467	1.652×10^{-6}
K_3	8.857087	24.065600	3.861×10^{-7}
K_4	0.099507	0.118358	1.811×10^{-8}

To report simulation uncertainty directly, a representative comparison is made at $\kappa = 0.099$, $\sigma = 0.129$, $T = 1$, $N = 252$, and $\mathscr{T} = 1.5$. The simulation from the exact return law uses the same $N_{\mathrm{MC}} = 50\,000$ and random seed 20260926. The simulation from the original log-price process uses random seed 20260927 and the nonconstant trend $\mu(t) = 2.5 + 0.05t$, with $Y_0 = \alpha(0)$. Standard-normal innovations are generated with NumPy's default PCG64 generator as an $N_{\mathrm{MC}} \times N$ array, with each row corresponding to one simulated path. For each replication, the exact transition law in Lemma 1 is used on the observation grid, the futures prices are then constructed from (19), and the realized moments are computed from the resulting futures returns. This second experiment tests the construction of futures prices and returns from the original log-price process in addition to the final moment calculation. Table 2 reports both sets of estimates and their Monte Carlo standard errors. In every case, the analytical value lies inside the corresponding 95% interval formed as the estimate plus or minus 1.96 standard errors.

The next experiment investigates the effect of the futures delivery date $\mathscr{T}$ on the fair delivery prices of the second-, third-, and fourth-order futures-based moment swaps. The analytical prices in (32)–(34) are evaluated across delivery dates through the closed-form power sums in (30). For Fig. 2, $\kappa = 0.099$, $\sigma = 0.129$, $T = 1$, and $N = 252$ are fixed, while $\mathscr{T}$ ranges from 1 to 20.

Fig. 2(a–c) illustrates the dependence of the fair delivery prices on the futures delivery date. As the delivery date increases, the second- and fourth-order moment swap prices decrease monotonically, whereas the third-order moment swap remains negative and gradually approaches zero.

These observations follow from the variance increment $v_i(\mathscr{T})$, which decreases as the futures delivery date moves further away from the monitoring period. Because the analytical prices are sums of powers of $v_i(\mathscr{T})$, a smaller variance increment produces lower second- and fourth-order prices and reduces the magnitude of the negative third-order price, consistently with Corollary 6.

Overall, the results demonstrate that the futures delivery date has a significant influence on the magnitude of the fair delivery prices, while preserving the qualitative characteristics of each moment swap.

The effect of the futures delivery date is also visible in the risk-neutral payoff variances and covariances. Table 3 reports the variances of the second-, third-, and fourth-order realized moments together with the corresponding cross-order correlations for selected delivery dates. The values are calculated directly from (40). All three variances decrease as the delivery date moves away from the monitoring horizon. The second- and fourth-order realized moments remain positively correlated, while the correlations involving the third-order realized moment are negative under the present risk-neutral Gaussian model.

The sampling-frequency experiment investigates the effect of the number of monitoring intervals on the fair delivery prices. The exact pricing formula in Theorem 7 is evaluated for

$$N = 12,\ 24,\ 52,\ 126,\ 252, \text{ and } 504,$$

while the corresponding leading-order asymptotic approximations are computed from Corollary 7. For this experiment, $\kappa = 0.099$, $\sigma = 0.129$, $T = 1$, and $\mathscr{T} = 1.5$ are fixed.

Fig. 3(a–c) compares the exact fair delivery prices with their leading-order asymptotic approximations for the second-, third-, and fourth-order futures-based moment swaps. As the number of monitoring intervals increases, the exact second-order price converges toward the nonzero continuous-sampling limit predicted by (46). In contrast, the third- and fourth-order prices converge to zero at rate $1/N$, from below and

Table 2 Monte Carlo estimates at the representative parameter values. Standard errors are shown in parentheses.

Order	Exact price	Exact-return-law simulation	Log-price-process simulation
2	1.3674272×10^{-2}	1.3674970×10^{-2} (5.4444×10^{-6})	1.3679370×10^{-2} (5.4463×10^{-6})
3	$-1.1166187 \times 10^{-6}$	$-1.2086996 \times 10^{-6}$ (1.1067×10^{-7})	$-1.2043844 \times 10^{-6}$ (1.1031×10^{-7})
4	2.2332881×10^{-6}	2.2354423×10^{-6} (2.0677×10^{-9})	2.2339489×10^{-6} (2.0541×10^{-9})

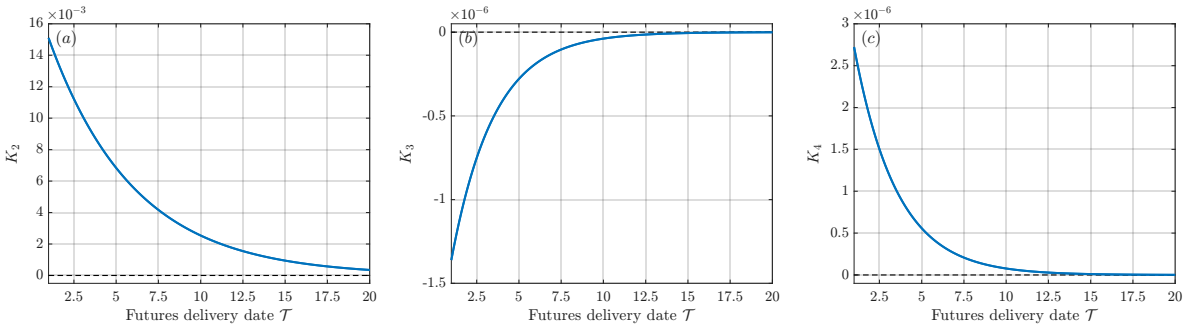

Fig. 2 Maturity effects on the fair delivery prices of futures-based moment swaps. Panels (a–c) correspond to the second-, third-, and fourth-order contracts, respectively. The dashed horizontal line indicates zero.

Table 4 Percentage relative errors of the leading-order asymptotic approximations.

N	K_2 (%)	K_3 (%)	K_4 (%)
12	0.028572	0.007288	0.055047
24	0.014288	0.004211	0.028099
52	0.006595	0.002085	0.013111
126	0.002722	0.000890	0.005441
252	0.001361	0.000450	0.002726
504	0.000680	0.000226	0.001364

Table 5 Numerical recovery of the mean-reversion and volatility parameters.

Method	Price perturbation	$\widehat{\kappa}$	$\widehat{\sigma}$
Two delivery dates	none	0.099000	0.129000
Two delivery dates	$+0.1\%, -0.1\%$	0.100000	0.129127
Finite-grid NLS	Gaussian, SD 0.1%	0.099074	0.129034

above, respectively, in agreement with the asymptotic expansions in (47) and (48).

The corresponding percentage relative errors are reported in Table 4, while their convergence behavior is illustrated in Fig. 3(d). The PREs decrease monotonically for all three moment orders as the sampling frequency increases, confirming that the leading-order asymptotic approximations provide increasingly accurate estimates of the exact fair delivery prices. At $N = 504$, all PREs are below 0.002%, providing numerical support for the asymptotic results established in Theorem 9 and Corollary 7.

The final numerical experiment examines parameter recovery. First, the continuous-sampling formula (50) is applied with the true values $\kappa = 0.099$ and $\sigma = 0.129$ and delivery dates $\mathcal{T}_1 = 1$ and $\mathcal{T}_2 = 2$. With exact prices, the two parameters are recovered to numerical precision. To illustrate sensitivity, the first price is then increased by 0.1% and the second is decreased by 0.1%. The estimated value of κ becomes 0.1000000, which has a relative error of about 1.01%, while the estimated value of σ becomes 0.1291269. This result is consistent with (52). Second, the finite-grid estimator (51) is applied with $N = 252$ and six delivery dates $\mathcal{T} = 1, 1.5, 2, 2.5, 3, 4$. The perturbed observations are generated as $\widehat{K}_{2,j} = K_2(T, N, \mathcal{T}_j)(1 + \varepsilon_j)$, where the ε_j are independent $N(0, 0.001^2)$ random variables generated with seed 20260926. Equal weights are then used in the nonlinear least-squares fit. The resulting estimates are close to the true values, as reported in Table 5.

For the noisy two-delivery-date experiment, the

Table 3 Risk-neutral variances and cross-order correlations for selected futures delivery dates.

$\mathcal{T}$	$\mathrm{Var}(M^{(2)})$	$\mathrm{Var}(M^{(3)})$	$\mathrm{Var}(M^{(4)})$	ρ_{23}	ρ_{24}	ρ_{34}
1.0	1.815×10^{-6}	8.208×10^{-10}	3.178×10^{-13}	−0.00851	0.86464	−0.00984
1.5	1.489×10^{-6}	6.099×10^{-10}	2.139×10^{-13}	−0.00810	0.86464	−0.00936
2.0	1.221×10^{-6}	4.532×10^{-10}	1.439×10^{-13}	−0.00770	0.86464	−0.00891
3.0	8.220×10^{-7}	2.502×10^{-10}	6.519×10^{-14}	−0.00698	0.86464	−0.00807
5.0	3.723×10^{-7}	7.627×10^{-11}	1.337×10^{-14}	−0.00573	0.86464	−0.00662

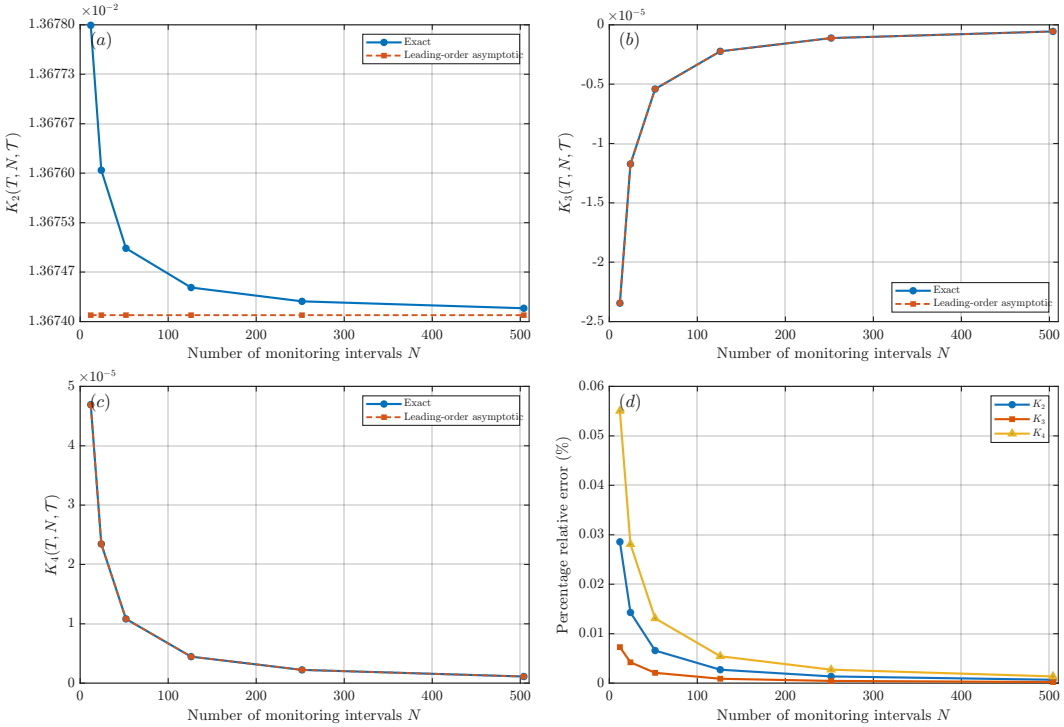

Fig. 3 Sampling-frequency effects on the fair delivery prices of futures-based moment swaps. Panels (a–c) compare the exact fair delivery prices with the corresponding leading-order asymptotic approximations for the second-, third-, and fourth-order moment swaps, respectively. Panel (d) presents the percentage relative errors (PREs) of the leading-order asymptotic approximations as functions of the number of monitoring intervals.

relative errors in κ and σ are 1.0101% and 0.0984%, respectively. For the finite-grid fit, the corresponding errors are 0.0751% and 0.0263%.

These numerical experiments support the analytical formulas from several directions. The simulation from the exact return law verifies the Gaussian moment calculations, the simulation from the original log-price process checks the construction of futures returns from the underlying model, the payoff-risk calculations illustrate the dependence of the risk-neutral variances and covariances on the futures delivery date, and the recovery experiment shows how prices across futures maturities can be used to estimate κ and σ. The sensitivity results also show why several well-separated delivery dates are preferable when market prices contain measurement error.

CONCLUSION

This paper studies moment swaps written on fixed-maturity commodity futures under a trending Ornstein–Uhlenbeck spot model. The futures delivery date is allowed to differ from the moment-swap maturity. The model-implied futures volatility is deterministic and equal to $\sigma\,\mathrm{e}^{-\kappa(\mathcal{T}-t)}$. As a result, nonoverlapping futures log returns are independent Gaussian random variables with explicit means and variances.

The deterministic trend affects the level of the futures price but cancels from the futures return law. This property reduces the moment-swap pricing problem to normal moments. The resulting pricing formula is consistent with earlier deterministic-volatility futures-based moment-swap calculations, while the separate delivery date in the present model makes it possible to study the maturity structure directly. Closed-form formulas are obtained for general integer orders, together with explicit second-, third-, and fourth-order prices, risk-neutral payoff variances and cross-order covariances, and sampling-frequency asymptotics.

The dependence on the futures delivery date also provides information about the model parameters. In the continuous-sampling limit, second-order prices at two delivery dates identify κ and σ explicitly. For a finite observation grid, the exact prices lead to a nonlinear least-squares estimator based on several delivery dates. The numerical results show the expected sensitivity when two delivery dates are close and illustrate the benefit of using several futures delivery dates.

The analysis is based on a one-factor Gaussian model with deterministic volatility after the futures transformation. It does not include stochastic volatility, jumps, or an independent stochastic convenience-yield factor. The payoff variances and covariances are also risk-neutral quantities rather than physical-measure

forecasts. Extensions to richer commodity models would allow the separate effects of non-Gaussian returns and additional state variables to be studied while retaining the distinction between the swap maturity and the futures delivery date.

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