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A novel generalization of (λ,μ)-Bernstein-Kantorovich operators and their approximation properties

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ABSTRACT: Using the shape parameter λ is of great interest nowadays. This parameter provides considerable flexibility, and improves the modeling possibilities in approximation theory. This paper mainly introduces a novel generalized (λ,μ)-Bernstein-Kantorovich operators. For these operators, we first examine the order of approximation using a classical approach, Lipschitz class, and the second modulus of continuity. Finally, to compare the convergence of such operators both to version of the (λ,μ)-Bernstein-Kantorovich operators and to themselves for different parameters, we provide some graphical and numerical examples.

KEYWORDS: λ-Bernstein operators, (λ,μ)-Bernstein-Kantorovich operators, Lipschitz continuous function, rate of convergence

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INTRODUCTION

Many improvements in CAGD have depended on the Bézier basis functions and their modifications (see for the details [1,2]). In a book [2], Farin provided many facts, discussions and historical backgrounds about how Bézier basis functions, Bézier curves and surfaces emerge. Bézier basis functions also gave a chance to contemplate an interdisciplinary approach. They have evolved into effective and essential tools not only in CAGD but also in Approximation Theory.

Researchers are currently very interested in creating new types of Bézier curves and Bernstein operators by using different mathematical bases. For example, in 2010, Ye et al published a paper [3] where they not only developed a new set of basis functions for Bézier curves that included a single shape parameter, but they also presented a practical algorithm for modeling these curves. They went on to detail the key characteristics of these basis functions and their corresponding curves. Ultimately, their work clarified why curves and surfaces that incorporate shape parameters are a valuable choice in design and modeling.

In 2018, inspired by a paper [3], Cai et al [4] introduced a new generalization of Bernstein operators as follows

$$B_{m,\lambda}(c;\kappa)=\sum_{\zeta=0}^{m}\tilde{b}_{m,\zeta}(\lambda;\kappa)c\left(\frac{\zeta}{m}\right), \tag{1}$$

where $\lambda\in[-1,1]$ and $\tilde{b}_{m,\zeta}$ Bézier basis for $\zeta=0,1,\ldots,$ are defined by

$$\begin{aligned}\tilde{b}_{m,0}(\lambda;\kappa)&=b_{m,0}(\kappa)-\frac{\lambda}{m+1}b_{m+1,1}(\kappa),\\ \tilde{b}_{m,i}(\lambda;\kappa)&=b_{m,i}(\kappa)+\lambda\bigg(\frac{m-2i+1}{m^2-1}b_{m+1,i}(\kappa)\\ &\qquad-\frac{m-2i-1}{m^2-1}b_{m+1,i+1}(\kappa)\bigg),\\ \tilde{b}_{m,m}(\lambda;\kappa)&=b_{m,m}(\kappa)-\frac{\lambda}{m+1}b_{m+1,m}(\kappa)\end{aligned} \tag{2}$$

with $b_{m,\zeta}(\kappa)=\binom{m}{\zeta}\kappa^{\zeta}(1-\kappa)^{m-\zeta}$. These operators become the classical Bernstein operators [5] for $\lambda=0$. There are several papers related to these operators and their generalizations. To mention a few, we firstly refer to [6–13].

In 2024, Zhou and Cai [14] introduced (λ,μ)-Bernstein operators as

$$B_m^{\lambda,\mu}(c;\kappa)=\sum_{\zeta=0}^{m}b_{m,\zeta}^{\lambda,\mu}(\kappa)c\left(\frac{\zeta}{m}\right), \tag{3}$$

where $b_{m,\zeta}^{\lambda,\mu}(\kappa)$ are the (λ,μ) Bézier bases functions given by

$$b_{m,\zeta}^{\lambda,\mu}(\kappa)=b_{m,\zeta}(\kappa)+\Lambda_\zeta b_{m+1,\zeta}(\kappa)-\Lambda_{\zeta+1}b_{m+1,\zeta+1}(\kappa), \tag{4}$$

for $\zeta=0,1,\ldots,m$ with

$$\begin{cases}\Lambda_0=\Lambda_{m+1}=0,\\ \Lambda_\zeta=(1-\mu)\frac{m-2\zeta+1}{2(m^2-1)}+\frac{1+\mu}{2(m+1)}\lambda,\ \zeta=1,2,\ldots,m,\end{cases} \tag{5}$$

$\lambda \in [-1,1]$ and μ is a constant between 1 and m. Some articles can be given about related work [15–19]. Also, in 2024, Cai proposed the (λ,μ)-Bernstein-Kantorovich operators as

$$\mathscr{K}_m^{\lambda,\mu}(c;\kappa)=\sum_{\zeta=0}^{m} b_{m,\zeta}^{\lambda,\mu}(\kappa)(m+1)\int_{\frac{\zeta}{m+1}}^{\frac{\zeta+1}{m+1}} c(t)\,\mathrm{d}t. \quad (6)$$

In 2017, Altomare et al [20] introduced a new sequence of positive linear operators depending on a given Markov operator, a positive real number, and a sequence of Borel probability measures. They generalized classical Kantorovich operators and provided a special case of generalized Kantorovich operators as

$$C_{m,a}(c;\kappa)=\sum_{\zeta=0}^{m} b_{m,\zeta}(\kappa)\int_{0}^{1} c\left(\frac{\zeta+as}{m+a}\right)\mathrm{d}s \quad (7)$$

for a fixed real number $a \geqslant 0$. In particular, if $a=1$, the given operators reduce to the classical Kantorovich operators [21] and $a=0$, the operators turn into the classical Bernstein operators [5]. This parameter a provides considerable flexibility, and improves approximation results. Eq. (7) can also be written as

$$C_{m,a}(c;\kappa)=\sum_{\zeta=0}^{m} b_{m,\zeta}(\kappa)\left(\frac{m+a}{a}\right)\int_{\frac{\zeta}{m+a}}^{\frac{\zeta+a}{m+a}} c(t)\,\mathrm{d}t \quad (8)$$

for $a>0$.

Inspired by these ideas, we introduce the novel (λ,μ)-Bernstein-Kantorovich operators, for $m\in\mathbb{N}$, $\lambda\in[-1,1]$ and μ is a constant between 1 and m as

$$G_{m,a}^{\lambda,\mu}(c;\kappa)=\sum_{\zeta=0}^{m} b_{m,\zeta}^{\lambda,\mu}(\kappa)\int_{0}^{1} c\left(\frac{\zeta+as}{m+a}\right)\mathrm{d}s, \quad (9)$$

where $a\geqslant 0$, $m\in\mathbb{N}$, $\kappa\in[0,1]$, $c\in C[0,1]$ and $b_{m,\zeta}^{\lambda,\mu}(\kappa)$ are given by (4). $G_{m,a}^{\lambda,\mu}$ is a positive linear operator from $C[0,1]$ into $C[0,1]$. For the case $a=1$, the introduced operators reduce to the (λ,μ)-Bernstein-Kantorovich operators (6) and for the case $a=0$, the operators give the (λ,μ)-Bernstein operators (3). (9) can also be defined as

$$G_{m,a}^{\lambda,\mu}(c;\kappa)=\sum_{\zeta=0}^{m} b_{m,\zeta}^{\lambda,\mu}(\kappa)\left(\frac{m+a}{a}\right)\int_{\frac{\zeta}{m+a}}^{\frac{\zeta+a}{m+a}} c(t)\,\mathrm{d}t \quad (10)$$

for $a>0$.

The anatomy of the paper is as follows: Preliminary results dedicate to the fundamental lemmas and remarks that are useful for proving the main theorems. Then, we investigate the rate of convergence results. The last section presents some examples and numerical results that provide some comparisons concerning the convergence of the operators to better understanding.

PRELIMINARY RESULTS

This section provides a quick overview on basic lemmas that will be necessary to prove the main results presented in the paper.

Lemma 1 *Let $e_r(t):=t^r$, $r=0,1,2$ and $m>1$. Then, $G_{m,a}^{\lambda,\mu}$ satisfy the following relations:*

$$G_{m,a}^{\lambda,\mu}(e_0;\kappa)=1,$$

$$\begin{aligned}G_{m,a}^{\lambda,\mu}(e_1;\kappa)&=\frac{2m\kappa+a}{2(m+a)}+(1-\mu)\frac{1-2\kappa+\kappa^{m+1}-(1-\kappa)^{m+1}}{2(m+a)(m-1)}\\&\quad+(1+\mu)\frac{1-\kappa^{m+1}-(1-\kappa)^{m+1}}{2(m+a)(m+1)}\lambda,\end{aligned}$$

$$\begin{aligned}G_{m,a}^{\lambda,\mu}(e_2;\kappa)&=\frac{3m\kappa^2(m-1)+3m\kappa(1+a)+a^2}{(m+a)^2}\\&\quad+(1-\mu)\Big(\frac{2m(\kappa-2\kappa^2+\kappa^{m+1})-1+\kappa^{m+1}}{2(m+a)^2(m-1)}\\&\quad+\frac{(1-\kappa)^{m+1}+a(1-2\kappa+\kappa^{m+1}-(1-\kappa)^{m+1})}{2(m+a)^2(m-1)}\Big)\\&\quad+(1+\mu)\Big(\frac{2(m+1)(\kappa-\kappa^{m+1})-1+\kappa^{m+1}}{2(m+a)^2(m+1)}\\&\quad+\frac{(1-\kappa)^{m+1}+a(1-\kappa^{m+1}-(1-\kappa)^{m+1})}{2(m+a)^2(m+1)}\Big)\lambda.\end{aligned}$$

The next lemma follows from the above one.

Lemma 2 *Let $\kappa\in[0,1]$, $\lambda\in[-1,1]$, μ is a constant between 1 and $m,(m>1)$, then the following relations are satisfied.*

$$\begin{aligned}G_{m,a}^{\lambda,\mu}(t-\kappa;\kappa)&=\frac{a(1-2\kappa)}{2(m+a)}+(1-\mu)\frac{1-2\kappa+\kappa^{m+1}-(1-\kappa)^{m+1}}{2(m+a)(m-1)}\\&\quad+(1+\mu)\frac{1-\kappa^{m+1}-(1-\kappa)^{m+1}}{2(m+a)(m+1)}\lambda:=\psi_{m,a}^{\lambda,\mu}(\kappa), \quad (11)\end{aligned}$$

$$\begin{aligned}G_{m,a}^{\lambda,\mu}((t-\kappa)^2;\kappa)&=\frac{m\kappa^2(2m-3)+m\kappa(3+2a)+a^2(\kappa^2-\kappa+1)}{(m+a)^2}\\&+(1-\mu)\Big(\frac{4a\kappa(\kappa-1)+a-1+(2m+1+a-2\kappa(m+a))\kappa^{m+1}}{2(m+a)^2(m-1)}\\&+\frac{(2\kappa(m+a)-a+1)(1-\kappa)^{m+1}}{2(m+a)^2(m-1)}\Big)\\&+(1+\mu)\Big(\frac{(1-a)(2\kappa-1)+(-2m-1-a+2\kappa(m+a))\kappa^{m+1}}{2(m+a)^2(m+1)}\\&+\frac{(1-a+2\kappa(m+a))(1-\kappa)^{m+1}}{2(m+a)^2(m+1)}\Big)\lambda:=\rho_{m,a}^{\lambda,\mu}(\kappa) \quad (12)\end{aligned}$$

Lemma 3 *Consider an arbitrary function $c\in C[0,1]$. We have*

$$\|G_{m,a}^{\lambda,\mu}(c;\cdot)\|\leqslant\|c\|.$$

Proof: From the definition of $G_{m,a}^{\lambda,\mu}$ and considering Lemma 1, one can easily prove Lemma 3. □

MAIN RESULTS

This section basically incorporates the main results of the paper. At first, we mention a few concepts quickly and begin to present the results then. Consider the following set:

$$W^2[0,1] := \{h \in C[0,1] : h'' \in C[0,1]\}.$$

Given $\delta > 0$ and $c \in C[0,1]$, the appropriate Peetre's K-functional (see [22]) is given by

$$K_2(c;\delta) = \inf_{h\in W^2[0,1]} \{\|c-h\|_{C[0,1]} + \delta\|h'\|_{C[0,1]} + \delta^2\|h''\|_{C[0,1]}\}.$$

In [23], the authors showed that we may find an absolute constant $C > 0$ such that

$$K_2(c;\delta) \leqslant C\omega_2(c;\sqrt{\delta}),$$

where ω_2 which denotes the second order modulus of continuity of c has the following definition:

$$\omega_2(c;\delta) = \sup_{0<|\upsilon|\leqslant\delta} \sup_{\kappa,\kappa+\upsilon,\kappa+2\upsilon\in[0,1]} |c(\kappa+2\upsilon) - 2c(\kappa+\upsilon) + c(\kappa)|.$$

The classical modulus of continuity of the function c is denoted by $\omega(c;\delta)$ and defined as

$$\omega(c;\delta) = \sup_{0<|\upsilon|\leqslant\delta} \sup_{\kappa,\kappa+\upsilon\in[0,1]} |c(\kappa+\upsilon) - c(\kappa)|.$$

Now, we are ready to give the first theorem of the paper.

Theorem 1 *Let $c \in C[0,1]$, $\lambda \in [-1,1]$. Then, $\lim_{m\to\infty} G_{m,a}^{\lambda,\mu}(c) = c$ uniformly on $[0,1]$.*

Proof: We establish the result by applying the Korovkin theorem [24] and using the first three moments of our operators. □

Theorem 2 *For a continuous function c on $[0,1]$, $\kappa \in [0,1]$, $\lambda \in [-1,1]$, μ is a constant between 1 and m, the following is satisfied:*

$$\left|G_{m,a}^{\lambda,\mu}(c;\kappa) - c(\kappa)\right| \leqslant 2\omega(c;\delta),$$

where $\delta := \delta_{m,a}^{\lambda,\mu}(\kappa) := \sqrt{\rho_{m,a}^{\lambda,\mu}(\kappa)}$ and $\rho_{m,a}^{\lambda,\mu}(\kappa)$ is defined in (12).

Proof: The well-known property of modulus of continuity allows us to deduce

$$\left|G_{m,a}^{\lambda,\mu}(c;\kappa) - c(\kappa)\right| \leqslant \left[1 + \frac{1}{\delta} G_{m,a}^{\lambda,\mu}(|t-\kappa|;\kappa)\right]\omega(c;\delta).$$

Applying Cauchy-Schwarz inequality together with Lemma 2 and Lemma 3 give

$$\left|G_{m,a}^{\lambda,\mu}(c;\kappa) - c(\kappa)\right| \leqslant \left[1 + \tfrac{1}{\delta}\left(G_{m,a}^{\lambda,\mu}((t-\kappa)^2;\kappa)\right)^{1/2}\right]\omega(c;\delta).$$

Now, choosing $\delta := \delta_{m,a}^{\lambda,\mu}(\kappa) := \sqrt{\rho_{m,a}^{\lambda,\mu}(\kappa)}$ yields the desired result. □

Theorem 3 *Take a function $c \in C[0,1]$ and a point $\kappa \in [0,1]$. Then, the following inequality is satisfied for an absolute constant $C > 0$:*

$$\left|G_{m,a}^{\lambda,\mu}(c;\kappa) - c(\kappa)\right| \leqslant C\omega_2\left(c;\sqrt{\delta}\right),$$

where $\delta := \delta_{m,a}^{\lambda,\mu}(\kappa) := \sqrt{\rho_{m,a}^{\lambda,\mu}(\kappa)}$.

Proof: Let us take $l \in W^2$ and $t \in [0,1]$. From Taylor's expansion, we write

$$l(t) - l(\kappa) = l'(\kappa)(t-\kappa) + \int_\kappa^t (t-s)\, l''(s)\,\mathrm{d}s.$$

Then, we have

$$G_{m,a}^{\lambda,\mu}(l;\kappa) - l(\kappa) = l'(\kappa) G_{m,a}^{\lambda,\mu}(t-\kappa;\kappa) + G_{m,a}^{\lambda,\mu}\left(\int_\kappa^t (t-s)\, l''(s)\,\mathrm{d}s;\kappa\right).$$

Applying Cauchy-Schwarz inequality, Lemma 2 and Lemma 3, one can get

$$\begin{aligned}
&|G_{m,a}^{\lambda,\mu}(l;\kappa) - l(\kappa)| \\
&\leqslant |l'(\kappa)||G_{m,a}^{\lambda,\mu}(t-\kappa;\kappa)| + G_{m,a}^{\lambda,\mu}\left(\left|\int_\kappa^t (t-s)\, l''(s)\,\mathrm{d}s\right|;\kappa\right) \\
&\leqslant \|l'\|\left(G_{m,a}^{\lambda,\mu}(|t-\kappa|;\kappa)\right) + \frac{\|l''\|}{2} G_{m,a}^{\lambda,\mu}((t-\kappa)^2;\kappa) \\
&\leqslant \|l'\|\left(G_{m,a}^{\lambda,\mu}((t-\kappa)^2;\kappa)\right)^{1/2} + \frac{\|l''\|}{2} G_{m,a}^{\lambda,\mu}((t-\kappa)^2;\kappa) \\
&\leqslant \|l'\|\delta + \frac{\|l''\|}{2}\delta^2. \qquad (13)
\end{aligned}$$

For $c \in C[0,1]$ and $l \in W^2$, using Lemma 3 and (13) yields

$$\begin{aligned}
&|G_{m,a}^{\lambda,\mu}(c;\kappa) - c(\kappa)| \\
&\quad\leqslant |G_{m,a}^{\lambda,\mu}(c-l;\kappa)| + |G_{m,a}^{\lambda,\mu}(l;\kappa) - l(\kappa)| + |c(\kappa) - l(\kappa)| \\
&\quad\leqslant 2\|c-l\| + \|l'\|\delta + \frac{\|l''\|}{2}\delta^2.
\end{aligned}$$

Bearing in mind that $l \in W^2$ and taking the infimum of the right hand side of the above inequality, we find

$$|G_{m,a}^{\lambda,\mu}(c;\kappa) - c(\kappa)| \leqslant 2K_2\left(c;\sqrt{\delta}\right).$$

Eventually, taking account of the relation between the second order modulus of continuity and the appropriate K-functional concludes the proof. □

Theorem 4 *For any $c \in C^1[0,1]$ (the space of continuously differentiable functions) and $\kappa \in [0,1]$, we have*

$$\begin{aligned}
&\left|G_{m,a}^{\lambda,\mu}(c;\kappa) - c(\kappa)\right| \\
&\quad\leqslant \left|c'(\kappa)\right| \left|\psi_{m,a}^{\lambda,\mu}(\kappa)\right| + 2\sqrt{\rho_{m,a}^{\lambda,\mu}(\kappa)}\,\omega\left(c', \sqrt{\rho_{m,a}^{\lambda,\mu}(\kappa)}\right),
\end{aligned}$$

where $\psi^{\lambda,\mu}_{m,a}(\kappa)$ *and* $\rho^{\lambda,\mu}_{m,a}(\kappa)$ *are defined in* (11) *and* (12)*, respectively.*

Proof: Let $c \in C^1[0,1]$. For any $\kappa, t \in [0,1]$, we have

$$c(t)-c(\kappa) = c'(\kappa)(t-\kappa) + \int_{\kappa}^{t} \left(c'(u)-c'(\kappa)\right) \mathrm{d}u.$$

Applying the operators on the both sides of the above relation, we get

$$G^{\lambda,\mu}_{m,a}(c(t)-c(\kappa);\kappa) = c'(\kappa)G^{\lambda,\mu}_{m,a}(t-\kappa;\kappa) + G^{\lambda,\mu}_{m,a}\left(\int_{\kappa}^{t}(c'(u)-c'(\kappa))\,\mathrm{d}u;\kappa\right). \quad (14)$$

Using the well-known property of modulus of continuity

$$|c(u)-c(\kappa)| \leqslant \omega(c;\delta)\left(\frac{|u-\kappa|}{\delta}+1\right), \quad \delta > 0$$

it follows

$$\left|\int_{\kappa}^{t} |c'(u)-c'(\kappa)|\,\mathrm{d}u\right| \leqslant \omega(c';\delta)\left(\frac{(t-\kappa)^2}{\delta}+|t-\kappa|\right). \quad (15)$$

Hence, from (14) and (15),

$$|G^{\lambda,\mu}_{m,a}(c;\kappa)-c(\kappa)| \leqslant |c'(\kappa)||G^{\lambda,\mu}_{m,a}(t-\kappa;\kappa)| + \omega(c';\delta)\left(\frac{1}{\delta}G^{\lambda,\mu}_{m,a}((t-\kappa)^2;\kappa)+G^{\lambda,\mu}_{m,a}(|t-\kappa|;\kappa)\right).$$

From Cauchy-Schwarz inequality, we have

$$|G^{\lambda,\mu}_{m,a}(c;\kappa)-c(\kappa)| \leqslant |c'(\kappa)||G^{\lambda,\mu}_{m,a}(t-\kappa;\kappa)| +\omega(c';\delta)\left(\frac{1}{\delta}\sqrt{G^{\lambda,\mu}_{m,a}((t-\kappa)^2;\kappa)}+1\right)\left(\sqrt{G^{\lambda,\mu}_{m,a}((t-\kappa)^2;\kappa)}\right).$$

Choosing $\delta = \sqrt{\rho^{\lambda,\mu}_{m,a}(\kappa)}$ and keeping in mind that $G^{\lambda,\mu}_{m,a}(t-\kappa;\kappa) = \psi^{\lambda,\mu}_{m,a}(\kappa)$, then we get desired result. □

A necessary definition for the following theorem is of Lipschitz type space. A Lipschitz type space with two parameters $\eta_1 \geqslant 0$ and $\eta_2 > 0$ is defined as [25]

$$Lip^{(\eta_1,\eta_2)}_M(\sigma) := \left\{c \in C[0,1] : |c(t)-c(\kappa)| \leqslant M\frac{|t-\kappa|^\sigma}{\left(t+\eta_1\kappa^2+\eta_2\kappa\right)^{\sigma/2}};\ \kappa, t \in [0,1],\ \text{s.t. } 0<\sigma\leqslant 1\right\},$$

where M is any positive constant.

Theorem 5 *Let* $c \in Lip^{(\eta_1,\eta_2)}_M(\sigma)$*. Then, we have*

$$\left|G^{\lambda,\mu}_{m,a}(c;\kappa)-c(\kappa)\right| \leqslant M\left(\frac{\rho^{\lambda,\mu}_{m,a}(\kappa)}{\eta_1\kappa^2+\eta_2\kappa}\right)^{\sigma/2}.$$

Proof: Let us first consider the case $\sigma = 1$. We can write

$$\left|G^{\lambda,\mu}_{m,a}(c;\kappa)-c(\kappa)\right| \leqslant G^{\lambda,\mu}_{m,a}(|c(t)-c(\kappa)|;\kappa) \leqslant MG^{\lambda,\mu}_{m,a}\left(\frac{|t-\kappa|}{\sqrt{t+\eta_1\kappa^2+\eta_2\kappa}};\kappa\right).$$

Using the fact that $\frac{1}{\sqrt{t+\eta_1\kappa^2+\eta_2\kappa}} \leqslant \frac{1}{\sqrt{\eta_1\kappa^2+\eta_2\kappa}}$, the Cauchy-Schwarz inequality and applying Lemma 2 together with Lemma 3 give

$$\begin{aligned}\left|G^{\lambda,\mu}_{m,a}(c;\kappa)-c(\kappa)\right| &\leqslant M\frac{1}{\sqrt{\eta_1\kappa^2+\eta_2\kappa}}G^{\lambda,\mu}_{m,a}\left(|t-\kappa|;\kappa\right)\\ &\leqslant \frac{M}{\sqrt{\eta_1\kappa^2+\eta_2\kappa}}\left(G^{\lambda,\mu}_{m,a}((t-\kappa)^2;\kappa)\right)^{1/2}\\ &\leqslant M\left(\frac{\rho^{\lambda,\mu}_{m,a}(\kappa)}{\eta_1\kappa^2+\eta_2\kappa}\right)^{1/2}.\end{aligned}$$

Thus, the assertion holds for the case $\sigma = 1$. Let us now take $0<\sigma<1$. Then, using the Hölder inequality for $p = \dfrac{2}{\sigma}$ and $q = \dfrac{2}{2-\sigma}$, Lemma 2 and Lemma 3, we obtain

$$\begin{aligned}\left|G^{\lambda,\mu}_{m,a}(c;\kappa)-c(\kappa)\right| &\leqslant \sum_{\zeta=0}^{m} b^{\lambda,\mu}_{m,\zeta}(\kappa)\left(\frac{m+a}{a}\right)\int_{\frac{\zeta}{m+a}}^{\frac{\zeta+a}{m+a}} |c(t)-c(\kappa)|\,\mathrm{d}t\\ &\leqslant \left(\sum_{\zeta=0}^{m} b^{\lambda,\mu}_{m,\zeta}(\kappa)\left(\frac{m+a}{a}\right)\int_{\frac{\zeta}{m+a}}^{\frac{\zeta+a}{m+a}} (|c(t)-c(\kappa)|)^{2/\sigma}\,\mathrm{d}t\right)^{\sigma/2}\\ &\quad\times\left(\sum_{\zeta=0}^{m} b^{\lambda,\mu}_{m,\zeta}(\kappa)\left(\frac{m+a}{a}\right)\int_{\frac{\zeta}{m+a}}^{\frac{\zeta+a}{m+a}} 1^{2/2-\sigma}\,\mathrm{d}t\right)^{(2-\sigma)/2}\\ &\leqslant \frac{M}{(\eta_1\kappa^2+\eta_2\kappa)^{\sigma/2}}\left(G^{\lambda,\mu}_{m,a}((t-\kappa)^2;\kappa)\right)^{\sigma/2}\\ &\leqslant M\left(\frac{\rho^{\lambda,\mu}_{m,a}(\kappa)}{\eta_1\kappa^2+\eta_2\kappa}\right)^{\sigma/2}.\end{aligned}$$

This concludes the proof. □

Next, we study the local direct estimate of the operators defined in (10) applying the Lipschitz-type maximal function of order ϕ given by Lenze [26] as

$$\tilde{\omega}_\phi(c;\kappa) = \sup_{t\neq\kappa,\, t\in[0,1]} \frac{|c(t)-c(\kappa)|}{|t-\kappa|^\phi}, \quad (16)$$

for $\kappa \in [0,1]$ and $\phi \in (0,1]$.

Theorem 6 *Let* $c \in C[0,1]$ *and* $0<\phi\leqslant 1$*. Then, for all* $\kappa\in[0,1]$*, we have*

$$\left|G^{\lambda,\mu}_{m,a}(c;\kappa)-c(\kappa)\right| \leqslant \tilde{\omega}_\phi(c;\kappa)\left(\rho^{\lambda,\mu}_{m,a}(\kappa)\right)^{\phi/2}.$$

Proof: Considering (16), and

$$\left|G_{m,a}^{\lambda,\mu}(c;\kappa)-c(\kappa)\right| \leqslant G_{m,a}^{\lambda,\mu}(|c(t)-c(\kappa)|;\kappa) \leqslant \tilde{\omega}_{\phi}(c;\kappa)G_{m,a}^{\lambda,\mu}(|t-\kappa|^{\phi};\kappa).$$

Now, utilizing the Hölder's inequality with $p=\dfrac{2}{\phi}$ and $q=\dfrac{2}{2-\phi}$, Lemma 2 and Lemma 3, hence we get

$$\left|G_{m,a}^{\lambda,\mu}(c;\kappa)-c(\kappa)\right| \leqslant \tilde{\omega}_{\phi}(c;\kappa)G_{m,a}^{\lambda,\mu}((t-\kappa)^2;\kappa)^{\phi/2}.$$

Finally, the Theorem 6 is proved. □

EXAMPLES AND NUMERICAL RESULTS

This section presents illustrative and numerical examples to provide a comparative analysis of the convergence properties of the operators under consideration. Our aim is for these examples to facilitate a clearer understanding of how our operator approaches a target function.

Example 1 In this example, we give comparative analysis of our operators shown as $G_{m,a}^{\lambda,\mu}$ and (λ,μ)-Bernstein-Kantorovich operators, shown as $\mathscr{K}_m^{\lambda,\mu}$. Consider the function $c(\kappa)=\cos(2\pi\kappa)$ in the interval $[0,1]$. Choose the parameters $m=12$, $\lambda=1$, $\mu=4$, $a=0.2$, $a=1$ (correspond to the $\mathscr{K}_m^{\lambda,\mu}$) and $a=2.5$. The comparison is given in Fig. 1.

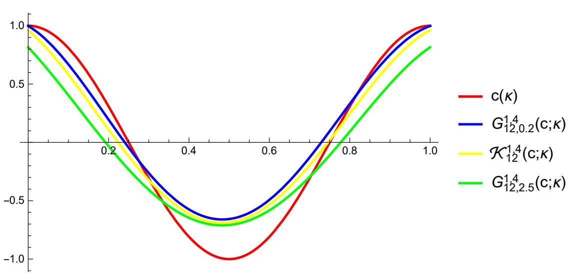

Fig. 1 The comparison of $\mathscr{K}_{12}^{1,4}$ and $G_{12,a}^{1,4}$.

As illustrated in the preceding figure, the approximation achieved by our operators surpasses that of the (λ,μ)-Bernstein-Kantorovich operators in certain subintervals of $[0,1]$ for the specified function. Concurrently, Table 1 unequivocally indicates that our operators yield a more accurate approximation of the given function across particular subintervals of $[0,1]$. To see this clearly, numerical values of error of approximation are presented in Table 1 for the function $\kappa(1-\kappa)$, $m=30$, $\lambda=1$, $\mu=10$ and $a=1.6$.

Table 1 Error of approximation for $\mathscr{K}_{30}^{1,10}(c;\kappa)$ and $G_{30,1.6}^{1,10}(c;\kappa)$.

κ	$\lvert\mathscr{K}_{30}^{1,10}(c;\kappa)-c(\kappa)\rvert$	$\lvert G_{30,1.6}^{1,10}(c;\kappa)-c(\kappa)\rvert$
0.30	0.00275062	0.00127278
0.33	0.00393002	0.00282533
0.36	0.00497385	0.00417971
0.39	0.00588212	0.00533592
0.42	0.00665482	0.00629396
0.45	0.00729194	0.00705381
0.48	0.00779348	0.00761557
0.51	0.00815949	0.00797867
0.54	0.00838991	0.00814433
0.57	0.00848475	0.00811148
0.60	0.00844403	0.00788046
0.63	0.00826773	0.00745126
0.66	0.00795585	0.00682386
0.69	0.00750833	0.00599823
0.72	0.00692503	0.00497422
0.75	0.00620543	0.00375132

Example 2 In this illustration, we will investigate how variations in the parameter m influence the convergence characteristics for predefined values of λ,μ and a. Let us consider the function $c(\kappa)=1-\sin(4\kappa^2)$ in the interval $[0,1]$. For the parameters $\lambda=1$, $\mu=2$, and $a=1.2$, the approximation process for the different values of m can be seen.

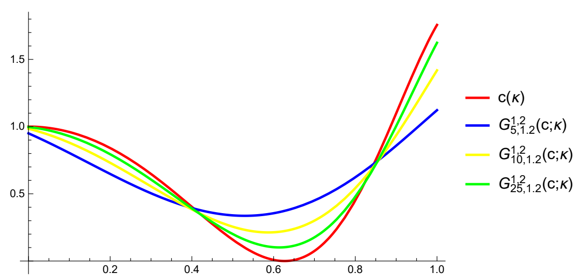

Fig. 2 Approximation process of $G_{m,1.2}^{1,2}$ for $m=5,10,25$.

Table 2 Error of approximation by $G_{m,0.7}^{1,10}(c;0.4)$ and $G_{m,0.7}^{1,10}(c;0.9)$ for different values of m.

m	$\lvert G_{m,0.7}^{1,10}(c;0.4)-c(0.4)\rvert$	$\lvert G_{m,0.7}^{1,10}(c;0.9)-c(0.9)\rvert$
15	0.155168	1.481582
25	0.088561	1.013787
35	0.060022	0.766171
50	0.039878	0.561956
75	0.025339	0.390777
100	0.018511	0.300266
200	0.008871	0.156545
300	0.005827	0.106018
500	0.003454	0.064463

As evident from Fig. 2, larger values of m lead to better approximation. For a quantitative assessment of the impact of m on the approximation of the function

$c(\kappa) = 2\kappa^2 \cos(3\pi\kappa^3)$, $\lambda = 1$, $\mu = 10$, $a = 0.7$ and two specific points $\kappa = 0.4$ and $\kappa = 0.9$, consult Table 2.

Example 3 Our last demonstration aid us to see the impact of the parameter μ. Consider the function $c(\kappa) = \sin(2\pi\kappa)$ in the interval $[0, 1]$. For the parameters $m = 10$, $\lambda = 1$ and $a = 0.5$, one can see in Fig. 3 the approximation process for the different values of μ.

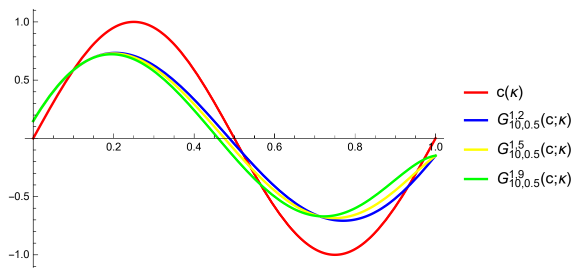

Fig. 3 Approximation process of $G^{1,\mu}_{10,0.5}$ for $\mu = 2, 5, 9$.

In Table 3 shown as below, the increasing values of μ lead to improved approximation for the specificed function. Here, $c(\kappa) = 1 - \cos(4e^{\kappa})$, $m = 40$, $\lambda = 1$, $a = 0.6$ and three specific points $\kappa = 0.3$, $\kappa = 0.4$ and $\kappa = 0.8$, respectively.

Table 3 Error of approximation by $G^{1,\mu}_{40,0.6}(c; 0.3)$, $G^{1,\mu}_{40,0.6}(c; 0.4)$ and $G^{1,\mu}_{40,0.6}(c; 0.8)$ for different values of μ.

μ	Error $\lvert G^{1,\mu}_{40,0.6}(c; x) - c(x)\rvert$		
	$x = 0.3$	$x = 0.4$	$x = 0.8$
5	0.0230967	0.0875893	0.1235887
10	0.0200714	0.0864445	0.1163168
15	0.0170461	0.0852997	0.1090437
20	0.0140208	0.0841548	0.1017713
25	0.0109955	0.0830132	0.0944985
30	0.0079701	0.0818651	0.0872258
35	0.0049448	0.0807203	0.0799531

CONCLUSION

In this paper, we introduced and rigorously analyzed a new class of operators, the (λ, μ)-Bernstein-Kantorovich operators, which extend the previously studied (λ, μ)-Bernstein-Kantorovich operators. Our investigation comprehensively explored their approximation properties. The accompanying graphical and numerical examples effectively illustrated the convergence characteristics of these novel operators. We showcased the operators' comparative performance against (λ, μ)-Bernstein-Kantorovich operators and demonstrated how the parameters λ and μ influence their convergence. Future research could explore the approximation of functions of two variables using similar constructions or investigate the shape-preserving properties of these operators.

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