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No $\mathbb{T}$-gain graph with the rank $r(\Phi)=2m(G)-2c(G)+1$

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ABSTRACT: Let $\Phi=(G,\varphi)$ be a $\mathbb{T}$-gain graph. In this paper, we will prove that there are no $\mathbb{T}$-gain graphs with the rank $2m(G)-2c(G)+1$, where $c(G)$ is the dimension of cycle space of G, $m(G)$ is the matching number of G. For a given $c(G)$, we also prove that there are infinitely many connected $\mathbb{T}$-gain graphs with the rank $2m(G)-2c(G)+s$, $(0\leqslant s\leqslant 3c(G), s\neq 1)$. These results can also apply to undirected graphs, signed graphs and mixed graphs.

KEYWORDS: $\mathbb{T}$-gain graph, rank, matching number

MSC2020: 05C50

INTRODUCTION

Let $G=(V(G),E(G))$ be an undirected graph, where $V(G)=\{v_1,v_2,\dots,v_n\}$ is the vertex set and $E(G)$ is the edge set of G, respectively. The *adjacency matrix* $A(G)$ of G is the symmetric $n\times n$ matrix with entries $A(i,j)=1$ (or written as $a_{ij}=1$) if and only if $v_iv_j\in E(G)$ and zeros elsewhere. Denote by $v_i\sim v_j$, if v_i is adjacent to v_j in G. Let $\overrightarrow{E}$ be the set of oriented edges. Let e_{ij} be the oriented edge from v_i to v_j, and $\varphi(e_{ij})$ be the gain of e_{ij}.

Let $\mathbb{T}=\{z\in\mathbb{C}:|z|=1\}$. A *complex unit gain graph* (or $\mathbb{T}$-gain graph) $\Phi=(G,\mathbb{T},\varphi)$ is a triple, which consisting of the *underlying graph* G, $\mathbb{T}$ and a gain function $\varphi:\overrightarrow{E}\to\mathbb{T}$ such that $\varphi(e_{ij})=\varphi(e_{ji})^{-1}=\overline{\varphi(e_{ji})}$. Sometimes, we use $\Phi=(G,\varphi)$ or G^{φ} instead of $\Phi=(G,\mathbb{T},\varphi)$. The *adjacency matrix* $A(\Phi)=(b_{ij})_{n\times n}$ of a $\mathbb{T}$-gain graph Φ, is defined as

$$b_{ij}=\begin{cases}\varphi(e_{ij}), & \text{if } v_i\sim v_j;\\ 0, & \text{otherwise.}\end{cases}$$

If $v_i\sim v_j$, then $b_{ji}\cdot b_{ij}=1$. The *rank* $r(\Phi)$ of Φ, is the number of non-zero eigenvalues of $A(\Phi)$.

If $V_1\subseteq V(G)$, $\Phi-V_1$ is the *induced subgraph* obtained from Φ by removing all vertices in V_1 and their incident edges. For $V(H^{\varphi})\subset V(\Phi)$, $H^{\varphi}+x$ is defined as the subgraph of Φ induced by the vertex set $V(H^{\varphi})\cup\{x\}$. Let Φ_1 and Φ_2 be two $\mathbb{T}$-gain graphs, where $V(\Phi_1)\cap V(\Phi_2)=\varnothing$, $E(\Phi_1)\cap E(\Phi_2)=\varnothing$. Denote by $\Phi=\Phi_1\cup\Phi_2$ the *disjoint union* graph of Φ_1 and Φ_2, where $V(\Phi)=V(\Phi_1)\cup V(\Phi_2)$, $E(\Phi)=E(\Phi_1)\cup E(\Phi_2)$.

A *pendant vertex* is defined as a vertex with degree 1, and its unique neighbour is called a *quasi-pendant vertex*. A *pendant edge* is an edge which is incident to a pendant vertex. A *pendant cycle* of G is a cycle which contains only a vertex of degree 3.

Denote by $m(G)$, the *matching number* of G. Let M be a matching of G and $v\in V(G)$, if there exists an edge $e\in M$ such that e is incident to v, then v is called *M-saturated*. Otherwise, v is called *M-unsaturated*. An *M-alternating path* of G is defined as a path whose edges are alternately in the edge sets $E\backslash M$ and M. An *M-augmenting path* is defined as an M-alternating path whose starting vertex and ending vertex are M-unsaturated.

The length of the shortest path from the vertex u to v is defined as the *distance* between u and v, denote by $d(u,v)$. The *girth* $g(G)$ of G, is the length of the shortest cycle of G. Let G be a graph with n vertices, m edges, and $\theta(G)$ connected components. Denote by $c(G)$ *the dimension of cycle space* of G, where $c(G)=m-n+\theta(G)$. If the cycles (if any) of G are pairwise vertex-disjoint, then the acyclic graph T_G is obtained from G by contracting each cycle of G into a vertex, which is called a *cyclic vertex*. Let W_G (resp., U) be the vertex set consisting of all cyclic vertices (resp., all non-cyclic vertices) in T_G, $V(T_G)=W_G\cup U$. Furthermore, denote by $[T_G]$ the graph obtained from T_G by deleting all cyclic vertices.

In general, let C_n, P_n and K_n be the *cycle*, *path* and *complete graph* have n vertices, respectively.

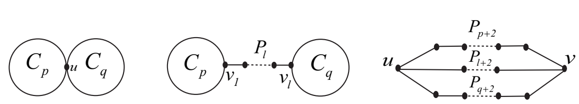

Fig. 1 $\infty(p,1,q)$, $\infty(p,l,q)$ and $\theta(p,l,q)$.

If $|E(G)|=|V(G)|+1$ for a connected graph G, then G is called *bicyclic*. If $|E(G)|=|V(G)|+2$ for a connected graph G, then G is called *tricyclic*. The connected bicyclic (or tricyclic) subgraph without pendant vertices of a bicyclic (or tricyclic) graph G is called the *base* of G. A connected bicyclic graph has two types of bases, they are $\infty(p,l,q)$ and $\theta(p,l,q)$ (as shown in Fig. 1). The bicyclic graph is called an ∞-graph (a θ-graph) if it contains $\infty(p,l,q)$ ($\theta(p,l,q)$) as its base. As shown in Fig. 2, denote by T_i, $i=1,2,\dots,8$, all the bases of tricyclic graphs.

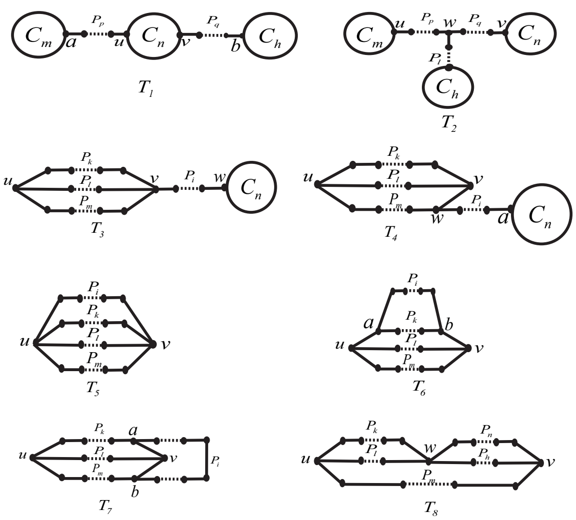

Fig. 2 T_1–T_8.

In chemistry, molecular stability corresponds to the singularity of graphs. Collatz and Sinogowitz [1] had wanted to solve the problem that is all graphs of order n with $r(G) < n$. Until today, this problem is also unsolved.

In recent years, the research on the relationship between $\mathbb{T}$-gain graph and other parameters has draw much attention. In 2012, Reff [2] gave some definitions of a $\mathbb{T}$-gain graph. In 2015, Yu, Qu and Tu [3] gave some results about the inertia indices of a $\mathbb{T}$-gain graph. In 2017, Lu, Wang and Xiao [4] characterized the $\mathbb{T}$-gain connected bicyclic graphs with rank 2, 3, or 4. In [5], the determinant of the Laplacian matrix of a $\mathbb{T}$-gain graph were characterized by Wang, Gong and Fan. In 2019, Lu, Wang and Zhou [6] obtained that

$$r(G)-2c(G)\leqslant r(\Phi)\leqslant r(G)+2c(G)$$

for a $\mathbb{T}$-gain graph. In 2020, Xu, Zhou, Wong and Tian [7] determine all the $\mathbb{T}$-gain graphs with rank 2. He, Hao and Yu [8] determined the bounds for the rank of a $\mathbb{T}$-gain graph in terms of its independence number. In [9], Lu and Wu obtained the relationship between the rank of a $\mathbb{T}$-gain graph and its maximum degree.

He, Hao and Dong [10] and Li, Yang [11] independently proved that for any $\mathbb{T}$-gain graph Φ,

$$2m(G)-2c(G)\leqslant r(\Phi)\leqslant 2m(G)+c(G).$$

The rank of a $\mathbb{T}$-gain graph attaining the bounds are also characterized by them. Motivated by this, in this paper, we will prove that there are no $\mathbb{T}$-gain graphs with the rank $2m(G)-2c(G)+1$. For a given $c(G)$, we also prove that there are infinitely many connected $\mathbb{T}$-gain graphs with the rank $2m(G)-2c(G)+s$, $(0\leqslant s\leqslant 3c(G), s\neq 1)$. These results can also apply to undirected graphs [12], signed graphs [13] and mixed graphs.

PRELIMINARIES

In this section, we will introduce some results about the undirected graph and $\mathbb{T}$-gain graph.

Lemma 1 ([14]) *A matching M of G is a maximum matching if and only if G contains no M-augmenting path.*

Let G be an undirected graph.

Lemma 2 ([15]) *If G has a pendant vertex u, and v is adjacent to u, then*

$$m(G)-1=m(G-v)=m(G-u-v).$$

Lemma 3 ([16]) *Let $v\in V(G)$, then*

$$m(G)-1\leqslant m(G-v)\leqslant m(G).$$

Lemma 4 ([17]) *Let x be a vertex of graph G.*
(i) *If x does not lie on any cycle of G, then $c(G-x)=c(G)$.*
(ii) *If x lies on a cycle of G, then $c(G-x)\leqslant c(G)-1$.*
(iii) *If x is the common vertex of distinct cycles of G, then $c(G-x)\leqslant c(G)-2$.*
(iv) *If the cycles of G are pairwise vertex-disjoint, then $c(G)$ is the number of cycles in G.*

Let Φ be a $\mathbb{T}$-gain graph.

Lemma 5 ([10]) *Let T^{φ} be an acyclic $\mathbb{T}$-gain graph, then*

$$r(T^{\varphi})=2m(T)=r(T).$$

Lemma 6 ([3]) *If Φ contains a pendant vertex u, $uv\in E(\Phi)$, then*

$$r(\Phi-u-v)=r(\Phi)-2.$$

Lemma 7 ([3]) *Let $x\in V(\Phi)$, then*

$$r(\Phi)-2\leqslant r(\Phi-x)\leqslant r(\Phi).$$

Lemma 8 ([10])

$$2m(G)-2c(G)\leqslant r(\Phi)\leqslant 2m(G)+c(G).$$

Lemma 9 ([3])
(i) *Let H^{φ} be an induced subgraph of Φ, then $r(H^{\varphi})\leqslant r(\Phi)$.*
(ii) *Let $\Phi=\Phi_1\cup\Phi_2\cup\cdots\cup\Phi_t$, where $\Phi_1,\Phi_2,\cdots,\Phi_t$ are connected components of Φ, then $r(\Phi)=\sum_{i=1}^{t} r(\Phi_i)$.*

Definition 1 ([4]) Let C_n^{φ} be a $\mathbb{T}$-gain cycle,

$$\begin{aligned}\varphi(C_n)&=\varphi(v_1v_2\cdots v_nv_1)\\&=\varphi(v_1v_2)\varphi(v_2v_3)\cdots\varphi(v_{n-1}v_n)\varphi(v_nv_1),\end{aligned}$$

then C_n^{φ} is one of the following Types:

Type A, if $\varphi(C_n) = (-1)^{n/2}$ and n is even,

Type B, if $\varphi(C_n) \neq (-1)^{n/2}$ and n is even,

Type C, if $\mathrm{Re}\left((-1)^{(n-1)/2}\varphi(C_n)\right) > 0$ and n is odd,

Type D, if $\mathrm{Re}\left((-1)^{(n-1)/2}\varphi(C_n)\right) < 0$ and n is odd,

Type E, if $\mathrm{Re}\left((-1)^{(n-1)/2}\varphi(C_n)\right) = 0$ and n is odd.

Lemma 10 ([3]) *Let C_n^{φ} be a $\mathbb{T}$-gain cycle, then*

$$r(C_n^{\varphi}) = \begin{cases} n-2, & \text{if } C_n^{\varphi} \text{ is of Type A,} \\ n, & \text{if } C_n^{\varphi} \text{ is of Type B,} \\ n, & \text{if } C_n^{\varphi} \text{ is of Type C,} \\ n, & \text{if } C_n^{\varphi} \text{ is of Type D,} \\ n-1, & \text{if } C_n^{\varphi} \text{ is of Type E.} \end{cases}$$

Lemma 11 ([11]) *Let Φ be a $\mathbb{T}$-gain graph, then $r(\Phi) = 2m(G) - 2c(G)$ if and only if Φ satisfies all of the following conditions:*
(i) *cycles of Φ are pairwise vertex-disjoint;*
(ii) *every $\mathbb{T}$-gain cycle (if any) of Φ is of Type A;*
(iii) $m(T_G) = m([T_G])$.

Lemma 12 ([11]) *Let Φ be a $\mathbb{T}$-gain graph, then $r(\Phi) = 2m(G) + c(G)$ if and only if Φ satisfies all of the following conditions:*
(i) *cycles of Φ are pairwise vertex-disjoint;*
(ii) *every $\mathbb{T}$-gain cycle (if any) of Φ is of either Type C or Type D;*
(iii) $m(T_G) = m([T_G])$.

NO $\mathbb{T}$-GAIN GRAPH Φ WITH THE RANK $2m(G) - 2c(G) + 1$

In this section, we will prove that there is no $\mathbb{T}$-gain graph Φ with the rank $2m(G) - 2c(G) + 1$. At first, we need the following results about $\mathbb{T}$-gain unicyclic graph.

Definition 2 ([13]) Let G be a unicyclic graph with a unique cycle C_q. Define
(i) E_1: the set of all edges of G between C_q and $[T_G]$.
(ii) F_1: the set of all matchings of G with $m(G)$ edges.
(iii) F_2: the set of all matchings of $[T_G]$ with $m([T_G])$ edges.
(iv) $F_1^{'}$: the set of all matchings of G with $m(G)$ edges, each of which has at least an edge in E_1.
(v) $F_1^{''}$: the set of all matchings of G with $m(G)$ edges, and $M \cap E_1 = \varnothing$ for all $M \in F_1$.

By Definition 2, we have $F_1 = F_1^{'} \cup F_1^{''}$.

Corollary 1 ([13]) *Let C_q be an even cycle.*
(i) *If $F_1^{'} = \varnothing$, the maximum matching of G is the union of a maximum matching of C_q and a maximum matching of $G - C_q$, then $|F_1| = |F_1^{''}| = 2|F_2|$.*
(ii) *If $F_1^{'} \neq \varnothing$, then $|F_1| = |F_1^{'}| + |F_1^{''}| > 2|F_2|$.*

If components are either K_2^{φ} or C_k^{φ} of the subgraph L of Φ, then L is called a *linear subgraph* of Φ. If $\varphi(C) \neq \mathrm{i}$ or $-\mathrm{i}$ ($\mathrm{i}^2 = -1$) for each cycle C (if any) in L, then L is called *basic*. Denote by $\mathbf{B_i}$ the set of all basic subgraphs with i vertices in Φ. The number of components and $\mathbb{T}$-gain cycles in L are defined as $p(L)$ and $c(L)$, respectively.

Lemma 13 ([7]) *Let Φ be a $\mathbb{T}$-gain graph of order n, and $f(\Phi, \lambda) = \sum_{i=0}^{n} a_i(\Phi) \cdot \lambda^{n-i}$ be the the characteristic polynomial of $A(\Phi)$. Then*

$$a_i(\Phi) = \sum_{L \in \mathbf{B_i}} (-1)^{p(L)} 2^{c(L)} \prod_{C \in L} Re(\varphi(C)),$$

$i \in \{1, 2, \ldots, n\}$, where L is over all basic subgraphs of Φ with i vertices.

For a $\mathbb{T}$-gain unicyclic graph with a unique cycle C_q^{φ}, there is the following theorem.

Theorem 1 *Let Φ be a $\mathbb{T}$-gain unicyclic graph with a unique cycle C_q^{φ}, then*
(i) *$r(\Phi) = 2m(G) + 1$, if $q \equiv 1(mod\ 2)$, $Re(\varphi(C_q)) \neq 0$ and $m(T_G) = m([T_G])$* ([10], Theorem 1.12]);
(ii) *$r(\Phi) = 2m(G) - 2$, if $q \equiv 0(mod\ 2)$, $\varphi(C_q) = (-1)^{q/2}$ and $m(T_G) = m([T_G])$* ([10], Theorem 1.11]);
(iii) *$r(\Phi) = 2m(G)$, otherwise* ([7], Theorems 3.1, 3.9]).

Xu et al [7] obtained the following result. Here, we will give a new proof using Lemma 13.

Lemma 14 (Theorem 3.9 [7]) *Let Φ be a $\mathbb{T}$-gain unicyclic graph with a unique cycle C_q^{φ}. If q is even, and there is an $M \in F_1$ such that $M \cap E_1 \neq \varnothing$, then $r(\Phi) = 2m(G)$.*

Proof: Let $m(G) = m$, $m([T_G]) = l$, $F_3 = \{L | L = C_q \cup M,\ M \in F_2\}$. Note that Φ is a bipartite graph. Using Lemma 13,

$$f(\Phi, \lambda) = \sum_{i=0}^{\lfloor n/2 \rfloor} b_i(\Phi) \lambda^{n-2i},$$
$$b_i(\Phi) = \sum_{L \in \mathbf{B_{2i}}} (-1)^{p(L)} 2^{c(L)} \prod_{C \in L} \mathrm{Re}(\varphi(C)),$$

for any $i \in \{1, 2, \ldots, \lfloor n/2 \rfloor\}$.

Note that $m(G) \geqslant m([T_G]) + m(C_q)$ and q is even, then $m(C_q) = q/2$ and $m \geqslant l + q/2$, that is, $2m \geqslant q + 2l$. If $i > m$, then Φ contains no basic subgraphs with $2i$ vertices and $b_i(\Phi) = 0$. Hence

$$\begin{aligned} f(\Phi, \lambda) &= \lambda^n + b_1(\Phi)\lambda^{n-2} + \cdots + b_m(\Phi)\lambda^{n-2m} \\ &= \lambda^{n-2m}(\lambda^{2m} + b_1(\Phi)\lambda^{2m-2} + \cdots + b_m(\Phi)). \end{aligned}$$

Therefore, $r(\Phi) \leqslant 2m$. In order to get the result $r(\Phi) = 2m$, we need to prove $b_m(\Phi) \neq 0$.

Case 1. $2m = q + 2l$.

There exists some basic subgraphs L with $2m$ vertices such that L contains C_q^{φ} as a subgraph. Then $\mathbf{B_{2m}} = F_1^{\varphi} \cup F_3^{\varphi}$. If $L \in F_1^{\varphi}$, then $p(L) = m, c(L) = 0$. If $L \in F_3^{\varphi}$, then $p(L) = l + 1, c(L) = 1$. Hence

$$\begin{aligned} b_m(\Phi) &= \sum_{L \in F_1^{\varphi}} (-1)^m + \sum_{L \in F_3^{\varphi}} (-1)^{l+1} 2^1 \operatorname{Re}(\varphi(C_q)) \\ &= (-1)^m |F_1| + (-1)^{l+1} 2 \operatorname{Re}(\varphi(C_q)) |F_3| \\ &= (-1)^l ((-1)^{m-l} |F_1| - 2 \operatorname{Re}(\varphi(C_q)) |F_3|) \\ &\geqslant (-1)^l ((-1)^{q/2} |F_1| - 2|F_3|), \end{aligned}$$

Since $\operatorname{Re}(\varphi(C_q)) \leqslant 1$, $m - l = q/2$.

Subcase 1.1. $q \equiv 2 (mod\ 4)$, then $q/2 \equiv 1 (mod\ 2)$. Hence,

$$b_m(\Phi) \geqslant (-1)^{l+1} (|F_1| + 2|F_3|) \neq 0.$$

Subcase 1.2. $q \equiv 0 (mod\ 4)$, then $q/2 \equiv 0 (mod\ 2)$. Hence,

$$b_m(\Phi) \geqslant (-1)^l (|F_1| - 2|F_3|), \quad |F_3| = |F_2|.$$

Note that $F_1^{'} \neq \varnothing$, by Corollary 1, we have $|F_1| > 2|F_2|$. Hence, $b_m(\Phi) > 0$.

Case 2. $2m > q + 2l$ and $M \cap E_1 \neq \varnothing$, $\exists\, M \in F_1$.

Then the basic subgraphs L with $2m$ vertices contains no $\mathbb{T}$-gain cycles, which shows that $F_3^{\varphi} = \varnothing$ and $\mathbf{B_{2m}} = F_1^{\varphi}$, then $p(L) = m$ and $c(L) = 0$. Hence,

$$b_m(\Phi) = \sum_{L \in F_1^{\varphi}} (-1)^m = (-1)^m |F_1| \neq 0.$$

Based on the above conclusions, we have $b_m(\Phi) \neq 0$. Thus $r(\Phi) = 2m(G)$. □

Let G be the graph with some pendant vertices and has at least a cycle. For any pendant vertex u, and v is adjacent to u, we will give the definitions of two types of the pendant vertex u.

Definition 3

(i) If v does not lie on a cycle, then u is of Type I.

(ii) If v lies on a cycle, then u is of Type II.

Lemma 15 *Let Φ be a $\mathbb{T}$-gain graph, u be a pendant vertex of Φ and v be adjacent to u. If u is of Type I, then $r(\Phi) = 2m(G) - 2c(G) + s$ if and only if $r(\Phi - u - v) = 2m(G - u - v) - 2c(G - u - v) + s$, $0 \leqslant s \leqslant 3c(G)$.*

Proof: By Lemmas 2 and 4,

$$\begin{aligned} m(G) - 1 &= m(G - u - v), \\ c(G) &= c(G - u - v). \end{aligned} \tag{1}$$

Sufficiency: By Lemma 6 and (1), $r(\Phi) = r(\Phi - u - v) + 2 = 2m(G - u - v) - 2c(G - u - v) + s + 2 = 2m(G) - 2c(G) + s$.

Necessity: By Lemma 6 and (1), $r(\Phi - u - v) = r(\Phi) - 2 = 2m(G) - 2c(G) + s - 2 = 2m(G - u - v) - 2c(G - u - v) + s$. □

Lemma 16 *Let Φ be a $\mathbb{T}$-gain graph with a pendant vertex u, and v be adjacent to u. If u is of Type II, then $r(\Phi) \geqslant 2m(G) - 2c(G) + 2$.*

Proof: By Lemma 8, suppose on the contrary, there exists some $\mathbb{T}$-gain graphs (H, φ) with the rank $r(H, \varphi) = 2m(H) - 2c(H) + s$, $s \in \{0, 1\}$. Let u and v be two vertices of (H, φ), u be a pendant vertex of (H, φ) and v be adjacent to u. Since u is of Type II, so v lies on a $\mathbb{T}$-gain cycle of (H, φ), by Lemmas 2 and 4,

$$\begin{aligned} m(H) - 1 &= m(H - u - v), \\ c(H) - 1 &\geqslant c(H - u - v). \end{aligned} \tag{2}$$

Combining with Lemma 6 and (2),

$$\begin{aligned} r(H - u - v, \varphi) &= r(H, \varphi) - 2 \\ &= 2m(H) - 2c(H) + s - 2 \\ &\leqslant 2m(H - u - v) - 2c(H - u - v) - 1, \end{aligned}$$

which contradicts Lemma 8. □

Denote by D^{φ} the $\mathbb{T}$-gain bicyclic graph obtained from the union of $\theta^{\varphi}(1, 1, 1)$ and some isolated vertices (if any).

Lemma 17 *For the $\mathbb{T}$-gain bicyclic graph D^{φ}, we have*

$$r(D^{\varphi}) \neq 2m(D) - 2c(D) + 1.$$

Proof: Let $|V(D^{\varphi})| = n$, $n \geqslant 5$. Note that $|E(D)| = 6$, $m(D) = 2$, $c(D) = 2$, and D^{φ} is a bipartite graph. By Lemma 13, $f(D^{\varphi}, \lambda) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} b_i(D^{\varphi}) \lambda^{n-2i}$. Then

$$b_i(D^{\varphi}) = \sum_{L \in \mathbf{B_{2i}}} (-1)^{p(L)} 2^{c(L)} \prod_{C \in L} \operatorname{Re}(\varphi(C)),$$

$i \in \{1, 2, \ldots, \lfloor n/2 \rfloor\}$. According to the concept of basic graph, if $i \geqslant 3$, then D^{φ} contains no basic subgraphs with $2i$ vertices and $b_i(D^{\varphi}) = 0$. Hence,

$$\begin{aligned} f(D^{\varphi}, \lambda) &= \lambda^n + b_1(D^{\varphi}) \lambda^{n-2} + b_2(D^{\varphi}) \lambda^{n-4} \\ &= \lambda^{n-4} (\lambda^4 + b_1(D^{\varphi}) \lambda^2 + b_2(D^{\varphi})). \end{aligned}$$

Since $b_1(D^{\varphi}) = \sum_{L \in \mathbf{B_2}} (-1)^{p(L)} = -|E(D^{\varphi})| = -6 \neq 0$. Hence, $2 \leqslant r(D^{\varphi}) \leqslant 4$.

Let F_4 be the matching set of D^{φ} with two edges. Let F_5 be the set of basic subgraph of Φ with four vertices and contains a $\mathbb{T}$-gain cycle C_4^{φ}. Then $\mathbf{B_4} = F_4 \cup F_5$, $|F_4| = 6$, and $|F_5| = 3$. If $L \in F_4$, then $p(L) = 2$

and $c(L)=0$. If $L \in F_5$, then $p(L)=1$ and $c(L)=1$. Hence,

$$b_2(D^\varphi)=\sum_{L\in F_4}(-1)^2+\sum_{L\in F_5}(-1)2\prod_{C\in L}\mathrm{Re}(\varphi(C))$$
$$=6-2\sum_{C\in F_5}\mathrm{Re}(\varphi(C)).$$

Therefore, $r(D^\varphi)=2$ if and only if $b_2(D^\varphi)=0$, if and only if $\mathrm{Re}(\varphi(C))=1$ for any $C\in F_5$. We can obtain that each C_4^φ in D^φ is of Type *A*. Otherwise, $r(D^\varphi)=4$.

Hence, $r(D^\varphi)=2$ or 4. On the other hand, $2m(D)-2c(D)+1=1$. Therefore, $r(D^\varphi)\neq 2m(D)-2c(D)+1$. □

Lemma 18 *Let $\Phi=(G,\varphi)$ $(G\neq D)$ be a $\mathbb{T}$-gain graph without pendant vertices. If $r(\Phi)\neq 2m(G)-2c(G)$, $c(G)\geqslant 2$, then there exists a vertex x on a cycle in Φ and $r(\Phi-x)\neq 2m(G-x)-2c(G-x)$.*

Proof: If $g(G)=3$ and $c(G)\geqslant 2$, let $C_q^\varphi(q=3)$ be a $\mathbb{T}$-gain cycle of Φ. Since $c(G)\geqslant 2$, there exists a vertex x on another cycle in Φ and C_q^φ is a subgraph of $\Phi-x$, this shows that $\Phi-x$ does not satisfy Lemma 11(ii), then

$$r(\Phi-x)\neq 2m(G-x)-2c(G-x).$$

If $g(G)\geqslant 4$ and $c(G)\geqslant 2$, since $r(\Phi)\neq 2m(G)-2c(G)$, so Φ does not satisfy at least one of the three conditions in Lemma 11.

Case 1. Φ does not satisfy Lemma 11(i).

Let C_k^φ, C_s^φ $(k,s\geqslant 4)$ be two vertex-joint cycles in Φ and $G[C_k^\varphi,C_s^\varphi]$ be the subgraph induced by $V(C_k^\varphi)$ and $V(C_s^\varphi)$.

Subcase 1.1. $c(G)=2$.

Note that Φ is a bicyclic graph, and Φ contains no pendant vertices. The definition of $G[C_k^\varphi,C_s^\varphi]$ implies that Φ is the union of $G[C_k^\varphi,C_s^\varphi]$ and some isolated vertices, where $G[C_k^\varphi,C_s^\varphi]$ is either an $\infty^\varphi(p,1,q)$ or a $\theta^\varphi(p,l,q)$. Note that $G\neq D$, as shown in Fig. 1, there has a vertex x on a cycle in Φ such that $\Phi-x$ contains a pendant vertex of Type II. By Lemma 16,

$$r(\Phi-x)\neq 2m(G-x)-2c(G-x).$$

Subcase 1.2. $c(G)\geqslant 3$.

For a given subgraph $G[C_k^\varphi,C_s^\varphi]$ of Φ, we mainly consider the following subcases.

Subcase 1.2.1. There exists at least a vertex x on a cycle of Φ, but not on the subgraph $G[C_k^\varphi,C_s^\varphi]$.

Let x on a cycle of Φ, $x\notin V(G[C_k^\varphi,C_s^\varphi])$, this implies that $G[C_k^\varphi,C_s^\varphi]$ is a subgraph of $\Phi-x$, so $\Phi-x$ does not satisfy Lemma 11(i). Hence,

$$r(\Phi-x)\neq 2m(G-x)-2c(G-x).$$

For example, as shown in Fig. 2, the $\mathbb{T}$-gain graph with $T_i(i=1,2,3,4)$ as an underlying graph contains a vertex x on a cycle and $x\notin V(G[C_k^\varphi,C_s^\varphi])$.

Subcase 1.2.2. Each vertex on a cycle of Φ is on the subgraph $G[C_k^\varphi,C_s^\varphi]$.

In this case, each cycle of Φ is the subgraph of $G[C_k^\varphi,C_s^\varphi]$. Since Φ contains no pendant vertices, then Φ is the union of $G[C_k^\varphi,C_s^\varphi]$ and some isolated vertices. Since $c(G)\geqslant 3$, Φ contains one of eight types of bases of tricyclic graphs as an underlying subgraph. As shown in Fig. 2, the graph T_j can be viewed as two vertex-joint cycles, where $j=5,6,7,8$, which implies that the tricyclic graph T_j is an underlying subgraph of $G[C_k^\varphi,C_s^\varphi]$. As shown in Fig. 2, there exists a vertex x of T_j and T_j-x also contains two vertex-joint cycles. Hence, there has a vertex x on a cycle of Φ and $\Phi-x$ does not satisfy Lemma 11(i), then

$$r(\Phi-x)\neq 2m(G-x)-2c(G-x).$$

Case 2. Φ satisfies Lemma 11(i) but does not satisfy Lemma 11(ii).

Note that all the cycles of Φ are pairwise vertex-disjoint and there exists at least a $\mathbb{T}$-gain cycle in Φ, say C_p^φ, is not of Type A. Since $c(G)\geqslant 2$, let x be a vertex on another cycle, $x\notin V(C_p^\varphi)$. Then, C_p^φ is a subgraph of $\Phi-x$, which shows that $\Phi-x$ does not satisfy Lemma 11(ii), then

$$r(\Phi-x)\neq 2m(G-x)-2c(G-x).$$

Case 3. Φ satisfies (i) and (ii) of Lemma 11, but dose not satisfy (iii) of Lemma 11.

In this case, $m(T_G)\geqslant m([T_G])+1$.

If $E(T_G)=\varnothing$, then Φ is the union of some vertex-disjoint cycles and isolated vertices. Hence, $m(T_G)=m([T_G])=0$, a contradiction. Therefore, we only consider $E(T_G)\neq\varnothing$. In T_G, each maximum matching must cover at least a pendant vertex. Otherwise, there exists an M-augmenting path in T_G, which contradicts Lemma 1. Let *su* be a pendant edge of T_G and u be a pendant vertex. Since Φ contains no pendant vertices, we have $u\in W_G$. Suppose that C_q^φ is the pendant cycle of Φ corresponding to the vertex u of T_G. Let u_0 be the unique vertex with degree three in C_q^φ, $u_0\in V(C_q^\varphi)$. Then, T_{G-x} is obtained from T_G and $C_q^\varphi-x$ by identifying u and u_0 as a vertex.

Subcase 3.1. Each maximum matching of T_G cover all pendant vertices.

Note that *su* is a pendant edge of T_G, u is a pendant vertex. Let $x\in V(C_q^\varphi)$, and x be adjacent to u_0. Since C_q^φ is an even cycle, then $C_q^\varphi-u_0-x$ is a path with length of odd and has a perfect matching. By the definition of T_{G-x}, which shows that the maximum matching of T_{G-x} is the union of the maximum matchings of T_G and $C_q^\varphi-u_0-x$. Then,

$$m(T_{G-x})=m(T_G)+m(C_q^\varphi-u_0-x).$$

Hence, each maximum matching of T_{G-x} must cover some vertices in W_{G-x}, we have $m(T_{G-x})\neq m([T_{G-x}])$, which shows that $\Phi - x$ does not satisfy Lemma 11(iii), then

$$r(\Phi - x) \neq 2m(G-x) - 2c(G-x).$$

Subcase 3.2. There exists some maximum matchings of T_G, denote by $M_i(T_G)$ $(i = 1, 2, \dots, r)$, such that the pendant edge $wv \notin M_i(T_G)$, v is a pendant vertex of T_G.

Let C_p^{φ} be the $\mathbb{T}$-gain cycle of Φ corresponding to the vertex v of T_G, and v_0 be the unique vertex with degree three in C_p^{φ}. Let y be a vertex on the $\mathbb{T}$-gain cycle C_p^{φ} and $d(v_0, y) = 2$. By the definition of T_{G-y}, which shows that the maximum matching of T_{G-y} is the union of $M_i(T_G)$ $(i \in \{1, 2, \dots, r\})$ and the maximum matching of $C_p^{\varphi} - y$. So,

$$m(T_{G-y}) = m(T_G) + m(C_p^{\varphi} - y).$$

Since $wv \notin M_i(T_G)$, for any $i \in \{1, 2, \dots, r\}$, then $M_i(T_G)$ must cover some vertices in W_{G-y} by Lemma 1, this implies that each maximum matching of T_{G-y} must cover some vertices in W_{G-y}. We have $m(T_{G-y}) \neq m([T_{G-y}])$, this shows that $\Phi - y$ does not satisfy Lemma 11(iii), then

$$r(\Phi - y) \neq 2m(G-y) - 2c(G-y).$$

□

Lemma 19 *Let Φ be a $\mathbb{T}$-gain graph has no pendant vertices, then $r(\Phi) \neq 2m(G) - 2c(G) + 1$.*

Proof: We apply induction on $c(G)$ to prove this lemma.

If $\Phi = D^{\varphi}$, by Lemma 17, $r(\Phi) \neq 2m(G) - 2c(G) + 1$. We only consider $G \neq D$ in the following.

$c(G) = 0$, i.e., $G = nK_1$, we can obtain the result.

$c(G) = 1$, i.e., $G = C_k^{\varphi} \cup (n-k)K_1$ $(3 \leqslant k \leqslant n)$. By Theorem 1,

$$r(\Phi) \neq 2m(G) - 2c(G) + 1.$$

If $c(G) \geqslant 2$, assume that the conclusion is true when $c(G) \leqslant k$. Next, we will prove the conclusion is true for $c(G) = k+1$. Suppose on the contrary, there exists a $\mathbb{T}$-gain graph (H, φ) with $c(H) = k+1$ such that $r(H, \varphi) = 2m(H) - 2c(H) + 1$.

Let x be any vertex on a cycle of (H, φ). For the $\mathbb{T}$-gain graph $(H - x, \varphi)$, combining with Lemmas 3 and 4,

$$\begin{aligned} m(H) &\leqslant m(H-x) + 1, \\ c(H) &\geqslant c(H-x) + 1. \end{aligned} \quad (3)$$

By Lemma 7 and (3),

$$\begin{aligned} r(H-x, \varphi) \leqslant r(H, \varphi) &= 2m(H) - 2c(H) + 1 \\ &\leqslant 2m(H-x) - 2c(H-x) + 1. \end{aligned}$$

By Lemma 8,

$$r(H-x, \varphi) = 2m(H-x) - 2c(H-x) + s,\ s \in \{0, 1\}. \quad (4)$$

Since (H, φ) contains no pendant vertices, then $(H - x, \varphi)$ contains either pendant vertices or no pendant vertices.

Case 1. $(H - x, \varphi)$ contains no pendant vertices.

Since $c(H - x) \leqslant c(H) - 1 = k$, so

$$r(H-x, \varphi) \neq 2m(H-x) - 2c(H-x) + 1. \quad (5)$$

Case 2. $(H - x, \varphi)$ contains some pendant vertices.

Subcase 2.1. $(H - x, \varphi)$ contains at least a pendant vertex of Type II.

By Lemma 16,

$$r(H-x, \varphi) \geqslant 2m(H-x) - 2c(H-x) + 2. \quad (6)$$

Subcase 2.2. All pendant vertices of $(H - x, \varphi)$ are of Type I.

Suppose that $(H - x, \varphi)$ contains p pendant vertices. For pendant vertices of Type I, by using Lemma 6 repeatedly, after p steps, we obtain a subgraph (H_1, φ) of (H, φ). If (H_1, φ) contains no pendant vertices or at least a pendant vertex of Type II, then (H_1, φ) is the graph we need in the following **(a)** and **(b)**. Otherwise, in (H_1, φ), for pendant vertices of Type I, we continue to use Lemma 6 repeatedly, we obtain a subgraph (H_2, φ) of (H_1, φ). If (H_2, φ) contains no pendant vertices or at least a pendant vertex of Type II, then (H_2, φ) is the graph we need in the following **(a)** and **(b)**. Otherwise, repeating the above steps until we obtain a $\mathbb{T}$-gain graph (H_0, φ) that meets the requirements.

(a). (H_0, φ) contains at least a pendant vertex of Type II.

By Lemma 16,

$$r(H_0, \varphi) \geqslant 2m(H_0) - 2c(H_0) + 2.$$

Next, since (H_0, φ) is obtained from $(H - x, \varphi)$ by removing a series of Type I pendant vertices and their corresponding adjacent vertices, by the sufficiency of Lemma 15, we have

$$r(H-x, \varphi) \geqslant 2m(H-x) - 2c(H-x) + 2. \quad (7)$$

(b). (H_0, φ) contains no pendant vertices.

Since $c(H_0) = c(H - x) \leqslant c(H) - 1 = k$, so $r(H_0, \varphi) \neq 2m(H_0) - 2c(H_0) + 1$.

Next, since (H_0, φ) is obtained from $(H - x, \varphi)$ by removing a series of Type I pendant vertices and their corresponding adjacent vertices, by the sufficiency of Lemma 15, we have

$$r(H-x, \varphi) \neq 2m(H-x) - 2c(H-x) + 1. \quad (8)$$

Based on the above results. Let x be any vertex on a cycle of (H, φ). Combining with Eqs. (5), (6), (7) and (8), either $r(H-x,\varphi) \neq 2m(H-x)-2c(H-x)+1$ or $r(H-x,\varphi) \geqslant 2m(H-x)-2c(H-x)+2$.

If $r(H-x,\varphi) \neq 2m(H-x)-2c(H-x)+1$, by Eq. (4), then $r(H-x,\varphi) = 2m(H-x)-2c(H-x)$. On the other hand, since $r(H,\varphi) \neq 2m(H)-2c(H)$, by Lemma 18, there exits a vertex y on a cycle of (H,φ) and $r(H-y,\varphi) \neq 2m(H-y)-2c(H-y)$, a contradiction.

If $r(H-x,\varphi) \geqslant 2m(H-x)-2c(H-x)+2$, which will contradicts Eq. (4).

Therefore, for any $\mathbb{T}$-gain graph Φ without pendant vertices, $r(\Phi) \neq 2m(G)-2c(G)+1$. □

Theorem 2 *For any $\mathbb{T}$-gain graph Φ, $r(\Phi) \neq 2m(G)-2c(G)+1$.*

Proof: If $c(G)=0$, using Lemma 5, $r(\Phi) = 2m(G) \neq 2m(G)-2c(G)+1$.

If $c(G)=1$, by Theorem 1, then $r(\Phi) \neq 2m(G)-2c(G)+1$. Next, we only consider $c(G) \geqslant 2$.

Case 1. Φ contains no pendant vertices, we can obtain the result by Lemma 19.

Case 2. Φ contains some pendant vertices.

Subcase 2.1. There exists at least a pendant vertex of Type II.

Using Lemma 16,

$$r(\Phi) \geqslant 2m(G)-2c(G)+2.$$

Subcase 2.2. All pendant vertices are of Type I.

By the similar proof as in Subcase 2.2 of Lemma 19. Suppose that Φ contains p pendant vertices. For pendant vertices of Type I, by using Lemma 6 repeatedly, after p steps, we obtain a subgraph (G_1,φ) of Φ. If (G_1,φ) contains no pendant vertices or at least a pendant vertex of Type II, then (G_1,φ) is the graph we need in the following **(a)** and **(b)**. Otherwise, in (G_1,φ), for pendant vertices of Type I, we continue to use Lemma 6 repeatedly, we obtain a subgraph (G_2,φ) of (G_1,φ). If (G_2,φ) contains no pendant vertices or at least a pendant vertex of Type II, then (G_2,φ) is the graph we need in the following **(a)** and **(b)**, repeating the above steps until we obtain a $\mathbb{T}$-gain graph (G_0,φ) that meets the requirements.

(a). (G_0,φ) contains at least a pendant vertex of Type II.

Using Lemma 16,

$$r(G_0,\varphi) \geqslant 2m(G_0)-2c(G_0)+2. \tag{9}$$

(b). (G_0,φ) contains no pendant vertices.

By Lemma 19,

$$r(G_0,\varphi) \neq 2m(G_0)-2c(G_0)+1. \tag{10}$$

Next, since (G_0,φ) is obtained from Φ by removing a series of Type I pendant vertices and their corresponding adjacent vertices, by the sufficiency of Lemma 15, Eqs. (9) and (10), we have $r(\Phi) \neq 2m(G)-2c(G)+1$. □

Note that if $\varphi(\overrightarrow{E}) \subset \{1\}$, then Φ is the undirected graph G. If $\varphi(\overrightarrow{E}) \subset \{1,-1\}$, then Φ is the signed graph Γ. If $\varphi(\overrightarrow{E}) \subset \{1,i,-i\}$, then Φ is the mixed graph $\widetilde{G}$. Using Theorem 2, the following corollaries can be obtained.

Corollary 2 ([12]) *For any undirected graph G, $r(G) \neq 2m(G)-2c(G)+1$.*

Corollary 3 ([13]) *For any signed graph Γ, $r(\Gamma) \neq 2m(G)-2c(G)+1$.*

Corollary 4 *For any mixed graph $\widetilde{G}$, $r(\widetilde{G}) \neq 2m(G)-2c(G)+1$.*

Let $K^{\varphi}_{1,d+1}$ be a $\mathbb{T}$-gain star, x be the center vertex of $K^{\varphi}_{1,d+1}$ and $y_0, y_1, \cdots, y_d$ be pendant vertices of $K^{\varphi}_{1,d+1}$. Let $O^{\varphi}_1, O^{\varphi}_2, \cdots, O^{\varphi}_{l_1}$ be $\mathbb{T}$-gain cycles of Type C (or Type D), $|V(O_1)| = |V(O_2)| = \cdots = |V(O_{l_1})| = 2a+1$, $a \in \mathbf{Z}^+$. Let $O^{\varphi}_{l_1+1}, O^{\varphi}_{l_1+2}, \cdots, O^{\varphi}_{l'_1}$ be $\mathbb{T}$-gain cycles of Type A, $|V(O_{l_1+1})| = |V(O_{l_1+2})| = \cdots = |V(O_{l'_1})| = 2b$, $b \in \mathbf{Z}^+$ and $b \geqslant 2$. Let $O^{\varphi}_{l'_1+1}, O^{\varphi}_{l'_1+2}, \cdots, O^{\varphi}_d$ be $\mathbb{T}$-gain cycles of Type E, $|V(O_{l'_1+1})| = |V(O_{l'_1+2})| = \cdots = |V(O_d)| = 2c+1$. $c \in \mathbf{Z}^+$.

Next, we construct a new $\mathbb{T}$-gain graph G^{φ}, which is obtained from $K^{\varphi}_{1,d+1}$ and O^{φ}_i by identifying y_i with a vertex of O^{φ}_i, $i = 1,2,\ldots,d$.

Theorem 3 *If $c(G)$ is fixed, then there exists infinitely connected $\mathbb{T}$-gain graphs $\Phi = (G,\varphi)$, such that $r(\Phi) = 2m(G)-2c(G)+s$, where $0 \leqslant s \leqslant 3c(G)$, $s \neq 1$.*

Proof: Let $\Phi = G^{\varphi}$, according to the definition of G^{φ}, let $l_2 = l'_1 - l_1$, $l_3 = d - l'_1$. Note that y_0 is the unique pendant vertex of G^{φ}, then

$$\begin{aligned} m(G^{\varphi}) &= al_1 + bl_2 + cl_3 + 1, \\ c(G^{\varphi}) &= d = l_1 + l_2 + l_3. \end{aligned} \tag{11}$$

By Lemmas 6, 9, 10 and Eq. (11), we have

$$\begin{aligned} r(G^{\varphi}) = r(G^{\varphi}-x-y_0)+2 &= \sum_{i=1}^{d} r(O^{\varphi}_i)+2 \\ &= (2a+1)l_1+(2b-2)l_2+2cl_3+2 \\ &= 2(al_1+bl_2+cl_3+1)-2(l_1+l_2+l_3)+(3l_1+2l_3) \\ &= 2m(G^{\varphi})-2c(G^{\varphi})+(3l_1+2l_3). \end{aligned}$$

Since $l_1, l_2, l_3 \geqslant 0$ and $l_1+l_2+l_3 = c(G)$, then $0 \leqslant 3l_1+2l_3 \leqslant 3l_1+3l_2+3l_3 = 3c(G)$ and $3l_1+2l_3 \neq 1$. Hence, $3l_1+2l_3$ can take over any integer between 0 and $3c(G)$ except for 1. □

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