RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2023.s004
*ScienceAsia* 49S (2023): **78–88**

# Stability analysis of unemployment model in Thailand after COVID-19 outbreak

**Phachara Chanosot, Suttida Wongkaew***

Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200 Thailand

*Corresponding author, e-mail: suttida.wongkaew@cmu.ac.th

*Received 17 Jan 2023, Accepted 8 Jul 2023*
*Available online 27 Dec 2023*

**ABSTRACT**: In this paper, a nonlinear mathematical model is formulated to study the unemployment problem in Thailand after the COVID-19 outbreak by considering the number of unemployed, the number of employed people, and the number of new migrants to find work. Moreover, the stability of this system is analyzed. It is found that this system has two equilibriums: the employment-free equilibrium and the positive equilibrium. Additionally, two equilibrium points are stable under certain conditions. The corresponding parameters in the models are derived from data from the National Statistical Office, the Ministry of Digital Economy and Society, and the Foreign Workers Administration Office during COVID-19 in 2020. Finally, the numerical experiments are demonstrated to validate the theoretical results and model.

**KEYWORDS**: unemployment, nonlinear system, stability

**MSC2020**: 34D20 92D25

## INTRODUCTION

The COVID-19 outbreak is such a crisis that no country has planned for or prepared its health system to cope with such a situation. Since this virus is readily transmitted between individuals, as a consequence, the rate of infection spread is accelerating. Numerous countries take lockdown measures, suspending all activities that risk spreading the virus to restrict group movement and reduce close contact. It results in allowing foreigners into the country. As lockdowns affect numerous economic activities, lockdowns directly affect individuals involved in those activities and indirectly affect other activities that were not locked down, for example, tourism-related businesses, including hotels, restaurants, special events, leisure activities, and airlines. Other service businesses in the supply chain, such as special events, catering, laundry services, restaurants, and transportation, suffered an immediate decline due to the closed hotels and tourist attractions. For instance, most restaurants were forced to close, although a shift to delivery-only sales permitted a few to remain open. Numerous people have lost their jobs and become unemployed due to COVID-19. As a result, the rate of change in unemployed people has been steadily increasing. This work aims to investigate unemployment in Thailand after the outbreak of COVID-19 through mathematical models governed by a system of nonlinear ordinary differential equations.

In recent years, many researchers have been interested in modeling the unemployment problem [1–11]. In 2011 Misra and Singh [6] presented the mathematical model of unemployment where the number of unemployed, regularly employed, and temporarily persons are considered. Following that, it gets expanded into several varieties. In 2015, Pathan and Bhathawala [10] evaluated the impact of self-employment on the unemployment rate in 2015, and Daud and Ghozali [4] provided a mathematical model based on two variables: the number of employed and the number of jobless. Additionally, Pathan and Bhathawala extended their unemployment model in 2016 by incorporating four dynamic variables: the number of unemployed, the number of new migrant workers, the number of employed, and the number of newly created jobs in the public-private sectors. Moreover, in 2017 Misra and Singh [7] proposed the unemployment model by examining the impact of skill development opportunities offered to unemployed individuals by some academic institutions as a critical component of resolving the problem. This model extended their previous findings by including four variables: the number of unemployed, temporary/self-employed individuals, regularly employed individuals, and opportunities for skill development among the unemployed. They discovered that as skill development avenues become effective, the number of unemployed people decreases while temporary/self-employed people increase. In addition, Al-Maalwi et al [1–3] also investigated unemployment. Their work proposed that increasing the employment rate and decreasing the rate diminution can reduce the unemployment problem. Following that, they study the effect of government support on the unemployment rate in developed countries. Additionally, they found that expanding vocational institutes to develop technical skills and craftsmanship can significantly reduce the unemployed problem and that improving the quality of education can significantly reduce the unemployed problem.

According to the worsening COVID-19 situation in Thailand, tens of thousands of Cambodian, Laotian, and Myanmar migrant laborers reportedly returned home in March 2020. Some foreign workers remained in Thailand after the full-scale lockdown was implemented, and others outside Thailand could not return to work [12, 13]. The COVID-19 situation has a direct effect on foreign workers. In this work, our contribution to the mathematical investigation of the unemployment model focuses mainly on the number of unemployed, employed, and new migrants to find work.

## AN UNEMPLOYMENT MODEL

This section aims to develop a dynamic system to describe the evolution of the unemployment rate. In the modeling process, three dynamical variables are considered: the number of unemployed persons, the number of employed persons, and the number of migrants who have entered the labor force. The mathematical model for unemployment is presented as follows:

$$\begin{aligned} \dot{U}(t) &= A_U - k_1 U(t)V(t) + k_3 V(t) + k_4 M(t) - \gamma_U U(t), \\ \dot{V}(t) &= k_1 U(t)V(t) + k_2 M(t)V(t) - k_3 V(t) - \gamma_V V(t), \quad (1) \\ \dot{M}(t) &= A_M - k_2 M(t)V(t) - k_4 M(t) - \gamma_M M(t), \end{aligned}$$

where the variables $U(t)$, $V(t)$, and $M(t)$ represent the number of unemployed, employed, and employed persons at any given time $t$, respectively. The first equation represents the rate of change of the unemployed population, increasing with a constant rate of $A_U$ over time. The second term in the first equation models the unemployed person's gaining employment, which is proportional to $U(t)V(t)$ with a constant of proportionality $k_1$. A positive constant $\gamma_U$ represents the rate of migration or mortality rate of unemployed individuals. The positive constant $k_3$ represents the rates of people being fired from their jobs, and the rate $\gamma_M$ denotes the number of unemployed immigrants. The second equation models the rate of change of the number of employed persons. It increases as some unemployed persons, and immigrants get jobs, whereas the rate of change in the number of employed individuals decreases with the rates of laid-off workers, $k_3$ and the rates of retirement or death of employed persons $\gamma_V$. The third equation describes the rate of change the number of immigrant.

To rewrite our unemployment system to general form, we define

$$\mathbf{x}(t) = \begin{bmatrix} U(t) \\ V(t) \\ M(t) \end{bmatrix} \quad \text{and}$$

$$\mathbf{f}(t, \mathbf{x}) = \begin{bmatrix} A_U - k_1 U(t)V(t) + k_3 V(t) + k_4 M(t) - \gamma_U U(t) \\ k_1 U(t)V(t) + k_2 M(t)V(t) - k_3 V(t) - \gamma_V V(t) \\ A_M - k_2 M(t)V(t) - k_4 M(t) - \gamma_M M(t) \end{bmatrix},$$

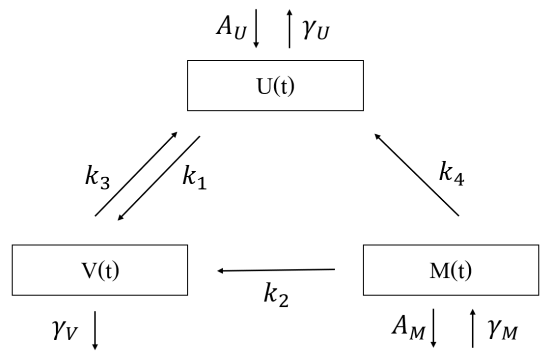


**Fig. 1** Schematic diagram of the mathematical model for unemployment in Thailand after the outbreak of COVID-19.

where $\mathbf{x}(t) \in \mathbb{R}^{3+} \cup \{0\}$. Therefore, the unemployment system can be written in the compact form

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x})$$

with given initial value $\mathbf{x}(t_0) = \mathbf{x}_0 = (U_0, V_0, M_0)^\top$.

Next, we shows that the solutions of the system (1) is non-negative and bounded as shown in Theorem 1.

**Theorem 1** *Let* $\mathbf{x}(t) = (U(t), V(t), M(t))^\top \in \mathbb{R}^{3+} \cup \{0\}$ *be a solution of the system* (1). *Then, the set defined by*

$$\Omega = \left\{ (U(t), V(t), M(t))^\top \in \mathbb{R}^{3+} \cup \{0\} \;\middle|\; 0 \leqslant U(t) + V(t) + M(t) \leqslant \frac{A}{\gamma} \right\},$$

*where* $A = A_U + A_M$ *and* $\gamma = \min(\gamma_U, \gamma_V, \gamma_M)$*, is positively invariant.*

*Proof*: Consider the the right hand-side of the system (1) with

$$\left.\frac{\mathrm{d}U(t)}{\mathrm{d}t}\right|_{U(t)=0} = A_U + k_3 V(t) + k_4 M(t),$$
$$\left.\frac{\mathrm{d}V(t)}{\mathrm{d}t}\right|_{V(t)=0} = 0,$$
$$\left.\frac{\mathrm{d}M(t)}{\mathrm{d}t}\right|_{M(t)=0} = A_M.$$

Since $A_U, k_3, k_4$ and $A_M > 0$, this implies that for $t \geqslant 0$, solutions of the system which are non-negative remain non-negative. If we combine all equations above, then it gets

$$\begin{aligned} \frac{\mathrm{d}U(t)}{\mathrm{d}t} + \frac{\mathrm{d}V(t)}{\mathrm{d}t} + \frac{\mathrm{d}M(t)}{\mathrm{d}t} &= A_U + A_M - \gamma_U U(t) - \gamma_V V(t) - \gamma_M M(t) \\ &= A_U + A_M - \gamma U(t) - \gamma V(t) - \gamma M(t) \\ &\leqslant A - \gamma[U(t) + V(t) + M(t)], \end{aligned}$$

where $A = A_U + A_M$ and $\gamma = min(\gamma_U, \gamma_V, \gamma_M)$. Then, we take the limit supremum implies that

$$\lim_{t\to\infty} \sup\Big(\frac{dU(t)}{dt} + \frac{dV(t)}{dt} + \frac{dM(t)}{dt}\Big) \leqslant \lim_{t\to\infty} \sup\,[A - \gamma(U(t) + V(t) + M(t)].$$

We know that $\frac{dU(t)}{dt} + \frac{dV(t)}{dt} + \frac{dM(t)}{dt} \geqslant 0$, then it yields that

$$\lim_{t\to\infty} \sup\Big(\frac{dU(t)}{dt} + \frac{dV(t)}{dt} + \frac{dM(t)}{dt}\Big) \geqslant \lim_{t\to\infty} \sup(0),$$
$$\lim_{t\to\infty} \sup\Big(\frac{dU(t)}{dt} + \frac{dV(t)}{dt} + \frac{dM(t)}{dt}\Big) \geqslant 0.$$

Hence,

$$\lim_{t\to\infty} \sup\,[A - \gamma(U(t) + V(t) + M(t))] \geqslant 0$$
$$\lim_{t\to\infty} \sup A - \lim_{t\to\infty} \sup\,(\gamma)(U(t) + V(t) + M(t)) \geqslant 0$$
$$A - \gamma \lim_{t\to\infty} \sup\,(U(t) + V(t) + M(t)) \geqslant 0$$
$$\lim_{t\to\infty} \sup\,(U(t) + V(t) + M(t)) \leqslant \frac{A}{\gamma}.$$

Therefore,

$$\lim_{t\to\infty} \sup\,[U(t) + V(t) + M(t)] \leqslant \frac{A}{\gamma}.$$

This proves that all solution of system (1) are bounded and do not exit the region Ω. As a consequence, Ω is positive invariant. □

Theorem 1 concludes that the system's solutions (1) are positive and bounded, i.e., the number of unemployed, employed, and migrant individuals are positive and bounded. The proposed model is therefore valid and well-defined.

## STABILITY ANALYSIS OF THE UNEMPLOYMENT SYSTEM

This section investigates the stability of the unemployment system. First, the equilibrium points of the system are determined by Theorem 2. Our system consists of two equilibria; free equilibrium, and positive equilibrium. We note that the definition of free equilibrium is the point that no one is employed. Further, both equilibrium points are classified, and the stability of the unemployment system is analyzed as detailed in the following subsection.

### Equilibria of the unemployment system

**Definition 1 (Free equilibrium)** Free equilibrium is defined as the point at which there are no employed people in the population, which is denoted in the model as $V_0^* = 0$.

**Theorem 2** *The unemployment system* (1) *has two equilibrium points:*

(i) *The employment-free equilibrium* $Q_0^* = (U_0^*, V_0^*, M_0^*)$ *which always exist.*
(ii) *The positive equilibrium* $Q_+^* = (U_+^*, V_+^*, M_+^*)$ *which exists when*

$$k_2 > k_1, \quad \gamma = \gamma_U = \gamma_V = \gamma_M, \quad \textit{and} \quad R > 1,$$

*where*

$$R = \frac{(k_4 + \gamma_M)(A_U k_1) + (k_1 k_4 + k_2 \gamma_U) A_M}{(k_4 + \gamma_M)(k_3 + \gamma_V)\gamma_U}.$$

*Proof*: Let $(U^*, V^*, M^*)$ be an equilibrium points of system (1), that is, the following equations are hold

$$A_U - k_1 U^* V^* + k_3 V^* + k_4 M^* - \gamma_U U^* = 0, \qquad (2a)$$
$$k_1 U^* V^* + k_2 M^* V^* - k_3 V^* - \gamma_V V^* = 0, \qquad (2b)$$
$$A_M - k_2 M^* V^* - k_4 M^* - \gamma_M M^* = 0. \qquad (2c)$$

Consider the equation (2b), it yields

$$(k_1 U^* + k_M^* - k_3 - \gamma_V) V^* = 0,$$

Since all parameters are positive, it implies that $V^* = 0$ and we denote $V^*$ as $V_0^* = 0$. As $V^* = 0$, therefore, from equations (2a) and (2c), we obtain

$$U_0^* = \frac{(k_4 + \gamma_M) A_U + A_M k_4}{(k_4 + \gamma_M)\gamma_U},$$
$$M_0^* = \frac{A_M}{k_4 + \gamma_M}.$$

Hence, the first equilibrium, called free equilibrium, is

$$Q_0^* = (U_0^*, V_0^*, M_0^*) = \Big(\frac{(k_4 + \gamma_M) A_U + A_M k_4}{(k_4 + \gamma_M)\gamma_U}, 0, \frac{A_M}{k_4 + \gamma_M}\Big).$$

Next, we find the second equilibrium. From equations (2a) and (2c), we write $U^*$ and $M^*$ in term of $V^*$ as follows.

$$U^* = \frac{(A_U + k_3 V^*)(k_2 V^* + k_4 + \gamma_M) + A_M k_4}{(k_2 V^* + k_4 + \gamma_M)(k_1 V^* + \gamma_U)},$$
$$M^* = \frac{A_M}{k_2 V^* + k_4 + \gamma_M}.$$

Therefore the second equilibrium is

$$Q^* = (U^*, V^*, M^*) = \Big(\frac{(A_U + k_3 V^*)(k_2 V^* + k_4 + \gamma_M) + A_M k_4}{(k_2 V^* + k_4 + \gamma_M)(k_1 V^* + \gamma_U)}, V^*, \frac{A_M}{k_2 V^* + k_4 + \gamma_M}\Big).$$

Substituting $U^*$ and $M^*$ into the equation (2b), then it gets

$$(k_1 V^*)\Big[\frac{(A_U + k_3 V^*)(k_2 V^* + k_4 + \gamma_M) + A_M k_4}{(k_2 V^* + k_4 + \gamma_M)(k_1 V^* + \gamma_U)}\Big] + (k_2 V^*)\Big[\frac{A_M}{k_2 V^* + k_4 + \gamma_M}\Big] - k_3 V^* - \gamma_V V^* = 0$$

$$\left[k_1\left[\frac{(A_U+k_3V^*)(k_2V^*+k_4+\gamma_M)+A_Mk_4}{(k_2V^*+k_4+\gamma_M)(k_1V^*+\gamma_U)}\right] + k_2\left[\frac{A_M}{k_2V^*+k_4+\gamma_M}\right]-k_3-\gamma_V\right]V^*=0$$

Since $V^*>0$,

$$k_1\left[\frac{(A_U+k_3V^*)(k_2V^*+k_4+\gamma_M)+A_Mk_4}{(k_2V^*+k_4+\gamma_M)(k_1V^*+\gamma_U)}\right] + k_2\left[\frac{A_M}{k_2V^*+k_4+\gamma_M}\right]-k_3-\gamma_V=0.$$

Multiplying $(k_2V^*+k_4+\gamma_M)(k_1V^*+\gamma_U)$ both side of equation, we get

$$k_1\left[(A_U+k_3V^*)(k_2V^*+k_4+\gamma_M)+A_Mk_4\right]+A_Mk_2(k_1V^*+\gamma_U) -(k_3+\gamma_V)(k_2V^*+k_4+\gamma_M)(k_1V^*+\gamma_U)=0$$

Consider the first term,

$$\begin{aligned} k_1&\left[(A_U+k_3V^*)(k_2V^*+k_4+\gamma_M)+A_Mk_4\right]\\ &=k_1(A_Uk_2V^*+A_Uk_4+A_U\gamma_M+k_2k_3(V^*)^2+k_3k_4V^*\\ &\quad +k_3\gamma_MV^*+A_Mk_4)\\ &=A_Uk_1k_2V^*+A_Uk_1k_4+A_Uk_1\gamma_M+k_1k_2k_3(V^*)^2\\ &\quad +k_1k_3k_4V^*+k_1k_3\gamma_MV^*+A_Mk_1k_4\\ &=k_1k_2k_3(V^*)^2+(A_Uk_1k_2+k_1k_3k_4+k_1k_3\gamma_M)V^*\\ &\quad +A_Uk_1k_4+A_Uk_1\gamma_M+A_Mk_1k_4. \end{aligned}$$

The second term is presented as

$$A_Mk_2(k_1V^*+\gamma_U)=A_Mk_1k_2V^*+A_Mk_2\gamma_U.$$

In addition, the third term is expressed as follows.

$$\begin{aligned} (k_3&+\gamma_V)(k_2V^*+k_4+\gamma_M)(k_1V^*+\gamma_U)\\ &=(k_3+\gamma_V)(k_1k_2(V^*)^2+k_2\gamma_UV^*+k_1k_4V^*+k_4\gamma_U\\ &\quad +k_1\gamma_MV^*+\gamma_U\gamma_M)\\ &=k_1k_2k_3(V^*)^2+k_2k_3\gamma_UV^*+k_1k_3k_4V^*+k_3k_4\gamma_U\\ &\quad +k_1k_3\gamma_MV^*+k_3\gamma_U\gamma_M+k_1k_2\gamma_V(V^*)^2+k_2\gamma_U\gamma_VV^*\\ &\quad +k_1k_4\gamma_VV^*+k_4\gamma_U\gamma_V+k_1\gamma_V\gamma_MV^*+\gamma_U\gamma_V\gamma_M\\ &=(k_1k_2k_3+k_1k_2\gamma_V)(V^*)^2+(k_2k_3\gamma_U+k_1k_3k_4\\ &\quad +k_1k_3\gamma_M+k_2\gamma_U\gamma_V+k_1k_4\gamma_V+k_1\gamma_V\gamma_M)V^*\\ &\quad +k_3k_4\gamma_U+k_3\gamma_U\gamma_M+k_4\gamma_U\gamma_V+\gamma_U\gamma_V\gamma_M. \end{aligned}$$

We combine all three terms, then

$$\begin{aligned} &(-k_1k_2\gamma_V)(V^*)^2+(A_Uk_1k_2+A_Mk_1k_2-k_2k_3\gamma_U-k_2\gamma_U\gamma_V\\ &\quad -k_1k_4\gamma_V-k_1\gamma_V\gamma_M)V^*+(A_Uk_1k_4+A_Uk_1\gamma_M+A_Mk_1k_4\\ &\quad +A_Mk_2\gamma_U-k_3k_4\gamma_U-k_3\gamma_U\gamma_M-k_4\gamma_U\gamma_V-\gamma_U\gamma_V\gamma_M)=0 \end{aligned}$$

Multiple $(-1)$ both-side of equation, then

$$\begin{aligned} &(k_1k_2\gamma_V)(V^*)^2+(k_1k_4\gamma_V+k_2k_3\gamma_U+k_2\gamma_U\gamma_V+k_1\gamma_V\gamma_M\\ &\quad -A_Uk_1k_2-A_Mk_1k_2)V^*+(k_3k_4\gamma_U+k_3\gamma_U\gamma_M+k_4\gamma_U\gamma_V\\ &\quad +\gamma_U\gamma_V\gamma_M-A_Uk_1k_4-A_Uk_1\gamma_M-A_Mk_1k_4-A_Mk_2\gamma_U)=0 \end{aligned}$$

The second term is rewritten as in the following,

$$\begin{aligned} &(k_1k_4\gamma_V+k_2k_3\gamma_U+k_2\gamma_U\gamma_V+k_1\gamma_V\gamma_M-A_Uk_1k_2-A_Mk_1k_2)V^*\\ &=\left[(k_4+\gamma_M)(k_1\gamma_V)+(k_3+\gamma_V)(k_2\gamma_U)-(A_U+A_M)(k_1k_2)\right]V^*. \end{aligned}$$

Moreover the third term are simplified as follows.

$$\begin{aligned} &(k_3k_4\gamma_U+k_3\gamma_U\gamma_M+k_4\gamma_U\gamma_V+\gamma_U\gamma_V\gamma_M-A_Uk_1k_4-A_Uk_1\gamma_M\\ &\quad -A_Mk_1k_4-A_Mk_2\gamma_U)\\ &=(k_4+\gamma_M)(k_3\gamma_U)+(k_4+\gamma_M)(\gamma_U\gamma_V)-(k_4+\gamma_M)(A_Uk_1)\\ &\quad -(k_1k_4+k_2\gamma_U)A_M\\ &=(k_4+\gamma_M)(k_3\gamma_U+\gamma_U\gamma_V)-(k_4+\gamma_M)(A_Uk_1)-(k_1k_4+k_2\gamma_U)A_M\\ &=\gamma_U(k_4+\gamma_M)(k_3+\gamma_V)-(k_4+\gamma_M)(A_Uk_1)-(k_1k_4+k_2\gamma_U)A_M\\ &=\gamma_U(k_4+\gamma_M)(k_3+\gamma_V)\left[1-\frac{(k_4+\gamma_M)(A_Uk_1)+(k_1k_4+k_2\gamma_U)A_M}{(k_4+\gamma_M)(k_3+\gamma_V)\gamma_U}\right]. \end{aligned}$$

Let

$$R=\frac{(k_4+\gamma_M)(A_Uk_1)+(k_1k_4+k_2\gamma_U)A_M}{(k_4+\gamma_M)(k_3+\gamma_V)\gamma_U}.$$

Hence, the third term is expressed as

$$\gamma_U(k_4+\gamma_M)(k_3+\gamma_V)\left[1-\frac{(k_4+\gamma_M)(A_Uk_1)+(k_1k_4+k_2\gamma_U)A_M}{(k_4+\gamma_M)(k_3+\gamma_V)\gamma_U}\right] =\gamma_U(k_4+\gamma_M)(k_3+\gamma_V)(1-R).$$

We combine the second term and the third term, then the following term

$$\begin{aligned} &(k_1k_2\gamma_V)(V^*)^2+\left[(k_4+\gamma_M)(k_1\gamma_V)+(k_3+\gamma_V)(k_2\gamma_U)\right.\\ &\quad \left.-(A_U+A_M)(k_1k_2)\right]V^*+\gamma_U(k_4+\gamma_M)(k_3+\gamma_V)(1-R) \end{aligned}$$

is equal to zero, which can be written as the quadratic equation

$$D_1(V^*)^2+D_2V^*+D_3=0, \tag{3}$$

where

$$\begin{aligned} D_1&=k_1k_2\gamma_V,\\ D_2&=(k_3+\gamma_V)(k_2\gamma_U)+(k_4+\gamma_M)(k_1\gamma_V)-(A_U+A_M)(k_1k_2),\\ D_3&=\gamma_U(k_4+\gamma_M)(k_3+\gamma_V)(1-R), \end{aligned}$$

and

$$R=\frac{(k_4+\gamma_M)(A_Uk_1)+(k_1k_4+k_2\gamma_U)A_M}{(k_4+\gamma_M)(k_3+\gamma_V)\gamma_U}.$$

The algebraic solution of the quadratic equation (3) is expressed in the following form

$$V_1^*=\frac{-D_2+\sqrt{D_2^2-4D_1D_3}}{2D_1},$$

$$V_2^*=\frac{-D_2-\sqrt{D_2^2-4D_1D_3}}{2D_1}.$$

Next, values of $V_1^*$ and $V_2^*$ are analyzed. It can be seen that the value $R$ affects to $D_3$. Therefore we consider three cases:

**Case I:** If $R > 1$, then $D_3 < 0$. Due to $D_1 > 0$, it yields $-4D_1D_3 > 0$. In addition, $D_2 \leqslant \sqrt{D_2^2}$, it implies that

$$D_2 < \sqrt{D_2^2 - 4D_1D_3},$$

That is,

$$V_1^* = \frac{-D_2 + \sqrt{D_2^2 - 4D_1D_3}}{2D_1} > 0,$$
$$V_2^* = \frac{-D_2 - \sqrt{D_2^2 - 4D_1D_3}}{2D_1} < 0.$$

Therefore, the system (1) has one positive equilibrium point $Q^* = (U^*, V_1^*, M^*)$.

**Case II:** If $R < 1$, then $D_3 > 0$. It is divided into two sub-cases

(a) If $D_2 \geqslant 0$. It implies that $-4D_1D_3 < 0$ and $D_2 > \sqrt{D_2^2 - 4D_1D_3}$. Therefore,

$$V_1^* = \frac{-D_2 + \sqrt{D_2^2 - 4D_1D_3}}{2D_1} < 0,$$
$$V_2^* = \frac{-D_2 - \sqrt{D_2^2 - 4D_1D_3}}{2D_1} < 0.$$

Hence, the system (1) has no positive equilibrium.

(b) If $D_2 < 0$ and $D_2^2 > 4D_1D_3$. We get that $-4D_1D_3 < 0$, and $D_2 > \sqrt{D_2^2 - 4D_1D_3}$. Therefore,

$$V_1^* = \frac{-D_2 + \sqrt{D_2^2 - 4D_1D_3}}{2D_1} > 0,$$
$$V_2^* = \frac{-D_2 - \sqrt{D_2^2 - 4D_1D_3}}{2D_1} > 0.$$

Then the system (1) has two positive equilibrium points

$$Q^* = (U^*, V_1^*, M^*) \quad \text{and} \quad Q^* = (U^*, V_2^*, M^*).$$

**Case III:** If $R = 1$, then $D_3 = 0$. We consider two sub-cases.

(a) If $D_2 \geqslant 0$. Then we get that

$$V_1^* = \frac{-D_2 + \sqrt{D_2^2 - 4D_1D_3}}{2D_1} = 0,$$
$$V_2^* = \frac{-D_2 - \sqrt{D_2^2 - 4D_1D_3}}{2D_1} < 0.$$

That is, the system (1) has no positive equilibrium.

(b) If $D_2 < 0$. Therefore, the system (1) has one positive equilibrium point $Q^* = (U^*, V_1^*, M^*)$ as follows.

$$V_1^* = \frac{-D_2 + \sqrt{D_2^2 - 4D_1D_3}}{2D_1} > 0,$$
$$V_2^* = \frac{-D_2 - \sqrt{D_2^2 - 4D_1D_3}}{2D_1} = 0.$$

□

**Remark 1** Note that the positive equilibrium points obtained in the part (b) in Case II and in Case III do not exist, since

$$D_2 < 0 \iff A_U k_1 > \frac{(k_3+\gamma_V)(k_2\gamma_U)+(k_4+\gamma_M)(k_1\gamma_V)-A_M k_1 k_2}{k_2},$$
$$R \leqslant 1 \iff A_U k_1 \leqslant \frac{(k_4+\gamma_M)(k_3+\gamma_V)\gamma_U-(k_1k_4+k_2\gamma_U)A_M}{k_4+\gamma_4}.$$

Combining these two inequality expressions yields the following relationship:

$$\frac{(k_3+\gamma_V)(k_2\gamma_U)+(k_4+\gamma_M)(k_1\gamma_V)-A_M k_1 k_2}{k_2} < \frac{(k_4+\gamma_M)(k_3+\gamma_V)\gamma_U-(k_1k_4+k_2\gamma_U)A_M}{k_4+\gamma_M}.$$

Multiplying $k_2(k_4+\gamma_M)$ both-side of this inequality yields

$$[(k_3+\gamma_V)(k_2\gamma_U)+(k_4+\gamma_M)(k_1\gamma_V)-A_M k_1 k_2](k_4+\gamma_M) < [(k_4+\gamma_M)(k_3+\gamma_V)\gamma_U-(k_1k_4+k_2\gamma_U)A_M]k_2.$$

Let us consider the left-hand side of this inequality.

$$\begin{aligned}&[(k_3+\gamma_V)(k_2\gamma_U)+(k_4+\gamma_M)(k_1\gamma_V)-A_M k_1 k_2](k_4+\gamma_M)\\&=(k_2k_3\gamma_U+k_2\gamma_U\gamma_V+k_1k_4\gamma_V+k_1\gamma_V\gamma_M-A_M k_1k_2)(k_4+\gamma_M)\\&=k_2k_3k_4\gamma_U+k_2k_4\gamma_U\gamma_V+k_1k_4^2\gamma_V+k_1k_4\gamma_V\gamma_M-A_M k_1k_2k_4\\&+k_2k_3\gamma_U\gamma_M+k_2\gamma_U\gamma_V\gamma_M+k_1k_4\gamma_V\gamma_M+k_1\gamma_V\gamma_M^2-A_M k_1k_2\gamma_M.\end{aligned}$$

Additionally, the right-hand side of the above inequality is presented as

$$\begin{aligned}&[(k_4+\gamma_M)(k_3+\gamma_V)\gamma_U-(k_1k_4+k_2\gamma_U)A_M]k_2\\&=(k_3k_4\gamma_U+k_4\gamma_U\gamma_V+k_3\gamma_U\gamma_M+\gamma_U\gamma_V\gamma_M-A_M k_1k_4\\&\quad-A_M k_2\gamma_U)k_2\\&=k_2k_3k_4\gamma_U-k_2k_4\gamma_U\gamma_V-k_2k_3\gamma_U\gamma_M-k_2\gamma_U\gamma_V\gamma_M\\&\quad+A_M k_1k_2k_4+A_M k_2^2\gamma_U.\end{aligned}$$

We combine the first term and the second term, then

$$k_1k_4^2\gamma_V+2k_1k_4\gamma_V\gamma_M+k_1\gamma_V\gamma_M^2+A_M k_2^2\gamma_U-A_M k_1k_2\gamma_M < 0,$$
$$(k_4^2+2k_4\gamma_M+\gamma_M^2)(k_1\gamma_V)+(k_2\gamma_U-k_1\gamma_M)(A_M k_2) < 0.$$

Hence,

$$(k_4+\gamma_M)^2(k_1\gamma_V)+(k_2\gamma_U-k_1\gamma_M)(A_M k_2)<0. \quad (4)$$

Since our assumptions that parameters are positive, $k_2 > k_1$, and $\gamma_U = \gamma_V = \gamma_M$, then the inequality (4) is greater than 0, that is,

$$(k_4+\gamma_M)^2(k_1\gamma_V)+(k_2\gamma_U-k_1\gamma_M)(A_M k_2)>0,$$

which contradicts. It is concluded that the system (1) has only one positive equilibrium when $R > 1$.

To sum up, the system (1) indicates the existence of two equilibrium point. The first one is the employment free equilibrium denoted by $Q_0^*$ and the second equilibrium is denoted by $Q_+^*$.

**Stability of an unemployment model**

The local stability behavior of the equilibrium point $Q_0^*$ and $Q_+^*$ is investigated by using the linearization method. Let $\mathbf{f}=(f_1,f_2,f_3)^\top$, where

$$f_1(U,V,M)=A_U-k_1U(t)V(t)+k_3V(t)+k_4M(t)-\gamma_U U(t),$$
$$f_2(U,V,M)=k_1U(t)V(t)+k_2M(t)V(t)-k_3V(t)-\gamma_V V(t),$$
$$f_3(U,V,M)=A_M-k_2M(t)V(t)-k_4M(t)-\gamma_M M(t).$$

To analyze the stability of the system (1), the linearization of the above system at equilibrium point $Q^*=(U^*,V^*,M^*)$ can be presented as follows.

$$J(Q^*)=\begin{bmatrix}\frac{\partial f_1}{\partial U}(Q^*) & \frac{\partial f_1}{\partial V}(Q^*) & \frac{\partial f_1}{\partial M}(Q^*)\\ \frac{\partial f_2}{\partial U}(Q^*) & \frac{\partial f_2}{\partial V}(Q^*) & \frac{\partial f_2}{\partial M}(Q^*)\\ \frac{\partial f_3}{\partial U}(Q^*) & \frac{\partial f_3}{\partial V}(Q^*) & \frac{\partial f_3}{\partial M}(Q^*)\end{bmatrix}$$
$$=\begin{bmatrix}-k_1V^*-\gamma_U & -k_1U^*+k_3 & k_4\\ k_1V^* & k_1U^*+k_2M^*-k_3-\gamma_V & k_2V^*\\ 0 & -k_2M^* & -k_2V^*-k_4-\gamma_M\end{bmatrix}.$$

Further, the characteristics of equilibrium points $Q_0^*$ and $Q_+^*$ are determined that are related to the values $R$ and $D_3$, which are defined in Theorem 1. To facilitate ease, we recall it again as follows.

$$R=\frac{(k_4+\gamma_M)(A_Uk_1)+(k_1k_4+k_2\gamma_U)A_M}{(k_4+\gamma_M)(k_3+\gamma_V)\gamma_U},$$
$$D_3=\gamma_U(k_4+\gamma_M)(k_3+\gamma_V)(1-R).$$

The stability of the equilibrium $Q_0^*=(U_0^*,V_0^*,M_0^*)$ is investigated as stated in the following Theorem 3.

**Theorem 3** *The employment free equilibrium*

$$Q_0^*=\left(\frac{(k_4+\gamma_M)A_U+A_Mk_4}{(k_4+\gamma_M)\gamma_U},0,\frac{A_M}{k_4+\gamma_M}\right)$$

*is locally asymptotically stable if $R<1$. Whereas, if $R>1$, it is unstable.*

*Proof*: The Jacobian matrix at the employment free equilibrium $Q_0^*$ is evaluated as follows

$$J(Q_0^*)=$$
$$\begin{bmatrix}-\gamma_U & -k_1\left(\frac{(k_4+\gamma_M)A_U+A_Mk_4}{(k_4+\gamma_M)\gamma_U}\right)+k_3 & k_4\\ 0 & k_1\left(\frac{(k_4+\gamma_M)A_U+A_Mk_4}{(k_4+\gamma_M)\gamma_U}\right)+k_2\left(\frac{A_M}{k_4+\gamma_M}\right)-k_3-\gamma_V & 0\\ 0 & -k_2\left(\frac{A_M}{k_4+\gamma_M}\right) & -k_4-\gamma_M\end{bmatrix}.$$

The characteristic equation of $J(Q_0^*)$ is given by

$$\det(J(Q_0^*)-\lambda I)=0, \quad \lambda \quad \text{is eigenvalue.}$$

$$\begin{vmatrix}-\gamma_U-\lambda & -k_1\left(\frac{(k_4+\gamma_M)A_U+A_Mk_4}{(k_4+\gamma_M)\gamma_U}\right)+k_3 & k_4\\ 0 & k_1\left(\frac{(k_4+\gamma_M)A_U+A_Mk_4}{(k_4+\gamma_M)\gamma_U}\right)+k_2\left(\frac{A_M}{k_4+\gamma_M}\right)-k_3-\gamma_V-\lambda & 0\\ 0 & -k_2\left(\frac{A_M}{k_4+\gamma_M}\right) & -k_4-\gamma_M-\lambda\end{vmatrix}=0,$$

$$(-\gamma_U-\lambda)\bigg[k_1\left(\frac{(k_4+\gamma_M)A_U+A_Mk_4}{(k_4+\gamma_M)\gamma_U}\right)$$
$$+k_2\left(\frac{A_M}{k_4+\gamma_M}\right)-k_3-\gamma_V-\lambda\bigg]\big(-k_4-\gamma_M-\lambda\big)=0.$$

Then we get

$$\lambda_1=-\gamma_U,$$
$$\lambda_2=\frac{(k_4+\gamma_M)A_Uk_1+A_Mk_4k_1+A_Mk_2\gamma_U-(k_3+\gamma_V)(k_4+\gamma_M)\gamma_U}{(k_4+\gamma_M)\gamma_U}$$
$$=-\gamma_U(k_4+\gamma_M)(k_3+\gamma_V)\left(\frac{1}{(k_4+\gamma_M)\gamma_U}\right)$$
$$\times\left(1-\frac{(k_4+\gamma_M)(A_Uk_1)+(k_1k_4+k_2\gamma_U)A_M}{(k_4+\gamma_M)(k_3+\gamma_V)\gamma_U}\right)$$
$$=-\gamma_U(k_4+\gamma_M)(k_3+\gamma_V)(1-R)\left(\frac{1}{(k_4+\gamma_M)\gamma_U}\right)$$
$$=-D_3\left(\frac{1}{(k_4+\gamma_M)\gamma_U}\right),$$
$$\lambda_3=-k_4-\gamma_M.$$

As $k_4$, $\gamma_U$, and $\gamma_M$ are positive, it can be seen that the first eigenvalue $\lambda_1$ and the third eigenvalue $\lambda_3$ are negative. If $R<1$, then the second eigenvalue ($\lambda_2$) is negative. We conclude that the employment free equilibrium $Q_0^*$ is locally asymptotically stable if $R<1$, whereas, it is unstable if $R>1$. □

It can be concluded from Theorem 3 that if the parameter $R<1$, then when $t\to\infty$, the solution of the system converges to a free-equilibrium point. In other words, the number of unemployed, employed, and migrant people converges to $U_0^*$, 0, and $M_0^*$, respectively. Next, we determine the stability of the equilibrium $Q_+^*$ as stated in the following theorem.

**Theorem 4** *If $R>1$, $k_3>k_4$, and $\gamma_U=\gamma_V=\gamma_M$, then the positive equilibrium $Q_+^*=(U^*,V_1^*,M^*)$ is locally asymptotically stable.*

*Proof*: The Jacobian matrix at the positive equilibrium $Q^*$ is obtained as follows.

$$J(Q^*) = \begin{bmatrix} -k_1V^*-\gamma_U & k_1U^*-k_3 & k_4 \\ k_1V^* & k_1U^*+k_2M^*-k_3-\gamma_V & k_2V^* \\ 0 & -k_2M^* & -k_2V^*-k_4-\gamma_M \end{bmatrix}.$$

The characteristic equation of $J(Q^*)$ is given by

$$\det(J(Q^*)-\lambda I)=0, \quad \lambda \quad \text{is eigenvalue.}$$

$$\begin{vmatrix} -k_1V^*-\gamma_U-\lambda & -k_1U^*+k_3 & k_4 \\ k_1V^* & k_1U^*+k_2M^*-k_3-\gamma_V-\lambda & k_2V^* \\ 0 & -k_2M^* & -k_2V^*-k_4-\gamma_M-\lambda \end{vmatrix}=0.$$

The characteristics equation of the above matrix is presented as the following equation.

$$\begin{aligned}(-k_1V^*-\gamma_U-\lambda)\big[&(k_1U^*+k_2M^*-k_3-\gamma_V-\lambda)\\ &\times(-k_2V^*-k_4-\gamma_M-\lambda)-(k_2V^*)(-k_2M^*)\big]\\ -(k_1V^*)[(-k_1U^*&+k_3)(-k_2V^*-k_4-\gamma_M-\lambda)-k_4(-k_2M^*)]=0.\end{aligned}$$

It can be written in the general form of polynomial of degree three as stated in the following

$$\lambda^3+a_1\lambda^2+a_2\lambda+a_3=0,$$

where

$$a_1=(V^*-U^*)k_1+(V^*-M^*)k_2+k_3+k_4+\gamma_U+\gamma_V+\gamma_M,$$

$$\begin{aligned}a_2&=(V^*-U^*-M^*)(k_1k_2V^*)+(V^*-U^*)(k_1k_4)\\ &+(V^*-U^*)(k_1\gamma_M)+(V^*-M^*)(k_2\gamma_U)+(\gamma_VV^*-\gamma_UU^*)k_1\\ &+(\gamma_UV^*-\gamma_MM^*)k_2+(k_3V^*-k_4M^*)k_2+k_3k_4+k_3\gamma_M\\ &+k_3\gamma_V+k_4\gamma_U+k_4\gamma_V+\gamma_U\gamma_M+\gamma_V\gamma_M+\gamma_U\gamma_V,\end{aligned}$$

$$\begin{aligned}a_3&=(V^*-U^*-M^*)(k_1k_2\gamma_VV^*)+(\gamma_VV^*-\gamma_UU^*)(k_1\gamma_M)\\ &+(\gamma_VV^*-\gamma_MM^*)(k_2\gamma_U)+(\gamma_VV^*-\gamma_UU^*)(k_1k_4)\\ &+(k_3V^*-k_4M^*)(k_2\gamma_U)+k_3k_4\gamma_U+k_3\gamma_U\gamma_M+k_4\gamma_U\gamma_V\\ &+\gamma_U\gamma_V\gamma_M.\end{aligned}$$

Since all parameters are positive and by assumptions $k_3>k_4$, $\gamma_U=\gamma_V=\gamma_M$, and the values $V^*>U^*+M^*$ satisfied under conditions depended on range of parameters. Then, $a_1, a_2$, and $a_3$ are positive.

Consider

$$\begin{aligned}a_1a_2&=(k_1V^*+\gamma_U-k_1U^*-k_2M^*+k_3+\gamma_V+k_2V^*+k_4+\gamma_M)a_2\\ &=a_2k_1V^*-a_2k_1U^*+a_2k_2V^*-a_2k_2M^*+a_2k_3+a_2k_4+a_2\gamma_U\\ &+a_2\gamma_V+a_2\gamma_M\\ &=(V^*-U^*)(a_2k_1)+(V^*-M^*)(a_2k_2)+(k_3+k_4+\gamma_U+\gamma_M)a_2\\ &+[(V^*-U^*-M^*)k_1k_2V^*+(V^*-U^*)(k_1k_4)+(V^*-U^*)k_1\gamma_M\\ &+(V^*-M^*)(k_2\gamma_U)+(\gamma_VV^*-\gamma_UU^*)k_1+(\gamma_U^*-\gamma_MM^*)k_2\\ &+(k_3V^*-k_4M^*)k_2+k_3k_4+k_3\gamma_M+k_3\gamma_V+k_4\gamma_U+k_4\gamma_V+\gamma_U\gamma_M\\ &+\gamma_V\gamma_M+\gamma_U\gamma_V]\gamma_V\\ &=[(V^*-U^*)k_1+(V^*-M^*)k_2+k_3+k_4+\gamma_U+\gamma_M]a_2\\ &+[(V^*-U^*)(k_1\gamma_M)+(V^*-M^*)(k_2\gamma_U)+k_3\gamma_V+k_4\gamma_V+\gamma_V\gamma_M\\ &+\gamma_U\gamma_V]\gamma_V+(V^*-U^*-M^*)(k_1k_2V^*)\gamma_V+(V^*-U^*)(k_1k_4)\gamma_V\\ &+(\gamma_VV^*-\gamma_UU^*)(k_1\gamma_V)+(\gamma_UV^*-\gamma_MM^*)(k_2\gamma_V)\\ &+(k_3V^*-k_4M^*)(k_2\gamma_V)+(k_3k_4)\gamma_V+(k_3\gamma_M)\gamma_V+(k_4\gamma_U)\gamma_V\\ &+(\gamma_U\gamma_M)\gamma_V\\ &=[(V^*-U^*)k_1+(V^*-M^*)k_2+k_3+k_4+\gamma_U+\gamma_M]a_2\\ &+[(V^*-U^*)k_1\gamma_M+(V^*-M^*)k_2\gamma_U+k_3\gamma_V+k_4\gamma_V+\gamma_V\gamma_M\\ &+\gamma_U\gamma_V]\gamma_V+(V^*-U^*-M^*)k_1k_2\gamma_VV^*+(\gamma_VV^*-\gamma_VU^*)k_1k_4\\ &+(\gamma_VV^*-\gamma_UU^*)(k_1\gamma_V)+(\gamma_UV^*-\gamma_MM^*)(k_2\gamma_V)\\ &+(k_3V^*-k_4M^*)k_2\gamma_V+k_3k_4\gamma_V+k_3\gamma_V\gamma_M+k_4\gamma_U\gamma_V+\gamma_U\gamma_V\gamma_M.\end{aligned}$$

By assumption $\gamma_U=\gamma_V=\gamma_M$, we get that

$$\begin{aligned}a_1a_2&=[(V^*-U^*)k_1+(V^*-M^*)k_2+k_3+k_4+\gamma_U+\gamma_M]a_2\\ &+[(V^*-U^*)k_1\gamma_M+(V^*-M^*)k_2\gamma_U+k_3\gamma_V+k_4\gamma_V+\gamma_V\gamma_M\\ &+\gamma_U\gamma_V]\gamma_V+(V^*-U^*-M^*)k_1k_2\gamma_VV^*+(\gamma_VV^*-\gamma_UU^*)k_1k_4\\ &+(\gamma_VV^*-\gamma_UU^*)(k_1\gamma_M)+(\gamma_VV^*-\gamma_MM^*)(k_2\gamma_U)\\ &+(k_3V^*-k_4M^*)k_2\gamma_U+k_3k_4\gamma_U+k_3\gamma_U\gamma_M+k_4\gamma_V\gamma_M+\gamma_U\gamma_V\gamma_M\\ &=[(V^*-U^*)k_1+(V^*-M^*)k_2+k_3+k_4+\gamma_U+\gamma_M]a_2\\ &+[(V^*-U^*)k_1\gamma_M+(V^*-M^*)k_2\gamma_U+k_3\gamma_V+k_4\gamma_V+\gamma_V\gamma_M\\ &+\gamma_U\gamma_V]\gamma_V+[(V^*-U^*-M^*)(k_1k_2\gamma_VV^*)+(\gamma_VV^*-\gamma_UU^*)\\ &+(\gamma_VV^*-\gamma_MM^*)(k_2\gamma_U)+(\gamma_VV^*-\gamma_UU^*)(k_1k_4)\\ &+(k_3V^*-k_4M^*)k_2\gamma_U+k_3k_4\gamma_U+k_3\gamma_U\gamma_M+k_4\gamma_U\gamma_V+\gamma_U\gamma_V\gamma_M].\end{aligned}$$

It is known that

$$\begin{aligned}a_3&=(V^*-U^*-M^*)(k_1k_2\gamma_VV^*)+(\gamma_VV^*-\gamma_UU^*)(k_1\gamma_M)\\ &+(\gamma_VV^*-\gamma_MM^*)(k_2\gamma_U)+(\gamma_VV^*-\gamma_UU^*)(k_1k_4)\\ &+(k_3V^*-k_4M^*)(k_2\gamma_U)+k_3k_4\gamma_U+k_3\gamma_U\gamma_M+k_4\gamma_U\gamma_V\\ &+\gamma_U\gamma_V\gamma_M,\end{aligned}$$

Therefore $a_1a_2$ can be expressed as the following.

$$\begin{aligned}a_1a_2&=[(V^*-U^*)k_1+(V^*-M^*)k_2+k_3+k_4+\gamma_U+\gamma_M]a_2\\ &+[(V^*-U^*)k_1\gamma_M+(V^*-M^*)k_2\gamma_U+k_3\gamma_V+k_4\gamma_V\\ &+\gamma_V\gamma_M+\gamma_U\gamma_V]\gamma_V+a_3.\end{aligned}$$

Since all parameters are positive, $a_2>0$, and $V^*>U^*+M^*$. Consequently, $a_1a_2>a_3$. According to Routh-Hurwitz test [9, 14], it is concluded that all eigenvalues are negative. In other words, the positive equilibrium $Q^*$ is locally asymptotically stable. □

**Remark 2** Notice that the condition $V^*>U^*+M^*$ holds depended on the range of parameters, for example $A_U,\ A_M\in[10000,\infty),\ k_1,k_2\in(0,1),\ k_3,k_4\in$

$(0,10)$, and $\gamma_U, \gamma_V, \gamma_M \in (0,10)$ as shown in the second test of the numerical simulations. It concludes from Theorem 4 that when $R > 1$ and with under restricted conditions of parameters, the number of the unemployed, employed and migrants converges to the equilibrium point $U^*, V^*$, and $M^*$, respectively. Moreover, more details about the calculation process can be seen in [15].

## NUMERICAL SIMULATIONS

This part demonstrates numerical simulations of the solution to the nonlinear unemployment model (1). The experiments are divided into two subsections. The first subsection aims to validate the asymptomatic local stability of the free equilibrium point $Q_0^*$ and the positive equilibrium point $Q_+^*$, as provided by Theorems 3 and 4, respectively. In addition, in the second subsection, we validate our unemployment model using data sets from the National Statistical Office, the Ministry of Digital Economy and Society, and the Office of Foreign Workers Administration, which examines the number of unemployed people in Thailand after the COVID-19 outbreak.

### Simulations of the unemployment model

In this subsection, we present results of numerical experiments that demonstrate the solution of our unemployment system. In the first test, the initial conditions are chosen with three sets as follows.

$$x_0^1 = \begin{bmatrix} U(0) \\ V(0) \\ M(0) \end{bmatrix} = \begin{bmatrix} 8000 \\ 10000 \\ 2000 \end{bmatrix}, \; x_0^2 = \begin{bmatrix} 6000 \\ 7500 \\ 4000 \end{bmatrix}, \; x_0^3 = \begin{bmatrix} 4000 \\ 5000 \\ 8000 \end{bmatrix}.$$

The parameters are chosen arbitrarily as shown in the Table 1. We calculate $R$ from the given parameters in Table 1, it yields

**Table 1** Parameter values for case $R < 1$.

| Parameter | Value |
|---|---|
| $A_U$ | 100000 |
| $A_M$ | 100000 |
| $k_1$ | 0.00001 |
| $k_2$ | 0.00005 |
| $k_3$ | 4 |
| $k_4$ | 2 |
| $\gamma_U$ | 4 |
| $\gamma_V$ | 4 |
| $\gamma_M$ | 4 |

$$R = \frac{(k_4 + \gamma_M)(A_U k_1) + (k_1 k_4 + k_2 \gamma_U)(A_M)}{(k_4 + \gamma_M)(k_3 + \gamma_V)(\gamma_U)} = 0.1458.$$

It can be seen from Theorem 3 that if $R < 1$ then the solution of the unemployment system converge to the equilibrium point

$$Q_0^* = (U_0, V_0, M_0)^\top = (33333.3333, 0, 16666.6667)^\top.$$

In particular, the equilibrium point $Q_0^* = (33333.3333, 0, 16666.6667)$ exhibits the local asymptomatic stability behavior.

The solutions of the unemployment system are presented in Fig. 2–Fig. 4 in time $t \in [0,3]$ where the number of unemployed people ($U(t)$), employed people ($V(t)$) and immigrant ($M(t)$) are shown in Fig. 2, Fig. 3, and Fig. 4, respectively. It is demonstrated that with different initial conditions, numerical solutions approach the free equilibrium point. These figures strongly support the local asymptotic stability behavior of the free equilibrium $Q_0^*$ analyzed and proved in Theorem 3.

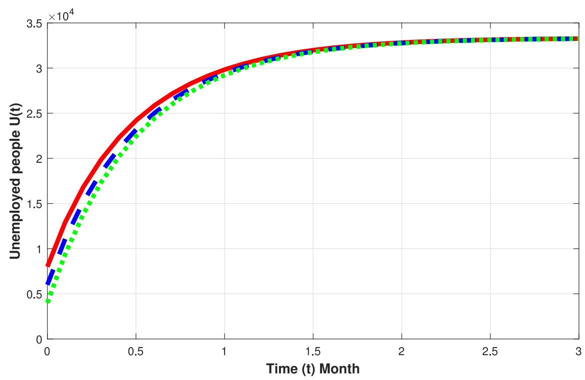


**Fig. 2** The number of the unemployed people versus time (months). Three lines represent the solution of the system with different initial conditions.

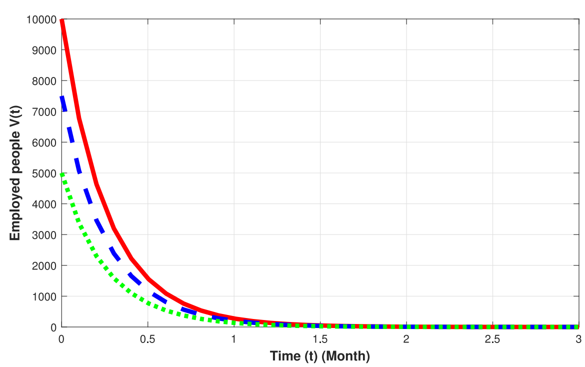


**Fig. 3** The number of the employed people versus time (months). Three lines represent the solution of the system with different initial conditions.

In the the second test, the initial conditions are chosen with three sets as follows.

$$x_0^1 = \begin{bmatrix} U(0) \\ V(0) \\ M(0) \end{bmatrix} = \begin{bmatrix} 6000 \\ 50000 \\ 9000 \end{bmatrix}, \; x_0^2 = \begin{bmatrix} 12000 \\ 30000 \\ 6000 \end{bmatrix}, \; x_0^3 = \begin{bmatrix} 18000 \\ 15000 \\ 3000 \end{bmatrix}.$$

The parameters are chosen arbitrarily as shown in Table 2. We calculate $R$ from the given parameters in

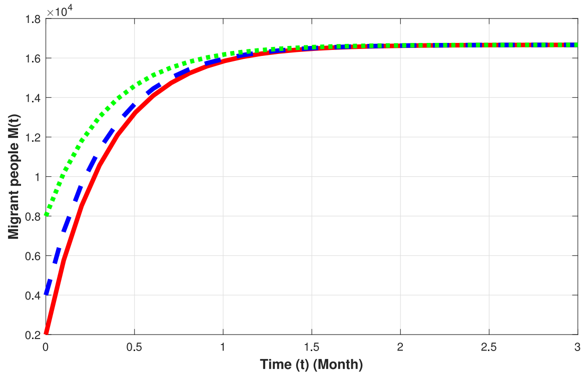


**Fig. 4** The number of the migrant people versus time (months). Three lines represent the solution of the system with different initial conditions.

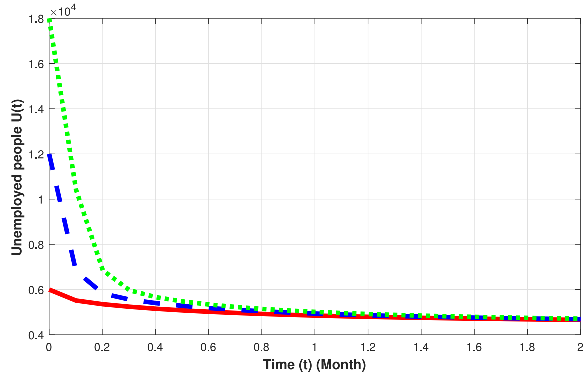


**Fig. 5** The number of the unemployed people versus time (months). Three lines represent the solution of the system with different initial conditions. The corresponding parameters $R$ is chosen as $R > 1$.

**Table 2** Parameter values for case $R > 1$.

| Parameter | Value |
|---|---|
| $A_U$ | 100000 |
| $A_M$ | 100000 |
| $k_1$ | 0.001 |
| $k_2$ | 0.005 |
| $k_3$ | 4 |
| $k_4$ | 2 |
| $\gamma_U$ | 1 |
| $\gamma_V$ | 1 |
| $\gamma_M$ | 1 |

Table 2, it yields

$$R = \frac{(k_4+\gamma_M)(A_U k_1) + (k_1 k_4 + k_2 \gamma_U)(A_M)}{(k_4+\gamma_M)(k_3+\gamma_V)(\gamma_U)} = 66.6667.$$

It can be seen from Theorem 4 that if $R > 1$, $k_3 > k_4$, and $\gamma_U = \gamma_V = \gamma_M$ then the solutions of the unemployment system converge to the equilibrium point.

$$Q_+^* = (U_+^*, V_+^*, M_+^*)^\top = (4492.12, 195408.15, 102.52)^\top.$$

The solutions of our model are illustrated in Fig. 5–Fig. 7 for time $t \in [0, 10]$. We remark that for the sake of easy checking the convergence of the solutions, the solution of the system are plotted in different frames of time where the solution of $U(t)$ is plotted in time $t \in [0, 2]$, the solution $V(t)$ in time $t \in [0, 8]$ and $V(t)$ in time $t \in [0, 0.5]$. It can be seen that with different choices of initial conditions, solutions approach their equilibrium points $Q_+^*$ that strongly support the local stability behavior of the positive equilibrium $Q_+^*$ analyzed and proved in Theorem 4.

## The unemployment in Thailand after COVID-19 outbreak

In this subsection, we study the unemployment problem in Thailand after the COVID-19 outbreak. We validate our unemployment model with data sets from

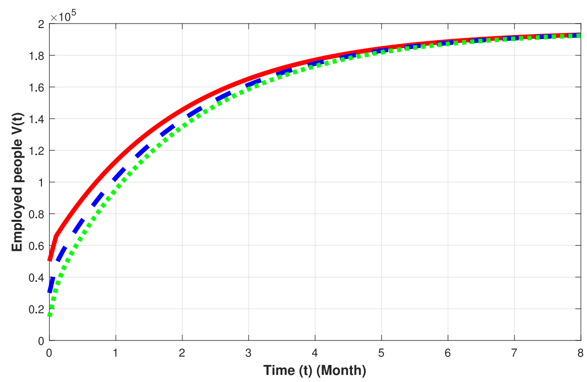


**Fig. 6** The number of the employed people versus time (months). Three lines represent the solution of the system with different initial conditions. The corresponding parameters $R$ is chosen as $R > 1$.

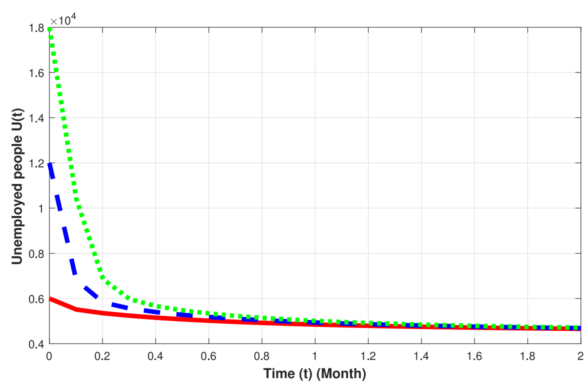


**Fig. 7** The number of the migrant people versus time (months). Three lines represent the solution of the system with different initial conditions. The corresponding parameters $R$ is chosen as $R > 1$.

the National Statistical Office and the Ministry of Digital Economy and Society, and the Office of Foreign Workers Administration from January to December 2020 and 2021. The numerical results of systems of unemployment are compared to the unemployment data, where the initial conditions were selected from the data in January 2020,

$$\mathbf{x}_0 = (U(0), V(0), M(0))^\top = (4.06\times10^5, 3.72\times10^7, 2.99\times10^6)^\top,$$

and the corresponding parameters are listed in Table 3.

**Table 3** Parameter values for the study of unemployment problem in Thailand.

| Parameter | Value |
|---|---|
| $A_U$ | 10,000 |
| $A_M$ | 5,000 |
| $k_1$ | 0.0002 |
| $k_2$ | 0.0001 |
| $k_3$ | 0.005 |
| $k_4$ | 0.002 |
| $\gamma_U$ | 2 |
| $\gamma_V$ | 2 |
| $\gamma_M$ | 2 |

The comparison of the number of unemployed, employed and migrant obtained in the simulation and the data are shown in the Table 4, Table 5, and Table 6, respectively.

**Table 4** The results of the number of unemployed people from model compared with the National Statistical Office's data.

| Month | Model | NSO(2020) | Model | NSO(2021) |
|---|---|---|---|---|
| January | 406,000 | 406,000 | 5,001,824 | |
| February | 2,367,408 | 419,000 | 5,002,181 | 758,093 |
| March | 3,641,855 | 392,000 | 5,002,348 | |
| April | 4,363,404 | 0 | 5,002,427 | |
| May | 4,709,059 | 0 | 5,002,464 | 731,828 |
| June | 4,867,135 | 0 | 5,002,482 | |
| July | 4,939,542 | 831,000 | 5,002,490 | |
| August | 4,973,067 | 724,000 | 5,002,494 | 871,266 |
| September | 4,988,699 | 693,000 | 5,002,495 | |
| October | 4,996,018 | 810,000 | 5,002,496 | |
| November | 4,999.452 | 784,000 | 5,002,497 | 631,887 |
| December | 5,001,066 | 590,000 | 5,002,497 | |

It can be seen in Table 4 that after the lockdown from April to June, the number of unemployed gradually increased, and there were some months when the number decreased. Meanwhile, the number of people employed steadily decreases and slightly increases in some months, as seen in Table 5. It is reported in Table 6 that the number of immigrants coming to work dropped within the first three months of the outbreak, and it continued to fall afterward, whereas, in 2021, the number of immigrants coming to work is increasing steadily as a result of reducing COVID-measure. The

**Table 5** The results of the number of employed people from the National Statistical Office's model and data.

| Month | Model | NSO(2020) | Model | NSO(2021) |
|---|---|---|---|---|
| January | 37,180,000 | 37,180,000 | 8,413 | |
| February | 30,341,082 | 37,630,000 | 3,955 | 37,578,919 |
| March | 23,935,479 | 37,330,000 | 1,859 | |
| April | 9,025,930 | 0 | 874 | |
| May | 3,835,729 | 0 | 411 | 37,821,800 |
| June | 1,722,295 | 0 | 193 | |
| July | 792,783 | 37,810,000 | 90 | |
| August | 369,113 | 38,050,000 | 42 | 37,705,741 |
| September | 172,768 | 37,860,000 | 20 | |
| October | 81,065 | 37,900,000 | 9 | |
| November | 38,080 | 38,270,000 | 4 | 37,898,725 |
| December | 17,897 | 38,760,000 | 2 | |

**Table 6** The results of the number of migrants who came to work in Thailand from model compared with the Office of Foreign Workers Administration's data.

| Month | Model | OFWA(2020) | Model | OFWA(2021) |
|---|---|---|---|---|
| January | 2,990,777 | 2,990,777 | 2,497,333 | 2,144,073 |
| February | 1,574,954 | 2,940,389 | 2,497,423 | 2,144,073 |
| March | 2,070,290 | 2,814,481 | 2,497,465 | 2,176,501 |
| April | 2,315,628 | 0 | 2,497,484 | 2,282,902 |
| May | 2,418,897 | 0 | 2,497,494 | 2,307,812 |
| June | 2,462,418 | 0 | 2,497,498 | 2,380,767 |
| July | 2,481,466 | 2,419,452 | 2,497,500 | 2,372,419 |
| August | 2,490,070 | 2,382,306 | 2,497,501 | 2,347,124 |
| September | 2,494,032 | 2,424,490 | 2,497,502 | 2,374,501 |
| October | 2,495,876 | 2,482,256 | 2,497,502 | 2,328,409 |
| November | 2,496,739 | 2,526,275 | 2,497,502 | 2,350,677 |
| December | 2,497,143 | 2,512,328 | 2,497,502 | 2,419,987 |

Office of Foreign Workers Administration permitted the monthly number of foreign workers to work in Thailand. Comparing with the data, it can be seen from the model results that the number of unemployed people steadily increases, the number of employed people drops, continues to fall, and tends to zero. While the number of migrants decreased in the first three months of the COVID outbreak and then slightly increased. We remark from the three above tables that our results depend on parameters. All relevant parameters influencing model results must be analyzed. Since the corresponding parameter $R = 0.6236 < 1$, the model results converge to the equilibrium points $Q_0^* = (U_0^*, 0, M_0^*)$, consistent with the theoretical results in Theorem 3. Further work, to obtain more precise results and fit with data, we suggest considering more variables such as the number of regular and temporarily employed people and the number of self-employed.

## CONCLUSION

This paper presented a mathematical model of unemployment in Thailand after the COVID-19 outbreak by considering the number of unemployed, the number of employed people, and the number of new migrants to find work. The theoretical and numerical solutions were investigated. Our unemployment system had two equilibria: the employment free equilibrium $Q_0^*$ and the positive equilibrium $Q_+^*$. Both equilibria's local stability was proved using the Routh-Hurwitz stability test. The set of parameters corresponding number $R$ plays a role in determining the stability of the equilibria. If $R < 1$, then $Q_0^*$ is asymptotically stable, whereas, if $R > 1$, then $Q_0^*$ is unstable, and $Q_+^*$ becomes asymptotically stable. Numerical simulations were done to support our theoretical results.

*Acknowledgements*: This research is supported by Faculty of Science, Chiang Mai University.

## REFERENCES

1. Al-Maalwi RM, Ashi H, Al-sheikh S (2018) Unemployment model. *Appl Math Sci* **12**, 989–1006.
2. Al-Sheikh S, Al-Maalwi R, Ashi H (2021) A mathematical model of unemployment with the effect of limited jobs. *Comptes Rendus Math* **359**, 283–290.
3. Ashi H, Al-Maalwi RM, Al-Sheikh S (2022) Study of the unemployment problem by mathematical modeling: predictions and controls. *J Math Sociol* **46**, 301–313.
4. Daud AAM, Ghozali AW(2015) Stability analysis of a simple mathematical model for unemployment. *Casp J Appl Sci Res* **4**, 15–18.
5. Galindro A, Torres DF (2017) A simple mathematical model for unemployment: a case study in Portugal with optimal control. *Stat Optim Inf Comput* **6**, 116–129.
6. Misra A, Singh AK (2011) A mathematical model for unemployment. *Nonlinear Anal Real World Appl* **12**, 128–136.
7. Misra A, Singh AK, Singh PK (2022) Modeling the role of skill development to control unemployment. *Differ Equ Dyn Syst* 30, 397–409.
8. Munoli SB, Gani S, Gani SR (2017) A mathematical approach to employment policies: An optimal control analysis. *Int J Stat Syst* **12**, 549–565.
9. Munoli SB, Gani S (2016) Optimal control analysis of a mathematical model for unemployment. *Optim Control Appl Methods* **37**, 798–806.
10. Pathan G, Bhathawala P (2017) A mathematical model for unemployment-taking an action without delay. *Adv Dyn Syst Appl* **12**, 41–48.
11. Sundar S, Tripathi A, Naresh R (2018) Does unemployment induce crime in society? a mathematical study. *Am J Appl Math Stat* **6**, 44–53.
12. Anna Engblom NL, Baruah N (2020) Covid-19: Impact on migrant workers and country response in Thailand. International Labour Organization, Country Office for Thailand Cambodia and Lao PDR. Available at: https://www.ilo.org/asia/publications/issue-briefs/WCMS_741920/lang--en/index.htm.
13. Casper Ferm LE, Thaichon P (2021) Successful covid-19 prevention factors and their effect on the economy: A comparison between Thailand, Vietnam and Australia. In: Ratten V, Thaichon P (eds) *COVID-19, Technology and Marketing*. Palgrave Macmillan, Singapore, pp 59–83.
14. Patil A (2021) Routh-Hurwitz criterion for stability: an overview and its implementation on characteristic equation vectors using Matlab. In: Hassanien AE, Bhattacharyya S, Chakrabati S, Bhattacharya A, Dutta S (eds) *Emerging Technologies in Data Mining and Information Security*, Advances in Intelligent Systems and Computing, vol 1286, Springer, pp 319–329.
15. Chanosot P (2022) Stability analysis of unemployment model in Thailand after Covid-19 outbreak. MSc thesis, Chiang Mai University, Chiang Mai, Thailand.