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An application of solutions of linear difference equations for obtaining the conditional moments of the trending Ornstein-Uhlenbeck processes

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ABSTRACT: This paper presents an application of solutions of linear difference equations for obtaining a closed-form formula for the γ-th conditional moment of the Ornstein-Uhlenbeck (O-U) process, for any positive real number γ. The partial differential equation associated with the O-U process is reduced to a system of ordinary differential equations, which can be solved analytically in Laplace-transformed space using solutions of linear difference equations. Our success in performing Laplace inverse transform leads to a simple closed-form formula for the conditional moment. Interestingly, several asymptotic properties of the conditional moment can easily be deduced using our closed-form formula. Secondly, the n-th conditional moment of the trending O-U process is derived in closed form, for any positive integer n. Finally, we derive the n-th unconditional moment of the O-U process and explore some asymptotic properties.

KEYWORDS: linear difference equations, O-U process, trending O-U process, moments, closed-form formula

MSC2020: 91G20

INTRODUCTION

The Ornstein-Uhlenbeck (O-U) process is a continuous-time stochastic process v_t described by the stochastic differential equation (SDE)

$$\mathrm{d}v_t = \kappa(\theta - v_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \tag{1}$$

where θ, $\kappa > 0$ and $\sigma > 0$ are parameters interpreting an equilibrium level or long-run mean level, speed to the equilibrium level, and variance of the state variable v_t, respectively. A standard Brownian motion W_t is driven under a probability space $(\Omega, \mathscr{F}, P)$ with a filtration $(\mathscr{F}_t)_{t\geqslant 0}$.

In terms of modeling, the O-U process is considerable potential as a building block for stochastic models of observational time series from a wide range of fields such as science, engineering, economics and finance. In particular, many researchers in financial markets have widely used the O-U process to describe the dynamics of state variables such as price, volatility, and interest rate, which stabilize at their equilibrium levels as time approaches infinity. Vasicek [1] employed the O-U process to capture the stochastic movement of the short term interest rate. The mean reversion property of the O-U process is particularly attractive because interest rates should not drift permanently upward the way stock prices do and this is commonly observed in practice.

In the O-U process (1), v_t tends towards a constant long-run mean level θ as time approaches infinity. This can in a beginning step be generalized to a deterministic trend $\mu(t)$ by introducing a continuous-time stochastic process p_t described by the SDE

$$\mathrm{d}(p_t - \mu(t)) = \kappa(\mu(t) - p_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t. \tag{2}$$

This demonstrates p_t, when it deviates from the trend $\mu(t)$, it is pulled back with a rate proportional to its deviation. The process (2) is therefore called the trending O-U process with the deterministic trend $\mu(t)$. In many applications [2–6], the deterministic trend $\mu(t)$ contains several parameters that need to be estimated from observational data. The method of moments is a simple technique that can be applied to estimate parameters of diffusion models as proposed in [7, 8], based on the idea that the sample moments are natural estimators of population moments. The main purpose of this paper is to contribute to the method of moments by providing a closed-form formula for the n-th conditional moment of the trending O-U process (2), for any positive integer n.

The paper starts by proposing a new analytical approach for obtaining a closed-form formula for the γ-th conditional moment of the O-U process (1), for any positive real number γ. Applying the Feynman-Kac theorem, the partial differential equation (PDE) associated with the O-U process (1) is derived. The PDE is reduced to a system of ordinary differential equations (ODEs) in the form of a recursion formula. Although the direct methods presented in literature [9–14] can be adopted to solve the system of ODEs, recursively, it consumes much computational time and effort due to the cumbersome nature of the resulting integral expression of the solutions of the ODEs. To avoid the problem, we apply Laplace transform to the system of ODEs and this gives us a linear difference equation (LDE) in Laplace-transformed space. The LDE is solved analytically utilizing the explicit solution of the LDE proposed by Mallik [15]. Our success in analytically performing Laplace inverse transform leads to our closed-form formula for the γ-th conditional moment. Using the current closed-form formula, the n-th conditional moment of the trending O-U process (2) is derived in closed form, for any positive integer n.

THE CONDITIONAL MOMENTS OF THE O-U PROCESS

We first derive a closed-form formula of the γ-th conditional moment of the O-U process (1), for any positive real number γ.

Theorem 1 *Suppose v_t follows the O-U process (1) and $\gamma > 0$. We define a real-valued function*

$$U^{(\gamma)}(v,\tau) := E^P[v_T^\gamma | (v_T^\gamma \in \mathbb{R}, v_t = v)] \tag{3}$$

for $v > 0$ and $\tau = T - t \geqslant 0$. Then, $U^{(0)}(v,\tau) = 1$ and $U^{(\gamma)}$ can be expressed as

$$U^{(\gamma)}(v,\tau) = \sum_{k=-1}^{\infty} A_{k+2}(\tau) v^{\gamma-k-1} \tag{4}$$

for all $(v,\tau) \in D(\gamma)$ where $D(\gamma)$ is a subset of $(0,\infty) \times [0,\infty)$,

$$A_1(\tau) = \mathrm{e}^{-\gamma\kappa\tau} \tag{5}$$

$$A_2(\tau) = \gamma\theta\,(\mathrm{e}^{\kappa\tau} - 1)\,\mathrm{e}^{-\gamma\kappa\tau} \tag{6}$$

$$A_{k+2}(\tau) = \sum_{j=1}^{2} \sum_{r=1}^{k+j-1} \sum_{L_{k,j,r} \in S(k,j,r)} (\gamma\kappa\theta)^{2-j} \times \left(\prod_{m=1}^{r} q^{(j,r)}_{k+l_m-\sum_{i=1}^m l_i, l_m} \right) \left(\sum_{m=j-2}^{r} C^{(j,r)}_{k,m}\, \mathrm{e}^{-w^{(j,r)}_{k,m}\tau} \right) \tag{7}$$

for $k = 1, 2, \ldots,$ where

$$q^{(j,r)}_{k+l_m-\sum_{i=1}^m l_i, l_m} := q^{(j,r)}_{k+l_m-\sum_{i=1}^m l_i, l_m}(L_{k,j,r}) = \begin{cases} a(l_m); & l_m = 1, \\ b(l_m); & l_m = 2, \end{cases} \tag{8}$$

where $a(l_m) = (\gamma - k - l_m + \sum_{i=1}^m l_i)\kappa\theta$, $b(l_m) = \frac{1}{2}(\gamma - k - l_m + \sum_{i=1}^m l_i)(\gamma - k - l_m + \sum_{i=1}^m l_i + 1)\sigma^2$ and for $j = 1, 2$, $m = -1, 0$,

$$w^{(1,r)}_{k,-1} = \gamma\kappa, \quad w^{(1,r)}_{k,0} = (\gamma - 1)\kappa, \quad w^{(2,r)}_{k,0} = \gamma\kappa \tag{9}$$

and for $j = 1, 2$, $m \geqslant 1$,

$$w^{(j,r)}_{k,m} := w^{(j,r)}_{k,m}(L_{k,j,r}) = (\gamma - k - l_m + \sum_{i=1}^m l_i - 1)\kappa \tag{10}$$

and for $j = 1, 2$, $m \geqslant j - 2$,

$$C^{(j,r)}_{k,m} := C^{(j,r)}_{k,m}(L_{k,j,r}) = \prod_{h=j-2, h\neq m}^{r} \frac{1}{w^{(j,r)}_{k,h} - w^{(j,r)}_{k,m}} \tag{11}$$

and $S(k,j,r)$ is a set of r-tuples $L_{k,j,r} = (l_1, \ldots, l_r)$ of positive integers defined by

$$S(k,j,r) := \Big\{ (l_1,\ldots,l_r) \Big| 1 \leqslant l_1,\ldots,l_r \leqslant 2, l_r \geqslant j, \sum_{i=1}^r l_i = k + j - 1 \Big\} \tag{12}$$

for $j = 1, 2$ and $r = 1, \ldots, k + j - 1$.

Proof: The Feynman-Kac theorem provides that $U^{(\gamma)}$ satisfies the PDE:

$$-\frac{\partial U^{(\gamma)}}{\partial \tau} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 U^{(\gamma)}}{\partial v^2} + \kappa(\theta - v)\frac{\partial U^{(\gamma)}}{\partial v} = 0 \tag{13}$$

for all $v > 0$ and $0 < \tau \leqslant T$, subject to the initial condition

$$U^{(\gamma)}(v, 0) = v^\gamma \tag{14}$$

for all $v > 0$.

Applying the procedure proposed in the proof of Theorem 2.1 of Rujivan [16] to the solution form (4), one can show that the coefficient functions, $A_{k+2}(\tau)$, $k = -1, 0, \ldots,$ must solve the system of ODEs:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} A_1 = -\gamma\kappa A_1 \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} A_2 = -(\gamma - 1)\kappa A_2 + \gamma\kappa\theta A_1 \tag{16}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\tau} A_{k+2} = -(\gamma - k - 1)\kappa A_{k+2} + (\gamma - k)\kappa\theta A_{k+1} + \frac{1}{2}(\gamma - k)(\gamma - k + 1)\sigma^2 A_k \tag{17}$$

for $k = 1, 2, \ldots$. The initial condition (14) implies that

$$A_1(0) = 1, \quad A_k(0) = 0 \tag{18}$$

for $k = 2, 3, \ldots$.

Providing the solutions of the ODEs (15)–(16) as expressed in (5)–(6), a closed-form formula for $U^{(\gamma)}$ can be obtained by solving the system of ODEs (17) subject to the initial conditions (18), recursively. However, this direct method consumes much computational time and effort due to the cumbersome nature of the resulting integral expression of the solution of the ODEs (17), derived later on as written in (44).

To reduce computational time and effort, this paper proposes an alternative analytical method to solve the system of ODEs (17) by using Laplace transforms and solutions of LDEs as follows.

Let

$$\hat{y}_k(s) := \mathscr{L}\{A_k(\tau)\} = \int_0^\infty e^{-s\tau} A_k(\tau)\,d\tau \tag{19}$$

where s is a complex number, providing that the improper integral on the RHS of (19) exists for any $k = 1, 2, \ldots$. Utilizing the Laplace transform (19), the initial value problems (17)–(18) can be written in terms of a LDE in Laplace space as

$$\hat{y}_{k+2}(s) = \hat{a}_{k,1}(s)\hat{y}_{k+1}(s) + \hat{a}_{k,2}(s)\hat{y}_k(s) \tag{20}$$

for $k = 1, 2, \ldots$, with the initial conditions

$$\hat{y}_1(s) = \frac{1}{s+\gamma\kappa} \tag{21}$$

$$\hat{y}_2(s) = \frac{\gamma\kappa\theta}{(s+(\gamma-1)\kappa)(s+\gamma\kappa)} \tag{22}$$

where the coefficient functions on the RHS of (20) are given by

$$\hat{a}_{k,1}(s) = \frac{(\gamma-k)\kappa\theta}{s+(\gamma-k-1)\kappa} \tag{23}$$

$$\hat{a}_{k,2}(s) = \frac{\frac{1}{2}(\gamma-k)(\gamma-k+1)\sigma^2}{s+(\gamma-k-1)\kappa}. \tag{24}$$

With a contribution of Mallik [15] work, the explicit solution of the LDE (20) subject to the initial conditions (21)–(22) can be expressed as

$$\hat{y}_{k+2}(s) = \hat{d}_{k,1}(s)\hat{y}_2(s) + \hat{d}_{k,2}(s)\hat{y}_1(s) \tag{25}$$

for $k = 1, 2, \ldots$, where

$$\hat{d}_{k,j}(s) = \sum_{r=1}^{k+j-1} \sum_{L_{k,j,r}\in S(k,j,r)} \left\{ \prod_{m=1}^{r} \hat{a}_{k+l_m-\sum_{i=1}^m l_i, l_m}(s) \right\} \tag{26}$$

for $j = 1, 2$, and $S(k, j, r)$ is defined in (12).

Next, we apply Laplace inversion to (25) to obtain

$$A_{k+2}(\tau) = \mathscr{L}^{-1}\left\{\hat{d}_{k,1}(s)\hat{y}_2(s) + \hat{d}_{k,2}(s)\hat{y}_1(s)\right\} \tag{27}$$

for $k = 1, 2, \ldots$, where $\mathscr{L}^{-1}\{f(s)\}$ denotes an inverse Laplace transform of a function f.

In order to obtain an explicit form of the inverse Laplace transform on the RHS of (27), for $L_{k,j,r} = (l_1, \ldots, l_r) \in S(k, j, r)$, we notice that

$$\prod_{m=1}^{r} \hat{a}_{k+l_m-\sum_{i=1}^m l_i, l_m}(s) = \left(\prod_{m=1}^{r} q^{(j,r)}_{k+l_m-\sum_{i=1}^m l_i, l_m}\right)\left(\prod_{m=1}^{r} \frac{1}{s+w^{(j,r)}_{k,m}}\right) \tag{28}$$

where $q^{(j,r)}_{k+l_m-\sum_{i=1}^m l_i, l_m}$ and $w^{(j,r)}_{k,m}$, $j = 1, 2$, are defined in (8) and (10), respectively.

From (27)–(28), we have

$$\begin{aligned}
A_{k+2}(\tau) &= \sum_{r=1}^{k} \sum_{L_{k,1,r}\in S(k,1,r)} \left(\gamma\kappa\theta \prod_{m=1}^{r} q^{(1,r)}_{k+l_m-\sum_{i=1}^m l_i, l_m}\right) \\
&\quad\times \mathscr{L}^{-1}\left\{\left(\prod_{m=1}^{r} \frac{1}{s+w^{(1,r)}_{k,m}}\right)\frac{1}{(s+(\gamma-1)\kappa)(s+\gamma\kappa)}\right\} \\
&\quad+ \sum_{r=1}^{k+1} \sum_{L_{k,2,r}\in S(k,2,r)} \left(\prod_{m=1}^{r} q^{(2,r)}_{k+l_m-\sum_{i=1}^m l_i, l_m}\right) \\
&\quad\times \mathscr{L}^{-1}\left\{\left(\prod_{m=1}^{r} \frac{1}{s+w^{(2,r)}_{k,m}}\right)\frac{1}{s+\gamma\kappa}\right\}.
\end{aligned} \tag{29}$$

Using the partial fraction method, we have

$$\left(\prod_{m=1}^{r} \frac{1}{s+w^{(1,r)}_{k,m}}\right)\frac{1}{(s+(\gamma-1)\kappa)(s+\gamma\kappa)} = \sum_{m=-1}^{r} \frac{C^{(1,r)}_{k,m}}{s+w^{(1,r)}_{k,m}} \tag{30}$$

$$\left(\prod_{m=1}^{r} \frac{1}{s+w^{(2,r)}_{k,m}}\right)\frac{1}{s+\gamma\kappa} = \sum_{m=0}^{r} \frac{C^{(2,r)}_{k,m}}{s+w^{(2,r)}_{k,m}} \tag{31}$$

where $C^{(j,r)}_{k,m}$, $j = 1, 2$, are defined in (11).

From (30)–(31) and straightforward but lengthy calculations, the explicit forms of the inverse Laplace transforms on RHS of (29) are specified as

$$\mathscr{L}^{-1}\left\{\left(\prod_{m=1}^{r} \frac{1}{s+w^{(1,r)}_{k,m}}\right)\frac{1}{(s+(\gamma-1)\kappa)(s+\gamma\kappa)}\right\} = \sum_{m=-1}^{r} C^{(1,r)}_{k,m}\, e^{-w^{(1,r)}_{k,m}\tau} \tag{32}$$

$$\mathscr{L}^{-1}\left\{\left(\prod_{m=1}^{r} \frac{1}{s+w^{(2,r)}_{k,m}}\right)\frac{1}{(s+\gamma\kappa)}\right\} = \sum_{m=0}^{r} C^{(2,r)}_{k,m}\, e^{-w^{(2,r)}_{k,m}\tau}. \tag{33}$$

Inserting (32)–(33) into (29) gives us the solution of the system of ODEs (17) as

$$\begin{aligned}
A_{k+2}(\tau) &= \sum_{r=1}^{k} \sum_{L_{k,1,r}\in S(k,1,r)} \left(\gamma\kappa\theta \prod_{m=1}^{r} q^{(1,r)}_{k+l_m-\sum_{i=1}^m l_i, l_m}\right) \\
&\quad\times \sum_{m=-1}^{r} C^{(1,r)}_{k,m}\, e^{-w^{(1,r)}_{k,m}\tau} + \sum_{r=1}^{k+1} \sum_{L_{k,2,r}\in S(k,2,r)} \left(\prod_{m=1}^{r} q^{(2,r)}_{k+l_m-\sum_{i=1}^m l_i, l_m}\right) \\
&\quad\times \sum_{m=0}^{r} C^{(2,r)}_{k,m}\, e^{-w^{(2,r)}_{k,m}\tau}
\end{aligned} \tag{34}$$

for $k \geqslant 1$. It is easy to show that the explicit formula as expressed in (34) can be reduced to a compact form as written in (7). □

As previously mentioned, the long-run mean level is a key parameter of the O-U process (1). The following corollary deduced from Theorem 1 shows that the γ-th conditional moment of the O-U process (1) can be written in terms of a power series in θ.

Corollary 1 *According to Theorem 1, the coefficient functions $A_{k+2}(\tau)$, $k = 1, 2, \ldots$, can be written as*

$$A_{k+2}(\tau) = \sum_{j=1}^{2} \gamma^{2-j} \sum_{r \geqslant \frac{k+j-1}{2}}^{k+j-1} \Bigg\{ a_{k,j,r}(\tau) \times \left(\frac{\sigma^2}{2\kappa}\right)^{k-r+j-1} \theta^{2r-k-2j+3} \Bigg\} \quad (35)$$

for $k = 1, 2, \ldots$, where

$$a_{k,j,r}(\tau) = \sum_{L_{k,j,r} \in S(k,j,r)} \left(\prod_{m=1}^{r} \bar{q}^{(j,r)}_{k+l_m-\sum_{i=1}^{m} l_i, l_m} \right) \times \left(\sum_{m=j-2}^{r} \bar{C}^{(j,r)}_{k,m} \, \mathrm{e}^{-\kappa \bar{w}^{(j,r)}_{k,m} \tau} \right) \quad (36)$$

$$\bar{q}^{(j,r)}_{k+l_m-\sum_{i=1}^{m} l_i, l_m} := \bar{q}^{(j,r)}_{k+l_m-\sum_{i=1}^{m} l_i, l_m}(L_{k,j,r}) = \begin{cases} \bar{a}(l_m); & l_m = 1, \\ \bar{b}(l_m); & l_m = 2, \end{cases} \quad (37)$$

where $\bar{a}(l_m) = \gamma - k - l_m + \sum_{i=1}^{m} l_i$, $\bar{b}(l_m) = (\gamma - k - l_m + \sum_{i=1}^{m} l_i)(\gamma - k - l_m + \sum_{i=1}^{m} l_i + 1)$ and for $j = 1, 2$, $m = -1, 0$,

$$\bar{w}^{(1,r)}_{k,-1} = \gamma, \quad \bar{w}^{(1,r)}_{k,0} = (\gamma - 1), \quad \bar{w}^{(2,r)}_{k,0} = \gamma \quad (38)$$

and for $j = 1, 2$, $m \geqslant 1$,

$$\bar{w}^{(j,r)}_{k,m} = (\gamma - k - l_m + \sum_{i=1}^{m} l_i - 1) \quad (39)$$

and for $j = 1, 2$, $m \geqslant j - 2$,

$$\bar{C}^{(j,r)}_{k,m} = \prod_{h=j-2, h\neq m}^{r} \frac{1}{\bar{w}^{(j,r)}_{k,h} - \bar{w}^{(j,r)}_{k,m}}. \quad (40)$$

Proof: We note that $S(k,j,r) = \varnothing$ if $r < \frac{k+j-1}{2}$. Let $r \geqslant \frac{k+j-1}{2}$. For $L_{k,j,r} \in S(k,j,r)$, $j = 1, 2$, one can deduce the following relations

$$(\gamma\kappa\theta)^{2-j} \prod_{m=1}^{r} q^{(j,r)}_{k+l_m-\sum_{i=1}^{m} l_i, l_m} = \gamma^{2-j} (\kappa\theta)^{(2-j)+2r-(k+j-1)} \left(\frac{\sigma^2}{2}\right)^{(k+j-1)-r} \times \prod_{m=1}^{r} \bar{q}^{(j,r)}_{k+l_m-\sum_{i=1}^{m} l_i, l_m} \quad (41)$$

and

$$C^{(j,r)}_{k,m} = \frac{1}{\kappa^{r-(j-2)}} \bar{C}^{(j,r)}_{k,m}. \quad (42)$$

Inserting (41) and(42) into the RHS of (7), we immediately obtain (35). □

Remark 1 The calculations of the coefficient functions $A_{k+2}(\tau)$, $k = -1, 0, \ldots$, in (4) based on the formula (35) and $a_{k,j,r}(\tau)$ based on the formula (36) can easily be done with the aid of a symbolic package, such as Maple, Matlab, and Mathematica. For the reader's convenience, all Mathematica codes used in our examples are available from the author upon reasonable request.

In the case that $\gamma = n$ is a non-negative integer, $U^{(n)}(v, \tau)$ can be written as a power series in v, which terminates at order n, for all $(v, \tau) \in D(n) = (0, \infty) \times [0, \infty)$ as shown in the following theorem.

Theorem 2 *Suppose v_t follows the O-U process (1) and n is a non-negative integer. Then,*

$$\begin{aligned} U^{(n)}(v, \tau) &:= E^P[v_T^n | (v_T^n \in \mathbb{R}, v_t = v)] \\ &= E^P[v_T^n | v_t = v] \\ &= \sum_{k=-1}^{n-1} A_{k+2}(\tau) v^{n-k-1} \end{aligned} \quad (43)$$

for $v > 0$ and $\tau = T - t \geqslant 0$ and the coefficient functions $A_{k+2}(\tau)$, $k = -1, 0, \ldots, n-1$, computed using (5)–(7) by setting $\gamma = n$, are strictly positive for all $\tau > 0$. Moreover, $U^{(n)}(v, \tau)$ is strictly increasing with respect to v for any $\tau > 0$.

Proof: To prove the theorem we shall consider the ODE (17) and the initial condition $A_k(0) = 0$ for $k \geqslant 2$. When $\gamma = n$ is a non-negative integer and $k = n$, we have $A_{n+2}(\tau) = 0$ for all $\tau \geqslant 0$. Consider the ODE (17) again but $k = n + 1$. Since $A_{n+2}(\tau)$ vanishes, $A_{n+3}(\tau) = 0$ for all $\tau \geqslant 0$. This implies that $A_{k+2}(\tau) = 0$ for all $\tau \geqslant 0$ and $k \geqslant n$. Next, we write a general solution of the ODE (17) when $\gamma = n$ in terms of an integral expression as

$$A_{k+2}(\tau) = \mathrm{e}^{-(n-k-1)\kappa\tau} \int_0^{\tau} \mathrm{e}^{(n-k-1)\kappa\eta} \Big((n-k)\kappa\theta A_{k+1}(\eta) + \tfrac{1}{2}(n-k)(n-k+1)\sigma^2 A_k(\eta) \Big) \,\mathrm{d}\eta \quad (44)$$

for $k \geqslant 1$. Since $A_1(\tau)$ and $A_2(\tau)$, written in (5)–(6), are strictly positive for all $\tau > 0$. This implies from (44) that $A_{k+2}(\tau)$, $k = 1, \ldots, n-1$, must be strictly positive for all $\tau > 0$ and we now obtain the last assertion of the theorem. □

The n-th conditional moment of the O-U process (1) when $\theta = 0$ can be deduced from Theorem 2 and Corollary 1 and will be used later on.

Corollary 2 *Define*

$$W^{(n)}(v,\tau) := \lim_{\theta\to 0^+} U^{(n)}(v,\tau;\theta) = \sum_{k=-1}^{n-1} \Big(\lim_{\theta\to 0^+} A_{k+2}(\tau;\theta)\Big) v^{n-k-1} \tag{45}$$

for $v > 0$ and $\tau = T-t \geqslant 0$, where $U^{(n)}(v,\tau;\theta)$ is given by (43). Then,

$$W^{(n)}(v,\tau) = \mathrm{e}^{-n\kappa\tau} v^n + \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} a_{2k+1,2,k+1}(\tau)\Big(\frac{\sigma^2}{2\kappa}\Big)^{k+1} v^{n-2k-2} \tag{46}$$

for $n = 1, 2, \ldots$, where $a_{2k+1,2,k+1}(\tau)$, $k = 0, \ldots, \lfloor \frac{n-2}{2} \rfloor$, are defined in (36) with $\gamma = n$ and we let $\sum_{k=0}^{-1} = 0$.

Proof: From (5)–(6) and (35), we have $A_{k+2}(\tau)$, $k = -1, 0, 1, .., n-1$, can be written as a polynomial function with respect to the long-run mean level θ, i.e,

$$A_{k+2}(\tau;\theta) = c_{0,k}(\tau) + c_{1,k}(\tau)\theta + c_{2,k}(\tau)\theta^2 + \cdots + c_{k+1,k}(\tau)\theta^{k+1} \tag{47}$$

where $c_{i,k}(\tau)$, $i = 0, 1, \ldots, k+1$, are coefficient functions. To verify the limit on the RHS of (45), we shall determine the coefficients $c_{0,k}(\tau)$, $k = -1, 0, 1, \ldots, n-1$. From (5)–(6), $c_{0,-1}(\tau) = A_1(\tau)$ and $c_{0,0}(\tau) = 0$. From (35), we consider the term $\theta^{2r-k-2j+3}$. To obtain $c_{0,k}(\tau)$, we set $r = \frac{2j+k-3}{2}$. Next, we consider the sets $S(k, j, \frac{2j+k-3}{2})$, $j = 1, 2$ defined in (12). For $j = 1$, $r = \frac{k-1}{2}$ and it is easy to show that $S(k, 1, \frac{k-1}{2}) = \varnothing$. For $j = 2$, $r = \frac{k+1}{2}$. If k is even, $\frac{k+1}{2}$ is not an integer. Thus, the summation $\sum_{r\geqslant \frac{k+j-1}{2}}^{k+j-1}$ in (35) must start from the index $r^* = \frac{k+1}{2} + \frac{1}{2} > r$. Hence, $\lim_{\theta\to 0^+} \theta^{2r^*-k-2j+3} = 0$. On the other hand, if k is odd, the summation starts from the index $r^* = \frac{k+1}{2} = r$. Moreover, we have $S(k, 2, \frac{k+1}{2}) = \{(2, 2, \ldots, 2)\} \neq \varnothing$. These results imply that

$$\lim_{\theta\to 0^+} A_{k+2}(\tau;\theta) = c_{0,k}(\tau) = \begin{cases} a_{k,2,\frac{k+1}{2}}(\tau)\left(\frac{\sigma^2}{2\kappa}\right)^{\frac{k+1}{2}} & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even,} \end{cases} \tag{48}$$

for $k = 1, 2, \ldots, n-1$. Applying (48) to the definition of $W^{(n)}(v,\tau)$ in (45) gives (46) as desired. □

By applying Theorem 2, an interesting property of the n-th conditional moment of the O-U process (1) when the initial variance approaches zero can be derived. The following corollary shows that if the O-U process (1) starts with either $v = c > 0$ or $v = 0$ at time t, it attains neither negative nor zero at any final time $T > t$.

Corollary 3 *Suppose v_t follows the O-U process (1), n is a non-negative integer, and $c > 0$. Then,*

$$\lim_{v\to c^+} U^{(n)}(v,\tau) > \lim_{v\to 0^+} E^P[v_T^n | v_t = v] = \lim_{v\to 0^+} U^{(n)}(v,\tau) = A_{n+1}(\tau) > 0 \tag{49}$$

for any $T > t \geqslant 0$ where $\tau = T - t$.

Proof: The result shown in (49) is a consequence of Theorem 2 in which the proof is rather trivial, omitted here. □

THE CONDITIONAL MOMENTS OF THE TRENDING O-U PROCESS

As previously introduced, the trending O-U process (2) can be written in the form of the following SDE

$$\mathrm{d}p_t = (\kappa(\mu(t) - p_t) + \mu'(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \tag{50}$$

where $\mu(t)$ is a deterministic function and its derivative $\mu'(t)$ with respect to t. In financial applications, many researchers assumed p_t as a log-price process which follows an arithmetic random walk with independently and identically distributed Gaussian increments. This log-price process is the sum of a zero-mean stationary auto-regressive Gaussian process, an O-U process and a deterministic trend $\mu(t)$. When we assume $\mu(t) = \theta$ is a constant, the trending O-U process (50) reduces to the O-U process (1).

The trending O-U process and its variants have a wide array of potential applications. In equity markets, Lo and Wang [3] used $\mu(t) = \theta t$, a linear trend in the log-price process. In commodity markets, Lucia and Schwartz [4], Wilhelm and Winter [5], and Zhang et al [6], assumed $\mu(t)$ is a linear combination of sinusoidal functions like the cosine and sin functions, which contain several parameters, to reflect seasonality in commodity prices. Recently, Deng et al [2] supposed $\mu(t) = a((t+1)^b - 1) + \sqrt{\sigma_\mu^2/2r}$ to describe the trend of the degradation level process with parameters a, b, σ_μ, and r. To employ these diffusion models in the real world, the model parameters need to be estimated from observational data. Therefore, a closed-form formula for the n-th conditional moment of the trending O-U process (50) is very helpful in parameter estimation of the diffusion models based on the method of moments as presented in [7, 8].

As shown in [17, Section 3.1.2], $p_T|p_t$ is distributed according to a normal distribution with mean $m(t,T,p) := p\,\mathrm{e}^{-\kappa(T-t)} + \frac{\sigma^2}{2\kappa}\left(\mathrm{e}^{-\kappa(T-t)} - 1\right) + \kappa\,\mathrm{e}^{-\kappa T}\int_t^T (\mu(u) + \mu'(u)/\kappa)\,\mathrm{e}^{\kappa u}\,\mathrm{d}u$ and variance $v(t,T) := \frac{\sigma^2}{2\kappa}\left(1 - \mathrm{e}^{-2\kappa(T-t)}\right)$ for $p_t = p \in \mathbb{R}$ and $0 \leqslant t < T$. As a result, the transition probability density function of p_T can be expressed as $f(\hat{p}, T|p, t) = \frac{1}{\sqrt{2\pi v(t,T)}}\,\mathrm{e}^{-\frac{(\hat{p}-m(t,T,p))^2}{2v(t,T)}}$ for all $\hat{p} \in \mathbb{R}$. This implies that the n-th conditional moment of the trending O-U process (50) defined by $E^P[p_T^n|p_t = p] := \int_{-\infty}^{\infty} \hat{p}^n f(\hat{p}, T|p,t)\,\mathrm{d}\hat{p}$ can be obtained by using either direct or numerical integration. For example, by applying [18, 19, Theorem 5], one can derive $E^P[p_T^n|p_t = p]$ in close form when $\mu(t)$ is a constant function. On the other hand, in order to obtain a closed-form formula for $E^P[p_T^n|p_t = p]$ when $\mu(t)$ is a non-constant function, a complicated numerical technique for computing multiple integrals may be required due to $m(t,T,p)$ contains the integral of $(\mu(u)+\mu'(u)/\kappa)\,\mathrm{e}^{\kappa u}$. To avoid this problem, we provide a closed-form formula for $E^P[p_T^n|p_t = p]$ by applying Corollary 2 as shown in the following theorem.

Theorem 3 *Suppose p_t follows the trending O-U process (50) and n is a positive integer. Then,*

$$E^P[p_T^n|p_t = p] = (\mu(T))^n + \sum_{k=1}^{n} \tfrac{n!}{(n-k)!k!}(\mu(T))^{n-k} W^{(k)}(p-\mu(t),\tau) \quad (51)$$

for $p \in \mathbb{R}$ and $\tau = T - t \geqslant 0$ where $W^{(k)}(p-\mu(t),\tau)$, $k = 1,2,\dots,n$, can be obtained by using (46).

Proof: We first consider the transformed process $v_t := p_t - \mu(t)$. Applying Itô formula, v_t is an O-U process with $\theta = 0$ satisfying the SDE; $\mathrm{d}v_t = -\kappa v_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$. Utilizing the Binomial theorem to compute the n-th conditional moment (51), we arrive

$$\begin{aligned} E^P[p_T^n|p_t = p] &= E^P[(v_T+\mu(T))^n|v_t = p-\mu(t)] \\ &= \sum_{k=0}^{n} \tfrac{n!}{(n-k)!k!}(\mu(T))^{n-k} E^P[v_T^k|v_t = p-\mu(t)]. \quad (52)\end{aligned}$$

Applying Corollary 2 to the O-U process v_t, we have

$$E^P[v_T^k|v_t = p-\mu(t)] = W^{(k)}(p-\mu(t),\tau). \quad (53)$$

Inserting (53) into (52), we immediately obtain (51). □

The formula for $W^{(n)}(v,\tau)$ as expressed in (46) is crucial in terms of speeding up computation of the conditional moment (51) required in the method of moments. Of course, on the RHS of (51), one can use the functions $U^{(k)}$ as written in (43) instead of $W^{(k)}$ for $k = 0,1,\dots,n$. However, this causes an increase in computational time because the coefficient functions A_k in (43) must be computed for all $k = 1,\dots,n+1$. In fact, we show in the proof of Corollary 2 that A_k vanishes for any even positive integer k. Hence, using $W^{(k)}$ to compute the conditional expectation in the method of moments is obviously more efficient than using $U^{(k)}$.

Example 1 By setting $\mu(t) = \theta t$, the trending O-U process (50) becomes

$$\mathrm{d}p_t = (\kappa(\theta t - p_t)+\theta)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t. \quad (54)$$

Applying Theorem 3, the n-th conditional moment of the trending O-U process with a linear trend (54) can be expressed as

$$E^P[p_T^n|p_t = p] = (\theta T)^n + \sum_{k=1}^{n} \tfrac{n!}{(n-k)!k!}(\theta T)^{n-k} W^{(k)}(p-\theta t,\tau) \quad (55)$$

for $p \in \mathbb{R}$ and $\tau = T - t \geqslant 0$ where $W^{(k)}(p-\theta t,\tau)$, $k = 1,2,\dots,n$, can be obtained by using (46). Furthermore, by setting $U^{(n)}_{\text{linear}}(p,\tau) := E^P[p_T^n|p_t = p]$, the n-th conditional moment of the trending O-U process (54) for $n = 1,2,3,4$ can be derived in explicit forms as follows:

$$U^{(1)}_{\text{linear}}(p,\tau) = \theta T + \mathrm{e}^{-\kappa\tau}(p-\theta t), \quad (56)$$

$$U^{(2)}_{\text{linear}}(p,\tau) = (\theta T)^2 + 2\theta T\,\mathrm{e}^{-\kappa\tau}(p-\theta t) + \mathrm{e}^{-2\kappa\tau}(p-\theta t)^2 - \tfrac{\sigma^2(\mathrm{e}^{-2\kappa\tau}-1)}{2\kappa} \quad (57)$$

$$\begin{aligned} U^{(3)}_{\text{linear}}(p,\tau) &= (\theta T)^3 + 3\,\mathrm{e}^{-\kappa\tau}(p-\theta t)(\theta T)^2 \\ &+ 3\Big(\tfrac{\sigma^2(1-\mathrm{e}^{-2\kappa\tau})}{2\kappa} + \mathrm{e}^{-2\kappa\tau}(p-\theta t)^2\Big)\theta T \\ &+ \mathrm{e}^{-3\kappa\tau}(p-\theta t)^3 + \tfrac{3\sigma^2\,\mathrm{e}^{-3\kappa\tau}(\mathrm{e}^{2\kappa\tau}-1)(p-\theta t)}{2\kappa} \quad (58)\end{aligned}$$

$$\begin{aligned} U^{(4)}_{\text{linear}}(p,\tau) &= (\theta T)^4 + 4\,\mathrm{e}^{-\kappa\tau}(p-\theta t)(\theta T)^3 \\ &+6\Big(\tfrac{\sigma^2(1-\mathrm{e}^{-2\kappa\tau})}{2\kappa} + \mathrm{e}^{-2\kappa\tau}(p-\theta t)^2\Big)(\theta T)^2 + \mathrm{e}^{-4\kappa\tau}(p-\theta t)^4 \\ &+ \tfrac{3\sigma^4\,\mathrm{e}^{-4\kappa\tau}(\mathrm{e}^{2\kappa\tau}-1)^2}{4\kappa^2} + \tfrac{3\sigma^2\,\mathrm{e}^{-4\kappa\tau}(\mathrm{e}^{2\kappa\tau}-1)(p-\theta t)^2}{\kappa} \quad (59)\end{aligned}$$

for $p \in \mathbb{R}$ and $\tau = T - t \geqslant 0$.

The following theorem is a consequence of Theorem 3.

Theorem 4 *According to Theorem 2, $U^{(n)}(v,\tau)$ can also be written as*

$$U^{(n)}(v,\tau) = \theta^n + \sum_{k=1}^{n} \tfrac{n!}{(n-k)!k!}\Big(\mathrm{e}^{-k\kappa\tau}(v-\theta)^k + \sum_{l=0}^{\lfloor\frac{k-2}{2}\rfloor} a_{2l+1,2,l+1}(\tau)\Big(\tfrac{\sigma^2}{2\kappa}\Big)^{l+1}(v-\theta)^{k-2l-2}\theta^{n-k}\Big) \quad (60)$$

for $n = 1,2,\dots$, where $a_{2l+1,2,l+1}(\tau)$, $l = 0,\dots,\lfloor\frac{k-2}{2}\rfloor$, are defined in (36) with $\gamma = k$ and we let $\sum_{l=0}^{-1} = 0$.

Moreover,

$$\lim_{v\to\theta} U^{(n)}(v,\tau) = \theta^n + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(n-2k)!(2k)!} \times \left(a_{2k-1,2,k}(\tau)\left(\frac{\sigma^2}{2\kappa}\right)^k\right)\theta^{n-2k} \quad (61)$$

$$\lim_{\sigma\to 0^+} U^{(n)}(v,\tau) = (\theta + (v-\theta)\,\mathrm{e}^{-\kappa\tau})^n \quad (62)$$

$$\lim_{\kappa\to\infty} U^{(n)}(v,\tau) = \theta^n \quad (63)$$

for $n = 1, 2, \dots$, *where* $a_{2k-1,2,k}(\tau), k = 1, \dots, \lfloor \frac{n}{2} \rfloor$, *are defined in (36) with* $\gamma = 2k$ *and we let* $\sum_{k=1}^{0} = 0$.

Proof: In order to obtain (60), we set $\mu(t) = \theta$ in (51), and hence, $p_t = v_t$. Next we consider the limits on the LHS of (61)–(63). Using (60) and straightforward but lengthy calculations, the limits can easily be obtained as shown on the RHS of (61)–(63). □

Example 2 By using (60), the n-th conditional moment of the O-U process (1) for $n = 1, 2, 3, 4$ can be derived as follows:

$$U^{(1)}(v,\tau) = \theta + \mathrm{e}^{-\kappa\tau}(v-\theta), \quad (64)$$

$$U^{(2)}(v,\tau) = \theta^2 + \mathrm{e}^{-2\kappa\tau}(v-\theta)^2 + 2\theta\,\mathrm{e}^{-\kappa\tau}(v-\theta) + \frac{\sigma^2\left(1-\mathrm{e}^{-2\kappa\tau}\right)}{2\kappa}, \quad (65)$$

$$U^{(3)}(v,\tau) = \theta^3 + 3\theta^2\,\mathrm{e}^{-\kappa\tau}(v-\theta) + \frac{3\sigma^2\,\mathrm{e}^{-3\kappa\tau}(\mathrm{e}^{2\kappa\tau}-1)(v-\theta)}{2\kappa} + 3\theta\left(\frac{\sigma^2(1-\mathrm{e}^{-2\kappa\tau})}{2\kappa} + \mathrm{e}^{-2\kappa\tau}(v-\theta)^2\right) + \mathrm{e}^{-3\kappa\tau}(v-\theta)^3, \quad (66)$$

$$U^{(4)}(v,\tau) = \theta^4 + 4\theta^3\,\mathrm{e}^{-\kappa\tau}(v-\theta) + \frac{3\sigma^2\,\mathrm{e}^{-4\kappa\tau}(\mathrm{e}^{2\kappa\tau}-1)(v-\theta)^2}{\kappa} + \mathrm{e}^{-4\kappa\tau}(v-\theta)^4 + \frac{3\sigma^4\,\mathrm{e}^{-4\kappa\tau}(\mathrm{e}^{2\kappa\tau}-1)^2}{4\kappa^2} + 6\theta^2\left(\frac{\sigma^2(1-\mathrm{e}^{-2\kappa\tau})}{2\kappa} + \mathrm{e}^{-2\kappa\tau}(v-\theta)^2\right) + 4\theta\left(\frac{3\sigma^2\,\mathrm{e}^{-3\kappa\tau}(\mathrm{e}^{2\kappa\tau}-1)(v-\theta)}{2\kappa} + \mathrm{e}^{-3\kappa\tau}(v-\theta)^3\right), \quad (67)$$

for $v > 0$ and $\tau = T - t \geqslant 0$.

THE UNCONDITIONAL MOMENTS OF THE O-U PROCESS

The following theorem presents a closed-form formula of the n-th unconditional moment of the O-U process (1), for any positive integer n.

Theorem 5 *According to Theorem 1, we suppose* v_t *follows the O-U process (1) and* n *is a positive integer. We define*

$$M_n^{\infty} := \lim_{\tau\to\infty} U^{(n)}(v,\tau) = \lim_{\tau\to\infty} E^P[V_{t+\tau}^n | v_t = v] \quad (68)$$

for any $v > 0$ *and* $t \geqslant 0$. *Then,* M_n^{∞} *can be expressed in terms of a polynomial function of degree* n *with respect to the long-run mean level as*

$$M_n^{\infty}(\theta,\kappa,\sigma) = \theta^n + \sum_{j=1}^{2} n^{2-j} \sum_{r\geqslant \frac{n+j-2}{2}}^{n+j-3} a_{n-1,j,r}(\infty) \times \left(\frac{\sigma^2}{2\kappa}\right)^{n-r+j-2} \theta^{2r-2j-n+4} \quad (69)$$

for $n \geqslant 1$, *where*

$$a_{n-1,j,r}(\infty) := \sum_{L_{n-1,j,r}\in S(n-1,j,r)} \left(\prod_{m=1}^{r} \bar{q}^{(j,r)}_{n-1+l_m-\sum_{i=1}^{m} l_i, l_m}\right) \bar{C}^{(j,r)}_{n-1,1} \quad (70)$$

for $j = 1, 2$.

Proof: For $n = 1$, the proof is trivial because $S(0, j, r) = \varnothing$ for $j = 1, 2$. Hence, $M_1^{\infty} = \theta$. Next, we consider for the case $n \geqslant 2$. According to (43) in Theorem 2, we have

$$M_n^{\infty} = \sum_{k=-1}^{n-1} \left(\lim_{\tau\to\infty} A_{k+2}(\tau)\right) v^{n-k-1}. \quad (71)$$

Utilizing (35) in Corollary 1 with $\gamma = n$, we arrive

$$\lim_{\tau\to\infty} A_{k+2}(\tau) = \sum_{j=1}^{2} n^{2-j} \sum_{r\geqslant \frac{k+j-1}{2}}^{k+j-1} \left(\lim_{\tau\to\infty} a_{k,j,r}(\tau)\right) \times \left(\frac{\sigma^2}{2\kappa}\right)^{k-r+j-1} \theta^{2r-k-2j+3}. \quad (72)$$

Notice from (38)–(39) that, for $j = 1, 2$, $m = -1, 0$,

$$\bar{w}^{(1,r)}_{k,-1} = n > 0,\ \bar{w}^{(1,r)}_{k,0} = (n-1) > 0,\ \bar{w}^{(2,r)}_{k,0} = n > 0, \quad (73)$$

and for $j = 1, 2$, $m \geqslant 1$,

$$\bar{w}^{(j,r)}_{k,m} = \begin{cases} 0 & \text{if } k = n-1 \text{ and } m = 1, \\ p_k > 0 & \text{if } k \leqslant n-2, \end{cases} \quad (74)$$

where $p_k = (n - k - l_m + \sum_{i=1}^{m} l_i - 1) > 0$. From (73)–(74), we conclude that

$$\lim_{\tau\to\infty} \mathrm{e}^{-\bar{w}^{(j,r)}_{k,m}\tau} = \begin{cases} 1 & \text{if } k = n-1 \text{ and } m = 1, \\ 0 & \text{otherwise,} \end{cases} \quad (75)$$

and from (36)

$$\lim_{\tau\to\infty} a_{k,j,r}(\tau) = \sum_{L_{k,j,r}\in S(k,j,r)} \left(\prod_{m=1}^{r} \bar{q}^{(j,r)}_{k+l_m-\sum_{i=1}^{m} l_i, l_m}\right) \times \left(\sum_{m=j-2}^{r} \bar{C}^{(j,r)}_{k,m} \lim_{\tau\to\infty} \mathrm{e}^{-\kappa\bar{w}^{(j,r)}_{k,m}\tau}\right) = a_{n-1,j,r}(\infty). \quad (76)$$

Using (76) and (72) to calculate the limits on the RHS of (71) yields

$$M_n^{\infty}=\sum_{j=1}^{2} n^{2-j}\sum_{r\geqslant\frac{n+j-2}{2}}^{n+j-2} a_{n-1,j,r}(\infty)\times\left(\frac{\sigma^2}{2\kappa}\right)^{n-r+j-2}\theta^{2r-2j-n+4}. \quad (77)$$

Next, we consider the RHS of (77) when $r=n+j-2$ for $j=1,2$. Since $S(n-1,2,n)=\varnothing$, we rewrite (77) to obtain

$$M_n^{\infty}=na_{n-1,1,n-1}(\infty)\theta^n+\sum_{j=1}^{2} n^{2-j}\sum_{r\geqslant\frac{n+j-2}{2}}^{n+j-3} a_{n-1,j,r}(\infty)\times\left(\frac{\sigma^2}{2\kappa}\right)^{n-r+j-2}\theta^{2r-2j-n+4}, \quad (78)$$

where

$$a_{n-1,1,n-1}(\infty)=\sum_{L_{n-1,1,n-1}\in S(n-1,1,n-1)}\left(\prod_{m=1}^{n-1}\bar{q}^{(1,n-1)}_{n-1+l_m-\sum_{i=1}^m l_i,l_m}\right)\bar{C}^{(1,n-1)}_{n-1,1}. \quad (79)$$

It should be noted that $S(n-1,1,n-1)=\{(1,1,\dots,1)\}$. Applying $S(n-1,1,n-1)$ to (37)–(40), we obtain the following results:

$$\bar{q}^{(1,n-1)}_{n-1+l_m-\sum_{i=1}^m l_i,l_m}=m, \quad (80)$$

$$\prod_{m=1}^{n-1}\bar{q}^{(1,n-1)}_{n-1+l_m-\sum_{i=1}^m l_i,l_m}=(n-1)!, \quad (81)$$

$$\bar{w}^{(1,n-1)}_{n-1,-1}-\bar{w}^{(1,n-1)}_{n-1,1}=n, \quad (82)$$

$$\bar{w}^{(1,n-1)}_{n-1,0}-\bar{w}^{(1,n-1)}_{n-1,1}=n-1, \quad (83)$$

$$\bar{w}^{(1,n-1)}_{n-1,h}-\bar{w}^{(1,n-1)}_{n-1,1}=h-1, \quad (84)$$

for $h=2,\dots,n-1$, and

$$\bar{C}^{(1,n-1)}_{n-1,1}=\prod_{h=-1,h\neq1}^{n-1}\frac{1}{\bar{w}^{(1,n-1)}_{n-1,h}-\bar{w}^{(1,n-1)}_{n-1,1}}=\frac{1}{n!}. \quad (85)$$

From (79)–(85), we conclude $a_{n-1,1,n-1}(\infty)=\frac{1}{n}$, and hence, we now obtain (69). □

Example 3 By applying (68) in Theorem 5 for $n=1,2,3,4$, we have

$$M_1^{\infty}=\theta, \quad (86)$$

$$M_2^{\infty}=\theta^2+a_{1,2,1}(\infty)\times\frac{\sigma^2}{2\kappa}=\theta^2+\frac{\sigma^2}{2\kappa}, \quad (87)$$

$$M_3^{\infty}=\theta^3+3^2\times a_{2,2,1}(\infty)\times\left(\frac{\sigma^2}{2\kappa}\right)\theta+3\times a_{2,2,2}(\infty)\times\left(\frac{\sigma^2}{2\kappa}\right)\theta=\theta^3+\frac{3\theta\sigma^2}{2\kappa} \quad (88)$$

$$M_4^{\infty}=\theta^4+4^3a_{3,1,2}(\infty)\left(\frac{\sigma^2}{2\kappa}\right)\theta^2+4^2a_{3,2,2}(\infty)\left(\frac{\sigma^2}{2\kappa}\right)^2+4^2a_{3,2,3}(\infty)\left(\frac{\sigma^2}{2\kappa}\right)\theta=\theta^4+\frac{3\theta^2\sigma^2}{\kappa}+\frac{3\sigma^4}{4\kappa^2} \quad (89)$$

where the coefficients $a_{n-1,j,r}(\infty)$ are computed by using (70). It should be noticed that the above results can be obtained by taking τ in (64)–(67) approaches infinity, respectively.

Utilizing the closed-form formula (69), some asymptotic properties of the n-th unconditional moment of the O-U process (1) can be derived as follows.

Corollary 4 *According to Theorem 5, we have*

$$\lim_{\sigma\to0^+}M_n^{\infty}(\theta,\kappa,\sigma)=\lim_{\kappa\to\infty}M_n^{\infty}(\theta,\kappa,\sigma)=\theta^n \quad (90)$$

and

$$\lim_{\theta\to0^+}M_n^{\infty}(\theta,\kappa,\sigma)=\begin{cases}0 & \text{if } n \text{ is odd,}\\ \prod_{m=1}^{\frac{n}{2}}(2m-1)\left(\frac{\sigma^2}{2\kappa}\right)^{\frac{n}{2}} & \text{if } n \text{ is even.}\end{cases} \quad (91)$$

Proof: Assertion (90) is trivial. To obtain Assertion (91), we shall determine the constant term of the polynomial function $M_n^{\infty}(\theta)$. Consider the term $\theta^{2r-2j-n+4}$ on the RHS of (69). We have the fact that $\lim_{\theta\to0^+}M_n^{\infty}(\theta)\neq0$ if $r=\frac{2j+n-4}{2}$. For $j=1$, $r=\frac{n-2}{2}$. It is easy to show that $S(n-1,1,\frac{n-2}{2})=\varnothing$ and $a_{n-1,1,\frac{n-2}{2}}(\infty)=0$. For $j=2$, $r=\frac{n}{2}$. If n is odd, r is not an integer. Hence, the summation $\sum_{r\geqslant\frac{n+j-2}{2}}^{n+j-3}$ in (69) must start from the index $r^*=\frac{n}{2}+\frac{1}{2}>r$. Hence, $\lim_{\theta\to0^+}\theta^{2r^*-2j-n+4}=0$. Thus, we consider the case $j=2$ and n is even. Note that $S(n-1,2,\frac{n}{2})=\{(2,2,\dots,2)\}$. To compute $a_{n-1,2,\frac{n}{2}}(\infty)$, we apply $S(n-1,2,\frac{n}{2})$ to (37)–(40) and the results are

$$\bar{q}^{(2,n-1)}_{n-1+l_m-\sum_{i=1}^m l_i,l_m}=(2m-1)(2m), \quad (92)$$

$$\prod_{m=1}^{\frac{n}{2}}\bar{q}^{(2,n-1)}_{n-1+l_m-\sum_{i=1}^m l_i,l_m}=2^{\frac{n}{2}}\left(\frac{n}{2}\right)!\prod_{m=1}^{\frac{n}{2}}(2m-1), \quad (93)$$

$$\bar{w}^{(2,\frac{n}{2})}_{n-1,0}-\bar{w}^{(2,\frac{n}{2})}_{n-1,1}=n, \quad (94)$$

$$\bar{w}^{(2,\frac{n}{2})}_{n-1,h}-\bar{w}^{(2,\frac{n}{2})}_{n-1,1}=2(h-1), \quad (95)$$

for $h=2,\dots,\frac{n}{2}$, and

$$\bar{C}^{(2,\frac{n}{2})}_{n-1,1}=\prod_{h=0,h\neq1}^{\frac{n}{2}}\frac{1}{\bar{w}^{(2,\frac{n}{2})}_{n-1,h}-\bar{w}^{(2,\frac{n}{2})}_{n-1,1}}=\frac{1}{2^{\frac{n}{2}}\left(\frac{n}{2}\right)!}. \quad (96)$$

From (69) and (92)–(96), for $j=2$, $r=\frac{2j+n-4}{2}=\frac{n}{2}$

and n is even, we conclude that

$$a_{n-1,2,\frac{n}{2}}(\infty)\left(\frac{\sigma^2}{2\kappa}\right)^{\frac{n}{2}} = \left(\prod_{m=1}^{\frac{n}{2}} \bar{q}^{(2,n-1)}_{n-1+l_m-\sum_{i=1}^{m} l_i, l_m}\right)\left(\bar{C}^{(2,\frac{n}{2})}_{n-1,1}\right)\left(\frac{\sigma^2}{2\kappa}\right)^{\frac{n}{2}} = \prod_{m=1}^{\frac{n}{2}}(2m-1)\left(\frac{\sigma^2}{2\kappa}\right)^{\frac{n}{2}} \tag{97}$$

is the constant term of the polynomial function $M_n^{\infty}(\theta)$. □

Example 4 According to the closed-form formulas for the unconditional moments (86)–(89) in Example 3, one can easily verify that (90)–(91) hold for $n = 1, 2, 3, 4$.

CONCLUSION

The paper has presented an application of solutions of linear difference equations for obtaining closed-form formulas for the γ-th conditional moment and n-th unconditional moment of the O-U process, for any positive real number γ and positive integer n. We have found that the n-th conditional and unconditional moments can be expressed in terms of a polynomial function of degree n with respect to the long-run mean level parameter of the O-U process. Several asymptotic properties of the n-th conditional and unconditional moments have been presented through the key parameters. Utilizing the current closed-form formula for the conditional moment, the n-th conditional moment of the trending O-U process has been deduced in closed form and also can be expressed in terms of a polynomial function of degree n with respect to the deterministic trend.

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