RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2023.s001
ScienceAsia 49S (2023): **50–58**

Hulls of codes from complete multipartite graphs

Jirapha Limbupasiriporn*, **Prasit Limbupasiriporn**

Department of Mathematics, Faculty of Science, Silpakorn University, Nakorn Pathom 73000 Thailand

*Corresponding author, e-mail: limbupasiriporn_j@su.ac.th

Received 17 Jan 2023, Accepted 26 Jun 2023
Available online 27 Dec 2023

ABSTRACT: Codes over a finite field of prime order that are row spaces of adjacency matrices of complete multipartite graphs are examined. The main parameters for the codes and their duals including the hulls of the codes are obtained. Bases of minimum-weight vectors for those codes are also constructed and the conditions for the codes to be self-orthogonal and self-dual are established.

KEYWORDS: linear codes, complete multipartite graphs, adjacency matrices

MSC2020: 05C50 94B05

INTRODUCTION

The binary codes from complete multipartite graphs were examined in [1, 2] with a view to permutation decoding. In this paper, codes over the finite field $\mathbb{F}_p$ of prime order p obtained from adjacency matrices of complete multipartite graphs are examined. The main parameters for the codes and their duals including the hulls are determined. Bases for those codes are found, some of which consists of vectors of minimum weight. The results are summarized in the following.

Theorem 1 *Let A be an adjacency matrix for the complete multipartite graph $K_{n_1,n_2,\dots,n_r}$ with vertex set V partitioned into parts $V_1, V_2, \dots, V_r$ of cardinalities $n_1, n_2, \dots, n_r$, respectively. Assume that $n_{i_1}, n_{i_2}, \dots, n_{i_m}$ are all congruent to zero modulo p and $n_{i_1} \leqslant n_{i_2} \leqslant \cdots \leqslant n_{i_m}$ where $2 \leqslant m < r$. Let $C_p(A)$ be the code obtained from the row span of A over the field $\mathbb{F}_p$.*

(i) *If $r \equiv 1 \pmod p$, then the hull of $C_p(A)$ is an $[n, m-1, n_{i_1} + n_{i_2}]$ code with a basis $\{v^{V_{i_j}} - v^{V_{i_m}} \mid 1 \leqslant j < m\}$.*
(ii) *If $r \not\equiv 1 \pmod p$, then the hull of $C_p(A)$ is an $[n, m, n_{i_1}]$ code with a basis $\{v^{V_{i_j}} \mid 1 \leqslant j \leqslant m\}$.*

The paper is organized as follows. First notations related to codes, incidence structures, and graphs are briefly reviewed. Then the dimension of the codes over the finite field of prime order from the graph $K_{n_1,n_2,\dots,n_r}$ is determined. Next in order to obtain the minimum weight of the codes, the dual codes from $K_{n_1,n_2,\dots,n_r}$ are described, and subsequently bases of vectors of minimum weight for the codes are constructed. Necessary and sufficient conditions for the codes to be self-orthogonal and self-dual are also provided. Finally the hulls of the codes are examined to obtain the parameters using the structure of the codes and their duals. In the end our conclusion is presented.

TERMINOLOGY

We introduce basic terminology regarding codes, incidence structures, and graphs. The reader is referred to [3, 4] for a full discussion.

All the codes here are linear codes of length n over the finite field $\mathbb{F}_p$ of prime order p which are subspaces of the vector space $\mathbb{F}_p^n$ of n-tuples over $\mathbb{F}_p$. The *support* of a vector x in $\mathbb{F}_p^n$, $\mathrm{supp}(x)$, is the set of nonzero coordinate positions of x, and the *weight* of x is the cardinality of its support. The minimum weight of nonzero vectors in a linear code is the minimum weight of the code. If C is a linear code of length n over $\mathbb{F}_p$ and dimension k and with minimum weight d, then it is said to be an $[n, k, d]_p$ code. The *dual code* $C^\perp$ of C is the orthogonal complement of C under the standard inner product $(,)$ on $\mathbb{F}_p^n$, i.e.

$$C^\perp = \{x \in \mathbb{F}_p^n \mid (x, c) = 0 \text{ for all } c \in C\}.$$

The dual code $C^\perp$ is also linear over $\mathbb{F}_p$ of dimension $n-k$. The *hull* of C, $\mathrm{Hull}(C)$, is $C \cap C^\perp$. If $\mathrm{Hull}(C) = \{0\}$, then C is a *linear code with complementary dual* or shortly an LCD code. Two linear codes of the same length and over the same field are *isomorphic* if they can be obtained from one another by permuting the coordinate positions. A code C is *self-orthogonal* if $C \subseteq C^\perp$ and *self-dual* if $C = C^\perp$. A code is *doubly even* if all the codewords have weight divisible by 4.

A (finite) incidence structure $\mathscr{D} = (\mathscr{P}, \mathscr{B}, \mathscr{I})$ consists of two finite disjoint sets $\mathscr{P}$ and $\mathscr{B}$ and a subset $\mathscr{I}$ of $\mathscr{P} \times \mathscr{B}$. The elements of $\mathscr{P}$ and $\mathscr{B}$ are called *points* and *blocks*, respectively. If all the points and blocks of $\mathscr{D}$ are labelled, then an *incidence matrix* A for $\mathscr{D}$ is a $|\mathscr{B}| \times |\mathscr{P}|$ matrix $[a_{B,b}]$ whose rows and columns correspond to blocks B and points b, respectively, where $a_{B,b}$ is 1 if $(b, B) \in I$ and is 0 otherwise. If F is a field, then we denote by $F^{\mathscr{P}}$ the full vector space of functions from the point set $\mathscr{P}$ to the field F. If $|\mathscr{P}| = n$, then a function u in the space $F^{\mathscr{P}}$ may be regarded as an n-tuple in the vector space F^n whose coordinate corresponding to a point b is the value of u at the point b. If $\mathscr{S}$ is any subset of $\mathscr{P}$, then the *incidence vector* of $\mathscr{S}$, denoted by $v^{\mathscr{S}}$, is a function in $F^{\mathscr{P}}$ defined by, for any point b in $\mathscr{P}$, $v^{\mathscr{S}}(b)$ is 1 if $b \in \mathscr{S}$ and 0 otherwise. If $\mathscr{S} = \{a\}$, then $v^{\mathscr{S}}$ will be written as v^a. Thus each row vector of an incidence matrix of the structure $\mathscr{D}$ can be considered as an incidence vector of the corresponding block. The code $C_F(\mathscr{D})$ of $\mathscr{D}$ over a field F is the row span of an incidence matrix of $\mathscr{D}$, i.e. $C_F(\mathscr{D}) = \langle v^B \mid B \in \mathscr{B} \rangle$. If the field F is of order p, then we will write $C_p(\mathscr{D})$ for $C_F(\mathscr{D})$.

All the graphs $\Gamma = (V, E)$ with vertex set V and edge set E discussed in this paper are simple, i.e. an undirected graph with no loops and no multiple edges. The *neighbour set* of a vertex x of Γ is the set of vertices adjacent to x, and the *valency* of x is the cardinality of its neighbour set. A graph is *k-regular* if all the vertices have the same valency k. A *strongly regular graph* of type (n, k, λ, μ) is a k-regular graph on n vertices, which is such that every pair of adjacent vertices has λ common neighbours and every pair of non-adjacent vertices has μ common neighbours. The *complement* of a graph Γ is a graph $\overline{\Gamma}$ with the same vertex set and with the property that two vertices are adjacent in $\overline{\Gamma}$ if and only if they are not adjacent in Γ. A *complete multipartite graph* $K_{n_1,n_2,\ldots,n_r}$ is a graph with vertex set partitioned into r parts of cardinalities $n_1, n_2, \ldots, n_r$ and two vertices being adjacent if they belong to different parts of V.

An *adjacency matrix* of a graph $\Gamma = (V, E)$ is a $|V| \times |V|$ matrix $A = [a_{xy}]$ whose rows and columns are indexed by vertices of Γ in the same order and which is such that a_{xy} is 1 if vertices x and y are adjacent and 0 otherwise. A linear code $C_p(A)$ of a graph Γ over the field $\mathbb{F}_p$ is the row space of an adjacency matrix A of Γ. A *neighbourhood structure* of Γ is an incidence structure $\mathscr{D}$ with point set V and block set the collection of the neighbour sets of vertices of Γ. Thus the incidence matrix of $\mathscr{D}$ is an adjacency matrix A of Γ, and the code $C_p(\mathscr{D})$ of $\mathscr{D}$ can be considered as the code $C_p(A)$ of Γ.

CODES FROM $K_{n_1,n_2,\ldots,n_r}$

We examine codes obtained from adjacency matrices of the complete multipartite graphs $K_{n_1,n_2,\ldots,n_r}$ with vertex set V partitioned into parts $V_1, V_2, \ldots, V_r$ of cardinalities $n_1, n_2, \ldots, n_r$, respectively. All the notations established here will be used throughout the paper.

It is noted that the complete multipartite graph $\Gamma = K_{n_1,n_2,\ldots,n_r}$ has $n = \sum_{i=1}^{r} n_i$ vertices and $\sum_{i=1}^{r-1} \Big(n_i \sum_{j=i+1}^{r} n_j \Big)$ edges. The complement $\overline{\Gamma}$ of Γ is a disconnected graph with r components of complete graphs $K_{n_1}, K_{n_2}, \ldots, K_{n_r}$. If the cardinalities of some parts of V are unequal, then Γ is not regular; otherwise Γ and $\overline{\Gamma}$ are strongly regular graph of type $\big(mr, m(r-1), m(r-2), m(r-1)\big)$ and $(mr, m-1, m-2, 0)$, respectively, where $m \geqslant 2$ and $n_i = m$ for all $1 \leqslant i \leqslant r$. In the latter case, Γ and $\overline{\Gamma}$ have the same automorphism group, which is isomorphic to the wreath product of the symmetric group S_m by S_r, i.e. $\mathrm{Aut}(\Gamma) \cong S_m \wr S_r$, see [2] for more details.

Now we look at the code spanned over the finite field $\mathbb{F}_p$ of prime order p by the row vectors of an adjacency matrix A of the complete multipartite graph $\Gamma = K_{n_1,n_2,\ldots,n_r}$. We note that if the vertices of Γ are labelled in the order $V = V_1 \cup V_2 \cup \cdots \cup V_r$, then those corresponding to the j^{th} vertex on the i-th part V_i of V will be written as the ordered pairs (i, j) for all $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant n_i$. Thus the matrix A can be partitioned into the following:

$$A = \left.\begin{bmatrix} 0_{n_1\times n_1} & J_{n_1\times n_2} & J_{n_1\times n_3} & \cdots & J_{n_1\times n_r} \\ J_{n_2\times n_1} & 0_{n_2\times n_2} & J_{n_2\times n_3} & \cdots & J_{n_2\times n_r} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ J_{n_r\times n_1} & J_{n_r\times n_2} & J_{n_r\times n_3} & \cdots & 0_{n_r\times n_r} \end{bmatrix}\right\} r \text{ blocks}$$

where $J_{n_i\times n_j}$ is an $n_i \times n_j$ matrix with entries equal to 1, and $0_{n_i\times n_j}$ is an $n_i \times n_j$ zero matrix for all i, j such that $1 \leqslant i, j \leqslant r$. Observe that for each $1 \leqslant i \leqslant r$, if $n_i \geqslant 2$, then the i-th block of A has n_i repeated rows. Thus we can construct an $r \times n$ matrix $\overline{A}$ obtained from A by removing $n_i - 1$ rows from each of the r blocks such that $n_i \geqslant 2$, leaving only the first rows in each block so that $\overline{A}$ is of the form

$$\overline{A} = \left.\begin{bmatrix} 0_{1\times n_1} & J_{1\times n_2} & J_{1\times n_3} & \cdots & J_{1\times n_r} \\ J_{1\times n_1} & 0_{1\times n_2} & J_{1\times n_3} & \cdots & J_{1\times n_r} \\ \vdots & \vdots & \vdots & & \vdots \\ J_{1\times n_1} & J_{1\times n_2} & J_{1\times n_3} & \cdots & 0_{1\times n_r} \end{bmatrix}\right\} r \text{ rows}. \quad (1)$$

The code $C_p(\overline{A})$ spanned over $\mathbb{F}_p$ by the row vectors of $\overline{A}$ is as the code of the neighbourhood structure $\mathscr{D}$ of Γ with point set V and block set $\mathscr{B} = \{B_i \mid 1 \leqslant i \leqslant r\}$, where B_i is defined to be $V \setminus V_i$ for all $1 \leqslant i \leqslant r$. Thus

$$C_p(A) = C_p(\overline{A}) = C_p(\mathscr{D}) = \langle v^{B_i} \mid 1 \leqslant i \leqslant r \rangle.$$

Note that since each point of $\mathscr{D}$ is on $r-1$ blocks, it follows that $\sum_{i=1}^{r} v^{B_i} = (r-1)\jmath$ where $\jmath$ is the all-one vector. Thus if $r \not\equiv 1 \pmod p$, then $\jmath$ is a codeword in $C_p(A)$. If $r \equiv 1 \pmod p$, then the incidence vectors v^{B_i} for $1 \leqslant i \leqslant r$ are linearly dependent so that the dimension of $C_p(\mathscr{D})$ is less than r. We will use the concept of the neighbourhood structure of Γ in the proofs of our results regarding codes from the graph.

The following result in [5] is required to determine the dimension of the code $C_p(A)$.

Result 1 *Let M be an integral matrix of order r with integral eigenvalues.*

(i) *If the eigenvalues of M are all nonzero modulo a prime p, then M has full p-rank r.*

(ii) *If M has precisely one eigenvalue λ divisible by p, then the p-rank of M is $r-m$ where m is a geometric multiplicity of λ.*

It is known that an adjacency matrix for the complete graph K_r on r vertices has two distinct eigenvalues, one of which is $r-1$ with multiplicity one and the other -1 with multiplicity $r-1$. Thus by Result 1, we obtain the following.

Corollary 1 *The p-rank, where p is a prime, of an adjacency matrix for the complete graph K_r is $r-1$ if $r \equiv 1 \pmod{p}$ and r if $r \not\equiv 1 \pmod{p}$.*

Proposition 1 *Let A be an adjacency matrix for $K_{n_1,n_2,\dots,n_r}$. Then the code $C_p(A)$ of the graph has dimension $r-1$ if $r \equiv 1 \pmod{p}$ and r if $r \not\equiv 1 \pmod{p}$.*

Proof: The dimension of the code $C_p(A)$ is the p-rank of the matrix $\overline{A}$ in (1), which equals the dimension of the column space of $\overline{A}$ over $\mathbb{F}_p$. Removing all duplicate columns from $\overline{A}$ gives an adjacency matrix for the complete graph K_r. Thus the dimension of $C_p(A)$ follows from Corollary 1. □

We note that since $K_{1,1,\dots,1}$ is the complete graph K_r, it follows that the code $C_p(A)$ of that graph is trivial, i.e. it is an $[r, r-1, 2]$ code if $r \equiv 1 \pmod{p}$ and an $[r, r, 1]$ code if $r \not\equiv 1 \pmod{p}$. Therefore, from now on we will consider only the codes $C_p(A)$ of $K_{n_1,n_2,\dots,n_r}$ where $n_i \neq 1$ for some $1 \leqslant i \leqslant r$.

What we need to do next is to determine the minimum weight for the code $C_p(A)$. To do this we look at the structure of its dual code $C_p(A)^{\perp}$ and then use it to obtain the minimum weight for $C_p(A)$ in the next two sections.

THE DUAL CODES

In order to obtain the minimum weight of the code from the complete multipartite graph $K_{n_1,n_2,\dots,n_r}$, we need to consider its dual code. In this section, the minimum weight for the dual code are determined and bases of minimum-weight vectors for the code are constructed.

Proposition 2 *Let A be an adjacency matrix for the graph $\Gamma = K_{n_1,n_2,\dots,n_r}$. For any vector u of weight 2, u is in the dual code $C_p(A)^{\perp}$ of Γ if and only if $u = \alpha(v^x - v^y)$ where $\alpha \in \mathbb{F}_p \backslash \{0\}$ and x and y are distinct vertices in the same part of the vertex set of Γ.*

Proof: Let u be any vector of weight 2. If $u = \alpha(v^x - v^y)$, where $\alpha \in \mathbb{F}_p \backslash \{0\}$ and x and y are distinct vertices in a part V_i of V for some $1 \leqslant i \leqslant r$, then for any $1 \leqslant j \leqslant r$, we have

$$\begin{aligned}\left(u, v^{B_j}\right) &= \sum_{a \in V} \alpha(v^x - v^y)(a)\, v^{B_j}(a) \\ &= \alpha\left(v^{B_j}(x) - v^{B_j}(y)\right) \\ &= 0 \mod p,\end{aligned}$$

so that $u \in C_p(A)^{\perp}$.

Conversely, assume that u is in $C_p(A)^{\perp}$ with $\mathrm{supp}(u) = \{x, y\}$. If x and y are in distinct parts of the vertex set V of Γ, say $x \in V_i$ and $y \in V_j$, where $1 \leqslant i, j \leqslant r$ and $i \neq j$, then

$$\left(u, v^{B_i}\right) = u(x)v^{B_i}(x) + u(y)v^{B_i}(y) = u(y) \neq 0,$$

a contradiction to the fact that $v^{B_i} \in C_p(A)$. Thus, x and y must be in the same part, say V_i, of V. This implies that for any j such that $1 \leqslant j \leqslant r$ with $j \neq i$, we have

$$0 = \left(u, v^{B_j}\right) = u(x) + u(y),$$

or $u(x) = -u(y) = \alpha \in \mathbb{F}_p \backslash \{0\}$. Thus,

$$u = u(x)v^x + u(y)v^y = \alpha v^x - \alpha v^y = \alpha(v^x - v^y).$$

□

We observe that if A is an adjacency matrix for $K_{n_1,n_2,\dots,n_r}$, then any vector of weight one is not in $C_p(A)^{\perp}$. For if w is any vector of weight one with nonzero entry at a vertex x in V, then we choose a part of V not containing x, say V_i, to obtain

$$\left(w, v^{B_i}\right) = w(x)v^{B_i}(x) = w(x) \neq 0.$$

Therefore, the minimum weight of $C_p(A)^{\perp}$ is 2 by Proposition 2. Thus by Proposition 1 we have the following.

Corollary 2 *Let A be an adjacency matrix for $K_{n_1,n_2,\dots,n_r}$ and $n = \sum_{i=1}^{r} n_i$. Then the dual code $C_p(A)^{\perp}$ of the graph is $[n, n-r+1, 2]_p$ if $r \equiv 1 \pmod{p}$ and $[n, n-r, 2]_p$ if $r \not\equiv 1 \pmod{p}$.*

We now establish bases of minimum-weight vectors for the dual code $C_p(A)^{\perp}$ of the graph $K_{n_1,n_2,\dots,n_r}$ with adjacency matrix A, where $r \not\equiv 1 \pmod{p}$. Such bases cannot be found for the case where $r \equiv 1 \pmod{p}$.

Recall that we denote by (i, j) the j-th vertex in a part V_i of the vertex set V of $K_{n_1,n_2,\dots,n_r}$ for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant n_i$. Thus a vector of the form $v^{(i,j)} - v^{(i,k)}$, where $1 \leqslant i \leqslant r$ and $1 \leqslant j, k \leqslant n_i$ with $j \neq k$, has weight 2 with support a subset of V_i. By Proposition 2, that vector is in $C_p(A)^{\perp}$.

Proposition 3 *Let A be an adjacency matrix for $K_{n_1,n_2,\dots,n_r}$ with $r \not\equiv 1 \pmod{p}$. Then for each i such that $1 \leqslant i \leqslant r$ and fixed k_i such that $1 \leqslant k_i \leqslant n_i$, the set*

$$\bigcup_{i=t+1}^{r} \left\{ v^{(i,j)} - v^{(i,k_i)} \mid 1 \leqslant j \leqslant n_i,\ j \neq k_i \right\}$$

is a basis of minimum-weight vectors for the dual code $C_p(A)^{\perp}$ where t is the number of n_i's equal to 1.

Proof: The set given in the theorem has cardinality

$$\sum_{i=t+1}^{r}(n_i-1)=n-r=\dim\left(C_p(A)^{\perp}\right),$$

where $n=\sum_{i=1}^{r} n_i$, and is linearly independent as

$$\left(\sum_{i=t+1}^{r}\sum_{\substack{j=1\\ j\neq k_i}}^{n_i} c_{i,j}\left(v^{(i,j)}-v^{(i,k_i)}\right)\right)(i_0,j_0) = \begin{cases}-c_{i_0,j_0} & \text{if } j_0=k_{i_0},\\ c_{i_0,j_0} & \text{if } j_0\neq k_{i_0},\end{cases}$$

for all i_0 and j_0 such that $t+1\leqslant i_0\leqslant r$ and $1\leqslant j_0\leqslant n_{i_0}$ with $j_0\neq k_{i_0}$, where $c_{i,j}\in\mathbb{F}_p$ for all $t+1\leqslant i\leqslant r$ and $1\leqslant j\leqslant n_i$ with $j\neq k_i$. Thus, that set is a basis for $C_p(A)^{\perp}$. □

Proposition 4 *Let A be an adjacency matrix for $K_{n_1,n_2,\dots,n_r}$ with $r\equiv 1 \pmod p$. Then the dual code of the graph has no bases of minimum-weight vectors.*

Proof: Let M be a matrix whose row vectors are all vectors of minimum weight in $C_p(A)^{\perp}$ of the form v^x-v^y for all distinct vertices x and y in the same part of the vertex set V of the graph. If the first t parts of V are all of cardinality one, where $0\leqslant t<r$, and the row vectors of M are indexed in such a way that those that have supports contained in the same part of V are in succession, then the entries in the first t columns of M are zeros and thus M is a block matrix of the form

$$M=\left[\begin{array}{c|c|c|c|c} 0 & M_{t+1} & 0 & \cdots & 0\\ \hline 0 & 0 & M_{t+2} & \cdots & 0\\ \hline \vdots & \vdots & \vdots & \ddots & \vdots\\ \hline 0 & 0 & 0 & \cdots & M_r\end{array}\right],$$

where each block M_i is an $\binom{n_i}{2}\times n_i$ matrix with two entries 1 and -1 in each row, where $t<i\leqslant r$. The transpose M^T of M is also a block matrix with each block of size $n_i\times\binom{n_i}{2}$ and with two entries 1 and -1 in each column. Thus the sum of the row vectors in each block of M^T is 0 so that each block has p-rank less than n_i for $t<i\leqslant r$. Therefore,

$$\begin{aligned}\operatorname{rank}_p(M)=\operatorname{rank}_p(M^T)&\leqslant\sum_{i=t+1}^{r}(n_i-1)\\ &=n-r\\ &<n-r+1\\ &=\dim(C_p(A)^{\perp}).\end{aligned}$$

This implies that any $(n-r+1)$-set of minimum-weight vectors in $C_p(A)^{\perp}$ are not linearly independent and so cannot be a basis for $C_p(A)^{\perp}$. □

Though the dual code $C_p(A)^{\perp}$ from $K_{n_1,n_2,\dots,n_r}$ with adjacency matrix A has no bases of minimum-weight vectors if $r\equiv 1 \pmod p$, we found its basis whose almost all vectors have minimum weight.

Proposition 5 *Let A be an adjacency matrix for $K_{n_1,n_2,\dots,n_r}$ with $r\equiv 1 \pmod p$.*
(i) *If $n_1=n_2=\cdots=n_t=1$, where $1\leqslant t<r$, and $n_i\geqslant 2$ for all $t<i\leqslant r$, then the set K given by*

$$K=\left\{\sum_{i=1}^{t}v^{V_i}+\sum_{i=t+1}^{r}v^{(i,n_i)}\right\} \cup\bigcup_{i=t+1}^{r}\left\{v^{(i,j)}-v^{(i,n_i)}\mid 1\leqslant j<n_i\right\}$$

is a basis for $C_p(A)^{\perp}$.
(ii) *If $n_i\geqslant 2$ for all $1\leqslant i\leqslant r$, then the set K' is a basis for $C_p(A)^{\perp}$ where*

$$K'=\left\{v^{(1,j)}+\sum_{i=2}^{r}v^{(i,n_i)}\mid 1\leqslant j\leqslant n_1\right\} \cup\bigcup_{i=2}^{r}\left\{v^{(i,j)}-v^{(i,n_i)}\mid 1\leqslant j<n_i\right\}.$$

Proof: To proof the first part, we note that

$$|K|=1+n-r=\dim\left(C_p(A)^{\perp}\right),$$

where $n=\sum_{i=1}^{r} n_i$, and each vector

$$\sum_{i=1}^{t}v^{V_i}+\sum_{i=t+1}^{r}v^{(i,n_i)}$$

in K is in $C_p(A)^{\perp}$ as

$$\begin{aligned}\left(v^{B_k},\sum_{i=1}^{t}v^{V_i}+\sum_{i=t+1}^{r}v^{(i,n_i)}\right) &=\sum_{i=1}^{t}v^{B_k}(i,1)+\sum_{i=t+1}^{r}v^{B_k}(i,n_i)\\ &=\begin{cases}(t-1)+(r-t)=0 \mod p, & \text{if } 1\leqslant k\leqslant t,\\ t+\left(r-t-1\right)=0 \mod p, & \text{otherwise,}\end{cases}\end{aligned}$$

for all $1\leqslant k\leqslant r$. Furthermore, it can be shown that if

$$x=c\left(\sum_{i=1}^{t}v^{V_i}+\sum_{i=t+1}^{r}v^{(i,n_i)}\right)+\sum_{i=t+1}^{r}\sum_{j=1}^{n_i-1}c_{ij}\left(v^{(i,j)}-v^{(i,n_i)}\right)=0,$$

where c and c_{ij} for $t+1\leqslant i\leqslant r$ and $1\leqslant j<n_i$ are scalars, then $x(1,1)=c=0$ and $x(i,j)=c_{ij}=0$ for all $t+1\leqslant i\leqslant r$ and $1\leqslant j<n_i$. So K is linearly independent, and thus, a basis for $C_p(A)^{\perp}$.

To prove the second part, we observe that for any j such that $1 \leqslant j \leqslant n_1$, the vector $v^{(1,j)} + \sum_{i=2}^{r} v^{(i,n_i)}$ in K' is in $C_p(A)^\perp$ since

$$\left(v^{B_k}, v^{(1,j)} + \sum_{i=2}^{r} v^{(i,n_i)}\right) = v^{B_k}(1,j) + \sum_{i=2}^{r} v^{B_k}(i,n_i) = 0 \mod p$$

for all $1 \leqslant k \leqslant r$. Moreover, if

$$x = \sum_{k=1}^{n_1} c_k \left(v^{(1,k)} + \sum_{i=2}^{r} v^{(i,n_i)}\right) + \sum_{i=2}^{r}\sum_{j=1}^{n_i-1} c_{ij}\left(v^{(i,j)} - v^{(i,n_i)}\right) = 0,$$

where c_k and c_{ij} for $1 \leqslant k \leqslant n_1$, $2 \leqslant i \leqslant r$ and $1 \leqslant j < n_i$ are scalars, then $x(1,k) = c_k = 0$ and $x(i,j) = c_{ij} = 0$ for all $1 \leqslant k \leqslant n_1$, $2 \leqslant i \leqslant r$ and $1 \leqslant j < n_i$, which implies that K' is linearly independent. Since $|K'| = n - r + 1 = \dim\left(C_p(A)^\perp\right)$, it follows that K' is a basis for $C_p(A)^\perp$. □

THE MINIMUM WEIGHT

We determine in this section the minimum weight of the codes from $K_{n_1,n_2,\ldots,n_r}$. Recall that we partition the vertex set V of the graph into sets $V_1, V_2, \ldots, V_r$ where $|V_i| = n_i$ for $1 \leqslant i \leqslant r$. The following is required to obtain the nature of codewords in the codes.

Lemma 1 *Let A be an adjacency matrix for $K_{n_1,n_2,\ldots,n_r}$.*
(i) *Any codeword in the code $C_p(A)$ has support a union of some parts of the vertex set of the graph.*
(ii) *$C_p(A)$ contains the all-one vector if and only if $r \not\equiv 1 \pmod p$.*

Proof: To prove the first part, we let u be any codeword in $C_p(A)$ with support S and let V_i for $1 \leqslant i \leqslant r$ be a part of the vertex set V of the graph such that $S \cap V_i \neq \varnothing$. Let $x \in S \cap V_i$. If there exists a vertex y in V_i that is not in S, then we have

$$(u, v^x - v^y) = u(x) - u(y) = u(x) \neq 0,$$

a contradiction to the fact that $v^x - v^y$ is a vector in $C_p(A)^\perp$ by Proposition 2. Thus, a part of V that contains any vertex in S is a subset of S.

For the second part, we have in the previous two sections that if $r \not\equiv 1 \pmod p$, then the all-one vector $\jmath$ is in $C_p(A)$. Conversely, if $r \equiv 1 \pmod p$, then by Proposition 5, we have that either the vector

$$u = \sum_{i=1}^{t} v^{V_i} + \sum_{i=t+1}^{r} v^{(i,n_i)}$$

or

$$u = v^{(1,1)} + \sum_{i=2}^{r} v^{(i,n_i)}$$

is in the dual code $C_p(A)^\perp$, and since $(u, \jmath) = r \neq \mod p$, it follows that $\jmath$ is not in $C_p(A)$. □

Proposition 6 *Let A be an adjacency matrix for $K_{n_1,n_2,\ldots,n_r}$ with $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_r$. Then the code $C_p(A)$ of the graph has minimum weight $n_1 + n_2$ if $r \equiv 1 \pmod p$ and n_1 if $r \not\equiv 1 \pmod p$.*

Proof: Note that for any distinct i and j such that $1 \leqslant i, j \leqslant r$, the vector $v^{B_i} - v^{B_j}$ in $C_p(A)$ is of the form $v^{V_j} - v^{V_i}$. Thus $C_p(A)$ contains codewords of weight $n_i + n_j$ for $1 \leqslant i, j \leqslant r$ with $i \neq j$. Note also that the all-one vector $\jmath$ can be in the form of $v^{B_i} + v^{V_i}$ for $1 \leqslant i \leqslant r$. Therefore, if $r \equiv 1 \pmod p$, then by Lemma 1, part (ii), $\jmath$ is not in $C_p(A)$, which implies that all the vectors v^{V_i} for $1 \leqslant i \leqslant r$ cannot be in $C_p(A)$, and hence Lemma 1, part (i), gives $n_1 + n_2$, the minimum weight of $C_p(A)$.

If $r \not\equiv 1 \pmod p$, then again by Lemma 1, $\jmath$ is in $C_p(A)$ and thus all the vectors v^{V_i}'s are also in $C_p(A)$ and those of weight n_1 are minimum-weight vectors in $C_p(A)$. □

From Proposition 1 and Proposition 6, the main parameters of the codes from $K_{n_1,n_2,\ldots,n_r}$ and the nature of their codewords of minimum weight are obtained.

Corollary 3 *Let A be an adjacency matrix for $K_{n_1,n_2,\ldots,n_r}$ where $r \equiv 1 \pmod p$. Then*
(i) *the set $\left\{v^{V_i} - v^{V_r} \mid 1 \leqslant i < r\right\}$ is a basis for $C_p(A)$;*
(ii) *any codeword in $C_p(A)$ is of the form $\sum_{i=1}^{r} \alpha_i v^{V_i}$ where $\alpha_i \in \mathbb{F}_p$ for $1 \leqslant i < r$ and $\alpha_r = -\sum_{i=1}^{r-1} \alpha_i$;*
(iii) *if $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_r$, then the code $C_p(A)$ is an $\left[\sum_{i=1}^{r} n_i, r-1, n_1+n_2\right]_p$ code and codewords of minimum weight in $C_p(A)$ are vectors of the form $\alpha\left(v^{V_i} - v^{V_j}\right)$ for $\alpha \in \mathbb{F}_p \backslash \{0\}$ and i and j such that $1 \leqslant i, j \leqslant r$, $i \neq j$, $n_i = n_1$ and $n_j = n_2$.*

Corollary 4 *Let A be an adjacency matrix for $K_{n_1,n_2,\ldots,n_r}$ where $r \equiv 1 \pmod p$ and $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_r$.*
(i) *The code $C_p(A)$ has a basis of minimum-weight vectors if and only if $n_r = n_2$.*
(ii) *If $n_1 < n_2 = n_3 = \cdots = n_r$, then $C_p(A)$ is an $[n_1+(r-1)n_2, r-1, n_1+n_2]_p$ code with a basis of minimum-weight vectors of the form $v^{V_1} - v^{V_j}$ for all $2 \leqslant j \leqslant r$.*
(iii) *If the n_i's are all equal, then $C_p(A)$ is an $[rn_1, r-1, 2n_1]_p$ code, and for any fixed i_0 such that $1 \leqslant i_0 \leqslant r$, the vectors of minimum weight of the form $v^{V_{i_0}} - v^{V_j}$ for $1 \leqslant j \leqslant r$ with $j \neq i_0$ form a basis for $C_p(A)$.*

Proof: We prove only Part (i) as Part (ii) and (iii) follow from that and Corollary 3.

It can be shown that the set S of minimum-weight vectors of the form $v^{V_1} - v^{V_j}$ for all j such that $2 \leqslant j \leqslant r$ and $n_j = n_2$ is linearly independent. Since $|S| \leqslant r-1$, it follows that S is a basis for $C_p(A)$ if and only if $n_r = n_2$. □

Corollary 5 *Let A be an adjacency matrix for the graph $K_{n_1,n_2,\ldots,n_r}$ where $r \not\equiv 1 \pmod p$ and $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_r$.*

(i) *The code* $C_p(A)$ *is an* $\Big[\sum_{i=1}^{r} n_i, r, n_1\Big]_p$ *code with a basis* $\{v^{V_i} \mid 1 \leqslant i \leqslant r\}$.
(ii) *Codewords of minimum weight in* $C_p(A)$ *are vectors of the form* αv^{V_i} *for* $\alpha \in \mathbb{F}_p \backslash \{0\}$ *and for i such that* $1 \leqslant i \leqslant r$ *and* $n_i = n_1$.
(iii) $C_p(A)$ *has a basis of minimum-weight vectors if and only if* $n_r = n_1$.

Corollary 6 *Let A be an adjacency matrix for* $K_{n_1,n_2,\dots,n_r}$ *with* $r \not\equiv 1 \pmod p$. *Then the code* $C_p(A)$ *is doubly-even if and only if all the* n_i*'s are divisible by four.*

Proof: By Corollary 5, the vector v^{V_i} is in $C_p(A)$ for all $1 \leqslant i \leqslant r$. If $C_p(A)$ is doubly-even, then $4 \mid n_i$ for all $1 \leqslant i \leqslant r$. Conversely, if $4 \mid n_i$ for all $1 \leqslant i \leqslant r$, then by Lemma 1, every codeword has weight, which is equal to the sum of the cardinalities of some parts of the vertex set and thus is divisible by four. □

We now establish necessary and sufficient conditions for the codes from $K_{n_1,n_2,\dots,n_r}$ to be self-orthogonal and self-dual.

Proposition 7 *Let A be an adjacency matrix for* $K_{n_1,n_2,\dots,n_r}$. *Then the code* $C_p(A)$ *of the graph is self-orthogonal if and only if* n_i *is divisible by p for all* $1 \leqslant i \leqslant r$.

Proof: Note first that for any $1 \leqslant i,j \leqslant r$, $\left(v^{B_i}, v^{B_j}\right)$ is $n - n_i$ if $i = j$ and $n - n_i - n_j$ otherwise, where $n = \sum_{i=1}^{r} n_i$. Thus, if $p \mid n_i$ for all $1 \leqslant i \leqslant r$, then $p \mid n$ giving $\left(v^{B_i}, v^{B_j}\right) = 0 \bmod p$ for all $1 \leqslant i,j \leqslant r$, and hence $C_p(A) \subseteq C_p(A)^{\perp}$.

Conversely, assume that $C_p(A)$ is self-orthogonal. Then $\left(v^{B_i}, v^{B_i}\right) = 0$ for all $1 \leqslant i \leqslant r$, obtaining

$$n_1 + n_2 + \cdots + n_{i-1} + n_{i+1} + \cdots + n_r = 0 \mod p$$

for all $1 \leqslant i \leqslant r$. This gives a system of linear equation $A(K_r)x = 0$ over $\mathbb{F}_p$ where $A(K_r)$ is an adjacency matrix of the complete graph K_r. If $r \not\equiv 1 \pmod p$, then $\operatorname{rank}_p(A(K_r)) = r$ by Corollary 1, so that the nullity of $A(K_r)$ is 0, yielding $n_i \equiv 0 \pmod p$ or $p \mid n_i$ for all $1 \leqslant i \leqslant r$.

In the case where $r \equiv 1 \pmod p$, we obtain $\operatorname{rank}_p(A(K_r)) = r - 1$, giving the nullity of $A(K_r)$ equal to 1. Note that in this case the null space of $A(K_r)$ is spanned by the vector $(1,1,\dots,1)$. Thus, $n_i \equiv \alpha \pmod p$ for all $1 \leqslant i \leqslant r$, where $\alpha \in \mathbb{F}_p$. By the assumption, we have

$$\left(v^{B_i}, v^{B_j}\right) = n - n_i - n_j \equiv (r-2)\alpha \equiv 0 \pmod p,$$

where $1 \leqslant i,j \leqslant r$ and $i \neq j$. This implies that $\alpha \equiv 0 \pmod p$. Therefore, $p \mid n_i$ for all $1 \leqslant i \leqslant r$. □

Proposition 8 *Let A be an adjacency matrix for* $K_{n_1,n_2,\dots,n_r}$ *with* $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_r$. *Then the code* $C_p(A)$ *is self-dual if and only if* $p = 2$, *r is even and* $n_i = 2$ *for all* $1 \leqslant i \leqslant r$.

Proof: Let $n = \sum_{i=1}^{r} n_i$. If $p = 2$, r is even and $n_i = 2$ for all $1 \leqslant i \leqslant r$, then by Proposition 7, we have $C_p(A) \subseteq C_p(A)^{\perp}$, and since $n = 2r$, it follows by Proposition 1 that

$$\dim(C_p(A)) = r = \frac{n}{2} = \dim(C_p(A)^{\perp}),$$

yielding $C_p(A) = C_p(A)^{\perp}$.

Conversely, assume that $C_p(A)$ is self-dual. Then by Proposition 2, the minimum weight of the code must be 2. If $r \equiv 1 \pmod p$, then by Proposition 6 we have $n_1 + n_2 = 2$ so $n_1 = n_2 = 1$, which contradics to the fact that $p \mid n_1$ by Proposition 7. Thus, $r \not\equiv 1 \pmod p$. By Corollary 5, $\dim(C_p(A)) = r$ and $n_1 = 2$. Again by Proposition 7, we obtain $p = 2$, and thus for each $2 \leqslant i \leqslant r$, $n_i = 2t_i$ for some positive integer t_i. By the assumption, we have $r = \frac{n}{2} = 1 + \sum_{i=2}^{r} t_i$, yielding $t_i = 1$ for all $2 \leqslant i \leqslant r$. Therefore, $n_i = 2$ for all $1 \leqslant i \leqslant r$. □

HULLS OF THE CODES

The hulls $\operatorname{Hull}(C_p(A))$ of the codes $C_p(A)$ from an adjacency matrices A of the graph $K_{n_1,n_2,\dots,n_r}$ are examined. Note from Proposition 7 that if $n_i \equiv 0 \pmod p$ for all $1 \leqslant i \leqslant r$, then $\operatorname{Hull}(C_p(A)) = C_p(A)$. Thus, we will consider only the graph where $n_i \not\equiv 0 \pmod p$ for some $1 \leqslant i \leqslant r$. We note here that a coordinate of a vector x corresponding to the j- th vertex of the i-th part of the vertex set V of the graph will be written as $x_{i,j}$ for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant n_i$.

Lemma 2 *Let A be an adjacency matrix for* $K_{n_1,n_2,\dots,n_r}$ *where* $r \equiv 1 \pmod p$. *Then any vector x in* $\operatorname{Hull}(C_p(A))$ *is of the form*

$$x = \Big(\underbrace{x_1,\dots,x_1}_{n_1\ terms},\underbrace{x_2\dots,x_2}_{n_2\ terms},\dots,\underbrace{x_r,\dots,x_r}_{n_r\ terms}\Big)$$

where $x_i \in \mathbb{F}_p$ *for* $1 \leqslant i \leqslant r$ *such that* $n_i x_i = n_j x_j$ *for* $1 \leqslant i,j \leqslant r$ *and* $\sum_{i=1}^{r} x_i = 0$.

Proof: Since any vector x in $\operatorname{Hull}(C_p(A))$ is orthogonal to all the vectors in $C_p(A)$ and $C_p(A)^{\perp}$, it follows by Corollary 3, part (i), that $(x, v^{V_i} - v^{V_r}) = 0$ or

$$\sum_{j=1}^{n_i} x_{i,j} = \sum_{j=1}^{n_r} x_{r,j} \qquad \text{for all } 1 \leqslant i < r. \tag{2}$$

By Proposition 5, we consider two cases of the n_i's.

If $n_i = 1$ for all $1 \leqslant i \leqslant t$ and $n_i > 1$ for $t < i \leqslant r$, then x is orthogonal to all the vectors in the set K of Proposition 5, part (i), and thus we have the following:

$$\sum_{i=1}^{t} x_{i,1} + \sum_{i=t+1}^{r} x_{i,n_i} = 0 \tag{3}$$

$$x_{i,j} = x_{i,n_i} \quad \text{for } t < i \leqslant r \text{ and } 1 \leqslant j < n_i. \tag{4}$$

In this case, we have by (4) that for each i such that $t < i \leqslant r$, all the coordinates of x corresponding to the vertices in the same part V_i of V are equal, and we assume that they are equal to some x_i in $\mathbb{F}_p$, i.e. $x_{ij} = x_i$ for all $t < i \leqslant r$ and $1 \leqslant j \leqslant n_i$. Thus, Eq. (2) asserts that $n_i x_i = n_r x_r$ for all $t < i < r$. For $1 \leqslant i \leqslant t$, we have again by (2) that $x_{i,1} = n_r x_r$ as $n_i = 1$. If we let $x_i = x_{i,1}$ for all $1 \leqslant i \leqslant t$, then we obtain $n_i x_i = n_r x_r$ for all $1 \leqslant i < r$. Also, Eq. (3) yields $\sum_{i=1}^{r} x_i = 0$, and hence we have the theorem.

In the case where $n_i \geqslant 2$ for all $1 \leqslant i \leqslant r$, we consider the set K' in Proposition 5, part (ii). Since all the vectors in K' are orthogonal to the vector x, it follows that

$$x_{1,j} + \sum_{i=2}^{r} x_{i,n_i} = 0 \quad \text{for } 1 \leqslant j \leqslant n_1, \tag{5}$$

$$x_{i,j} = x_{i,n_i} \quad \text{for } 2 \leqslant i \leqslant r \text{ and } 1 \leqslant j < n_i. \tag{6}$$

Then Eq. (6) gives all the coordinates of x which are in the same part V_i of V and are equal to some x_i in $\mathbb{F}_p$ for all $2 \leqslant i \leqslant r$. Also Eq. (5) provides $x_{1,j} = -\sum_{i=2}^{r} x_i$ for $1 \leqslant j \leqslant n_1$, and if we let $x_{1,j} = x_1$ for $1 \leqslant j \leqslant n_1$, then $\sum_{i=1}^{r} x_i = 0$. □

Now to determine the dimension of $\mathrm{Hull}(C_p(A))$, we consider cardinalities of the components of the graph $K_{n_1,n_2,\ldots,n_r}$ as we did in the previous sections entitled Codes from $K_{n_1,n_2,\ldots,n_r}$ and The dual codes. The following shows that if all of those not divisible by p, then $\mathrm{Hull}(C_p(A))$ is a trivial code.

Proposition 9 *Let A be an adjacency matrix for $K_{n_1,n_2,\ldots,n_r}$ where $r \equiv 1 \pmod p$ and $n_i \not\equiv 0 \pmod p$ for all $1 \leqslant i \leqslant r$. Then*

$$\dim\big(\mathrm{Hull}(C_p(A))\big) = \begin{cases} 1 & \text{if } \sum_{i=1}^{r} n_i^{-1} \equiv 0 \pmod p, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Let $x \in \mathrm{Hull}(C_p(A))$. We assume by Lemma 2 that the coordinates of x corresponding to the vertices in the same part of the vertex set are equal to $x_1, x_2, \ldots, x_r$, where $n_i x_i = n_r x_r$ for all $1 \leqslant i < r$ and $\sum_{i=1}^{r} x_i = 0$. Since $x_i = n_i^{-1}(n_r x_r) \mod p$ for all $1 \leqslant i < r$ and $n_r \not\equiv 0 \pmod p$, it follows that $\Big(\sum_{i=1}^{r} n_i^{-1}\Big) x_r = 0$. Thus, if $\sum_{i=1}^{r} n_i^{-1} \not\equiv 0 \pmod p$, then $x_r = 0$ which implies that $x = 0$. If $\sum_{i-1}^{r} n_i^{-1} \equiv 0 \pmod p$, then x can be written as $x_r u$ where

$$u = \Big(\underbrace{n_1^{-1}n_r, \ldots, n_1^{-1}n_r}_{n_1 \text{ terms}}, \underbrace{n_2^{-1}n_r \ldots, n_2^{-1}n_r}_{n_2 \text{ terms}}, \ldots, \underbrace{n_{r-1}^{-1}n_r, \ldots, n_{r-1}^{-1}n_r}_{n_{r-1} \text{ terms}}, \underbrace{1, 1, \ldots, 1}_{n_r \text{ terms}}\Big).$$

This shows that $\mathrm{Hull}(C_p(A))$ is $\mathrm{Span}\big(u\big)$ if $\sum_{i=1}^{r} n_i^{-1} \equiv 0 \pmod p$, and $\{0\}$ otherwise. □

Proposition 10 *Let A be an adjacency matrix for $K_{n_1,n_2,\ldots,n_r}$ with $r \equiv 1 \pmod p$. If $n_{i_1}, n_{i_2}, \ldots, n_{i_m}$ are all cardinalities of some parts of the vertex set such that $n_{i_j} \equiv 0 \pmod p$ for all $1 \leqslant j \leqslant m$, where $1 \leqslant m < r$, then*

(i) $\dim\big(\mathrm{Hull}(C_p(A))\big) = m - 1$*;*

(ii) *if $m \geqslant 2$, then the set $\big\{v^{V_{i_j}} - v^{V_{i_m}} \mid 1 \leqslant j < m\big\}$ is a basis for $\mathrm{Hull}(C_p(A))$ and any vector in $\mathrm{Hull}(C_p(A))$ is of the form $\sum_{j=1}^{m} \alpha_j v^{V_{i_j}}$, where $\alpha_j \in \mathbb{F}_p$ for $1 \leqslant j < m$ and $\alpha_m = -\sum_{j=1}^{m-1} \alpha_j$.*

Proof: We relabel the cardinalities n_i's so that $n_i \equiv 0 \pmod p$ for all $1 \leqslant i \leqslant m$. Now let $x \in \mathrm{Hull}(C_p(A))$. We again assume by Lemma 2 that the coordinates of x corresponding to the vertices in the same part of the vertex set are equal to $x_1, x_2, \ldots, x_r$, where $n_i x_i = n_r x_r$ for all $1 \leqslant i < r$ and $\sum_{i=1}^{r} x_i = 0$. Note that as $n_1 \equiv 0 \pmod p$, we obtain

$$n_r x_r = n_1 x_1 = 0 \mod p.$$

Since $n_r \not\equiv 0 \pmod p$, it follows that $x_r = 0$, and thus $x_{m+1} = \cdots = x_{r-1} = 0$. Therefore, if $m = 1$, then $x_i = 0$ for $2 \leqslant i < r$ so that $x_1 = 0$, and thus $x = 0$. In this case, $\mathrm{Hull}(C_p(A)) = \{0\}$.

Now we assume that $m \geqslant 2$. Then

$$x_m = -x_1 - x_2 - \cdots - x_{m-1},$$

and hence x can be written as

$$\begin{aligned} x &= x_1 v^{V_1} + x_2 v^{V_2} + \cdots + x_{m-1} v^{V_{m-1}} \\ &\qquad + (-x_1 - x_2 - \cdots - x_{m-1}) v^{V_m} \\ &= x_1 (v^{V_1} - v^{V_m}) + x_2 (v^{V_2} - v^{V_m}) \\ &\qquad + \cdots + x_{m-1} (v^{V_{m-1}} - v^{V_m}). \end{aligned} \tag{7}$$

Note that since $v^{V_i} - v^{V_m} = v^{B_m} - v^{B_i}$ for all $1 \leqslant i < m$, the vectors $\big(v^{V_i} - v^{V_m}\big)$'s are in the code $C_p(A)$. Those vectors are also in $C_p(A)^{\perp}$ as they are orthogonal to the vectors v^{B_k} for all $1 \leqslant k \leqslant r$. Furthermore, Eq. (7) shows that the set S of those vectors spans

$\mathrm{Hull}(C_p(A))$, and it can be shown that S is linearly independent. Therefore, S is a basis for $\mathrm{Hull}(C_p(A))$, giving $\dim\big(\mathrm{Hull}(C_p(A))\big) = m-1$. □

From Proposition 10, part (ii), we obtain the following.

Corollary 7 *Let A be an adjacency matrix for the graph $K_{n_1,n_2,\ldots,n_r}$ with $r \equiv 1 \pmod p$. If $n_{i_1}, n_{i_2}, \ldots, n_{i_m}$ are all cardinalities of some parts of the vertex set such that $n_{i_j} \equiv 0 \pmod p$ for all $1 \leqslant j \leqslant m$, where $2 \leqslant m < r$, and $n_{i_1} \leqslant n_{i_2} \leqslant \cdots \leqslant n_{i_m}$, then*
(i) *the minimum weight of $\mathrm{Hull}(C_p(A))$ is $n_{i_1} + n_{i_2}$;*
(ii) *all the vectors of minimum weight in $\mathrm{Hull}(C_p(A))$ are of the form $\alpha\big(v^{V_{i_1}} - v^{V_{i_j}}\big)$ for all $\alpha \in \mathbb{F}_p \backslash \{0\}$ and all j such that $1 < j \leqslant m$ and $n_{i_j} = n_{i_2}$.*

Proof: The vector $v^{V_{i_j}} - v^{V_{i_k}}$ of weight $n_{i_j} + n_{i_k}$ is in $\mathrm{Hull}(C_p(A))$ for any j and k such that $1 \leqslant j,k \leqslant m$ and $j \neq k$, and every vector of the form v^{V_i} for all $1 \leqslant i \leqslant r$ is not in the code $C_p(A)$. □

We summarize in the case of $r \equiv 1 \pmod p$ the structure of the hull of the code from $K_{n_1,n_2,\ldots,n_r}$.

Corollary 8 *Let A be an adjacency matrix for $K_{n_1,n_2,\ldots,n_r}$ with $r \equiv 1 \pmod p$.*
(i) *If $n_i \not\equiv 0 \pmod p$ for all $1 \leqslant i \leqslant r$ and $\sum_{i=1}^{r} n_i^{-1} \not\equiv 0 \pmod p$, then $C_p(A)$ is an LCD code.*
(ii) *If $n_i \not\equiv 0 \pmod p$ for all $1 \leqslant i \leqslant r$ and $\sum_{i=1}^{r} n_i^{-1} \equiv 0 \pmod p$, then $\mathrm{Hull}(C_p(A))$ is an $[n,1,n]$ code where $n = \sum_{i=1}^{r} n_i$.*
(iii) *If there exists only one n_i such that $n_i \equiv 0 \pmod p$, then $C_p(A)$ is an LCD code.*
(iv) *If $n_{i_1}, n_{i_2}, \ldots, n_{i_m}$ are all cardinalities of some parts of the vertex set such that $n_{i_j} \equiv 0 \pmod p$ for all $1 \leqslant j \leqslant m$, where $2 \leqslant m < r$ and $n_{i_1} \leqslant n_{i_2} \leqslant \cdots \leqslant n_{i_m}$, then $\mathrm{Hull}(C_p(A))$ is an $[n, m-1, n_{i_1} + n_{i_2}]$ code.*

We now examine further in the case where $r \not\equiv 1 \pmod p$.

Lemma 3 *Let A be an adjacency matrix for $K_{n_1,n_2,\ldots,n_r}$ with $r \not\equiv 1 \pmod p$. Then any vector x in $\mathrm{Hull}(C_p(A))$ is of the form*

$$x = \big(\underbrace{x_1,\ldots,x_1}_{n_1\ terms}, \underbrace{x_2\ldots,x_2}_{n_2\ terms}, \ldots, \underbrace{x_r,\ldots,x_r}_{n_r\ terms}\big)$$

where $x_i \in \mathbb{F}_p$ and $n_i x_i = 0 \mod p$ for all $1 \leqslant i \leqslant r$.

Proof: Let $x \in \mathrm{Hull}(C_p(A))$. Then by Corollary 5, part (i), we have $(x, v^{V_i}) = 0$ or

$$\sum_{j=1}^{n_i} x_{i,j} = 0 \mod p \qquad \text{for all } 1 \leqslant i \leqslant r, \tag{8}$$

and by Proposition 3, $x_{i,j} = x_{i,n_i}$ for all $t < i \leqslant r$ and $1 \leqslant j < n_i$, where t is the number of n_i's equal to 1. So the coordinates of x corresponding to the vertices in the same part of the vertex set are equal, and we assume that they are equal to x_i in $\mathbb{F}_p$ for $t < i \leqslant r$. By (8), we obtain

$$x_1 = x_2 = \cdots = x_t = n_{t+1}x_{t+1} = \cdots = n_r x_r = 0 \bmod p.$$

□

Proposition 11 *Let A be an adjacency matrix for $K_{n_1,n_2,\ldots,n_r}$ with $r \not\equiv 1 \pmod p$.*
(i) *If $n_i \not\equiv 0 \pmod p$ for all $1 \leqslant i \leqslant r$, then $C_p(A)$ is an LCD code.*
(ii) *If $n_{i_1}, n_{i_2}, \ldots, n_{i_m}$ are all cardinalities of some parts of the vertex set such that $n_{i_j} \equiv 0 \pmod p$ for all $1 \leqslant j \leqslant m$, where $1 \leqslant m < r$ and $n_{i_1} \leqslant n_{i_2} \leqslant \cdots \leqslant n_{i_m}$, then $\mathrm{Hull}(C_p(A))$ is an $[n, m, n_{i_1}]$ code with a basis $\{v^{V_{i_j}} \mid 1 \leqslant j \leqslant m\}$. Further, all the vectors of minimum weight in $\mathrm{Hull}(C_p(A))$ are of the form $\alpha v^{V_{i_j}}$ where $\alpha \in \mathbb{F}_p \backslash \{0\}$ and $1 \leqslant j \leqslant m$ such that $n_{i_j} = n_{i_1}$.*

Proof: Let $x \in \mathrm{Hull}(C_p(A))$. By Lemma 3, $n_i x_i = 0$ for all $1 \leqslant i \leqslant r$, where x_i is the coordinates of x corresponding to the vertices in the i^{th} part of the vertex set for all $1 \leqslant i \leqslant r$. If $n_i \not\equiv 0 \pmod p$ for all $1 \leqslant i \leqslant r$, then $x_i = 0$ for all $1 \leqslant i \leqslant r$, giving $x = 0$. Thus, in this case, $\mathrm{Hull}(C_p(A)) = \{0\}$.

Assume that $n_{i_j} \equiv 0 \pmod p$ for all $1 \leqslant j \leqslant m$ where $1 \leqslant m < r$. Then $x_k = 0$ for all $k \notin \{i_1, i_2, \ldots, i_m\}$ so that x can be in the form of $x = x_{i_1} v^{V_{i_1}} + x_{i_2} v^{V_{i_2}} + \cdots + x_{i_m} v^{V_{i_m}}$. Note that for each $1 \leqslant j \leqslant m$, we have $\big(v^{V_k}, v^{V_{i_j}}\big) = 0$ for all $1 \leqslant k \leqslant r$. So the vectors $v^{V_{i_j}}$'s are in $C_p(A)^{\perp}$. By Corollary 5, part (i), they are in $\mathrm{Hull}(C_p(A))$ and hence form a basis for $\mathrm{Hull}(C_p(A))$. □

CONCLUSION

This research examines linear codes obtained from the span of adjacency matrices of connected graphs, the complete multipartite graphs. The structure of those codes and their duals including the hulls has shown to be related to the properties of the graphs. Although the dual codes have low minimum weight, their structure can be exploited to determine the minimum weight and establish bases of minimum-weight vectors for the codes and the hulls.

Acknowledgements: The authors would like to thank the anonymous referee for their helpful comments and suggestions. They also acknowledges partial support from the Faculty of Science, Silpakorn University.

REFERENCES

1. Key JD, Seneviratne P (2007) Codes from the line graphs of complete multipartite graphs and PD-sets. *Discrete Math* **307**, 2217–2225.
2. Seneviratne P (2015) Codes from multipartite graphs and minimal permutation decoding sets. *Discrete Math Algorithm Appl* **7**, 1550060.
3. Assmus EF, Key JD (1992) *Designs and Their Codes*, Cambridge Tracts in Mathematics 103, Cambridge University Press, Cambridge.
4. Godsil C, Royle G (2001) *Algebraic Graph Theory*, Graduate Texts in Mathematics 207, Springer, New York.
5. Brouwer AE, van Eijl CA (1992) On the p-rank of the adjacency matrices of strongly regular graphs. *J Algebraic Combin* **1**, 329–346.