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Uniqueness of meromorphic functions and their differential-difference polynomials with shared small functions

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ABSTRACT: In this paper, we study the unicity of meromorphic functions and their differential-difference polynomials. Our results improve some results due to Chen-Yi [Results Math **63** (2013):557–565], Chen-Xu [Open Math **18** (2020):211–215], Banerjee-Maity [Bull Korean Math Soc **58** (2021):1175–1192], and Narasimha-Shilpa [Adv Pure Appl Math **13** (2022):53–61].

KEYWORDS: meromorphic functions, differential-difference polynomials, small functions, partially sharing

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INTRODUCTION AND MAIN RESULTS

In this paper, we assume that the reader is familiar with the basic notions of Nevanlinna's value distribution theory, see [1–4]. In the following, a meromorphic function always means meromorphic in the whole complex plane. By $S(r,f)$, we denote any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$ possible outside of an exceptional set E with finite measure. We say that two nonconstant meromorphic functions f and g share small function a CM(IM), if $f-a$ and $g-a$ have the same zeros counting multiplicities (ignoring multiplicities).

Denote the set of all zeros of $f-a$ by $E(a,f)$, where a zero with multiplicity m is counted m times. If $E(a,f)\subset E(a,g)$ $\left(\overline{E}(a,f)\subset\overline{E}(a,g)\right)$, then we say f and g partially share the value a CM(IM). Note that $E(a,f)=E(a,g)$ $\left(\overline{E}(a,f)=\overline{E}(a,g)\right)$ is equal to f and g share a CM(IM). Therefore, it is clear that the condition "partially shared value CM(IM)" is more general than the condition "shared value CM(IM)".

Let $f(z)$ be a nonconstant meromorphic function. Define

$$\rho(f)=\overline{\lim_{r\to\infty}}\frac{\log^+T(r,f)}{\log r},$$
$$\mu(f)=\underline{\lim_{r\to\infty}}\frac{\log^+T(r,f)}{\log r},$$
$$\rho_2(f)=\overline{\lim_{r\to\infty}}\frac{\log^+\log^+T(r,f)}{\log r},$$
$$\lambda(f)=\overline{\lim_{r\to\infty}}\frac{\log^+N\left(r,\frac{1}{f}\right)}{\log r},$$

by the order, lower order, the hyper-order of $f(z)$, and the exponent of convergence of zeros for $f(z)$, respectively.

Let $f(z)$ be a meromorphic function satisfying $\rho(f)=\mu(f)$, then $f(z)$ is called a function with regular growth.

Let $f(z)$ be a nonconstant meromorphic function and let a be a complex number. We define

$$\delta(a,f)=\underline{\lim_{r\to\infty}}\frac{m\left(r,\frac{1}{f-a}\right)}{T(r,f)}=1-\overline{\lim_{r\to\infty}}\frac{N\left(r,\frac{1}{f-a}\right)}{T(r,f)},$$
$$\Theta(a,f)=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}\left(r,\frac{1}{f-a}\right)}{T(r,f)}.$$

It is clear that $0\leqslant\delta(a,f)\leqslant 1$, $0\leqslant\Theta(a,f)\leqslant 1$. If $\delta(a,f)>0$, then a is called a deficient value of f or a Nevanlinna exceptional value of f.

Let $f(z)$ be a nonconstant meromorphic function. If

$$\overline{\lim_{r\to\infty}}\frac{\log^+N\left(r,\frac{1}{f-a}\right)}{\log r}<\rho(f),$$

for $\rho(f)>0$; and $N\left(r,\frac{1}{f-a}\right)=O(\log r)$ for $\rho(f)=0$, then a is called a Borel exceptional function of f. If a is a constant, then a is called a Borel exceptional value of f.

We say that a is a small function of f if $T(r,a)=S(r,f)$, and $\hat{S}(f)$ means $S(f)\cup\{\infty\}$, where $S(f)$ is the set of all small functions of f.

Let $f(z)$ be a meromorphic function, and let c be a nonzero finite complex number. We define the difference operators of $f(z)$ as $\Delta_c f(z)=f(z+c)-f(z)$ and $\Delta_c^n f(z)=\Delta_c(\Delta_c^{n-1}f(z))$, $n\geqslant 2$. In particular, for $c=1$, we denote $\Delta_c^n f(z)$ by $\Delta^n f(z)$.

We define the linear difference polynomial of f as follows:

$$L(f):=\sum_{i=1}^{n}m_i(z)f(z+c_i), \qquad (1)$$

where $m_i(z)(\not\equiv 0)$ $(i=1,2,\ldots,n)$ are small functions of f, and c_i $(i=1,2,\ldots,n)$ are distinct finite values.

Let $H(f)=H(f(z),f(z+c_1),\ldots,f(z+c_n))$ be a homogeneous difference polynomial of f with degree $m\geqslant 2$, where c_i $(i=1,2,\ldots,n)$ are distinct finite values, and coefficients $m_i(z)$ $(i=1,2,\ldots,n)$ are small functions of f.

Define

$$\psi(f):=\sum_{j_1\in J_1}A_{j_1}(z)f^{(k_{j_1})}(z)+\sum_{j_2\in J_2}B_{j_2}(z)f^{(k_{j_2})}(z+b_{j_2})+\sum_{j_3\in J_3}C_{j_3}(z)f(z+c_{j_3}), \quad (2)$$

where $A_{j_1}(z)$, $B_{j_2}(z)$, $C_{j_3}(z)$ are entire small functions of $f(z)$, $\{k_{j_1},k_{j_2}\}\in\mathbb{Z}^+$, b_{j_2},c_{j_3} are complex constants and $j_m\in J_m$, $m=\{1,2,3\}$ are finite indexed sets.

We define the differential-difference polynomial of f as follows:

$$W(f):=\sum_{j\in J}A_j(z)f^{(k_j)}(z+a_j), \quad (3)$$

where $A_j(z)$ are small functions of $f(z)$, k_j are non-negative integers, a_j are complex constants which satisfying (a_j,k_j) are distinct for each $j\in J$, where J is a finite indexed set.

Nevanlinna [3] proved the following famous five-value theorem.

Theorem A *Let f and g be two nonconstant meromorphic functions, and let a_j $(j=1,2,3,4,5)$ be five distinct values in the extended complex plane. If f and g share a_j $(j=1,2,3,4,5)$ IM, then $f\equiv g$.*

Li and Qiao [5] improved Theorem A as follows:

Theorem B *Let f and g be two nonconstant meromorphic functions, and let a_j $(j=1,2,3,4,5)$ (one of them can be identically infinite) be five distinct small functions of both f and g. If f and g share a_j $(j=1,2,3,4,5)$ IM, then $f\equiv g$.*

In 2013, Chen and Yi [6] proved the following result.

Theorem C *Let f be a transcendental meromorphic function such that $\rho(f)$ is not an integer or infinite. If $\Delta f(\not\equiv 0)$ and f share three distinct values a,b,∞ CM, then $\Delta f\equiv f$.*

In this paper, we extend Theorem C as follows:

Theorem 1 *Let f be a nonconstant meromorphic function such that $\rho(f)$ is not an integer or infinite, let a,b be two distinct small functions related to f, and let $L(f)$ be a linear difference polynomial of the form (1). If f and $L(f)$ share a,b,∞ CM, then $f\equiv L(f)$.*

In 2020, Chen [7] proved

Theorem D *Let $f(z)$ be a transcendental entire function with $\rho_2(f)<1$, and let $c\in\mathbb{C}\backslash\{0\}$ such that $\Delta_c^n f(z)\not\equiv 0$. If $f(z)$ and $\Delta_c^n f(z)$ share 0 CM and 1 IM, then $\Delta_c^n f(z)\equiv f(z)$.*

We extend Theorem D and prove the following result.

Theorem 2 *Let f be a nonconstant meromorphic function with $\rho_2(f)<1$ and $\overline{N}(r,f)=S(r,f)$, and let $a(\not\equiv 0)$ be a small function related to f. If f and $L(f)$ share a IM and $E(0,f)\subset E(0,L(f))$, $E(\infty,f)\supset E(\infty,L(f))$, then $f\equiv L(f)$.*

Corollary 1 *Let f be a nonconstant meromorphic function with $\rho_2(f)<1$ and $\overline{N}(r,f)=S(r,f)$, and let $a(\not\equiv 0)$ be a small function related to f. If f^m and $H(f)$ share a IM and $E(0,f^m)\subset E(0,H(f))$, $E(\infty,f^m)\supset E(\infty,H(f))$, then $f^m\equiv H(f)$.*

In 2021, Banerjee and Maity [8] proved the following result.

Theorem E *Let f be a nonconstant entire function with $\rho_2(f)<1$ and let $L_cf=\sum_{l=0}^{k}b_lf(z+lc)$, where $b_l\in\mathbb{C}$ and $b_k\neq 0$. For $c\in\mathbb{C}\backslash\{0\}$, let $a_j\in\widehat{S}_f$ $(j=1,2,3)$ be three distinct nonzero periodic functions with period c. If $L_cf\not\equiv 0$, $\overline{E}(a_j,f)\subseteq\overline{E}(a_j,L_cf)$ $(j=1,2,3)$ and $\delta(0,f)>0$, then $f\equiv L_cf$.*

In this paper, we remove the condition "a_j $(j=1,2,3)$ are periodic functions" and extend L_cf to $L(f)$.

Theorem 3 *Let f be a nonconstant meromorphic function with $\rho_2(f)<1$, let $c\in\mathbb{C}\backslash\{0\}$, and let $a_j\in\widehat{S}_f$ $(j=1,2,3)$ be three distinct nonzero small functions. If $L(f)\not\equiv 0$, $\overline{E}(a_j,f)\subseteq\overline{E}(a_j,L(f))$ $(j=1,2,3)$, $\delta(0,f)>0$ and $\delta(\infty,f)=1$, then $f\equiv L(f)$.*

In 2022, Narasimha and Shilpa [9] proved the following theorem.

Theorem F *Let f be a transcendental entire function of finite order and let $\psi(f)$ be defined as (2) such that $\sum_{j_3\in J_3}C_{j_3}\equiv 0$. Suppose that $\psi(f)$ and f share the finite value a CM and f has an exceptional value $\alpha(\neq a)$.*

(i) *If $a\neq 0$ and α is a Nevanlinna exceptional value of f, then*

$$\frac{\psi(f)-a}{f-a}=\tau(\neq 0).$$

(ii) *If α is a Borel exceptional value of f, then*

$$\frac{\psi(f)-a}{f-a}=\frac{a}{a-\alpha}.$$

In this paper, we extend Theorem F as follows:

Theorem 4 *Let f be a nonconstant meromorphic function with $\rho_2(f) < 1$, let a, α be two distinct small functions related to f, and let $W(f)$ be a differential-difference polynomial with $a \not\equiv W(\alpha)$. If f and $W(f)$ share a, ∞ CM, and α is a Nevanlinna exceptional small function of f, then*

$$\frac{W(f)-a}{f-a} = \tau(\neq 0).$$

The following example shows that the conditions "$a \not\equiv \alpha$" and "$a \not\equiv W(\alpha)$" are necessary in Theorem 4.

Example 1 Let $f(z) = \dfrac{e^{z^2}}{e^z+1} + 1$, and let $W(z,f) = f(z+2\pi i) = \dfrac{e^{z^2+4\pi iz-4\pi^2}}{e^z+1} + 1$. Then, we have f and $W(z,f)$ share $1, \infty$ CM, but

$$\frac{W(f)-a}{f-a} = \frac{\dfrac{e^{z^2+4\pi iz-4\pi^2}}{e^z+1}}{\dfrac{e^{z^2}}{e^z+1}} = e^{4\pi iz-4\pi^2}.$$

Theorem 5 *Let f and $W(f)$ be two nonconstant meromorphic functions of finite order, and let a, α be two distinct small functions related to f. If f and $W(f)$ share a IM, and α, ∞ are two Borel exceptional functions of f, then*

$$\frac{f-\alpha}{a-\alpha} \equiv \frac{W(f)-W(\alpha)}{a-W(\alpha)}.$$

By Theorem 5, we have the following corollary.

Corollary 2 *Let f be a transcendental entire function of finite order and $\psi(f)$ be defined as (2) such that $\sum\limits_{j_3 \in J_3} C_{j_3} \equiv 0$. Suppose that $\psi(f)$ and f share the finite value a IM and $\alpha(\neq a)$ is a Borel exceptional value of f, then*

$$\frac{\psi(f)-a}{f-a} = \frac{a}{a-\alpha}.$$

Remark 1 We change the condition "share a CM" of the second case in Theorem F to "share a IM".

LEMMAS

For the proof of our results, we need the following lemmas.

Lemma 1 ([1, 3, 4]) *Let f be a nonconstant meromorphic function of finite order and let $c \in \mathbb{C}\backslash\{0\}$, then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r,f).$$

If $\rho_2(f) = \rho_2 < 1$ and $\varepsilon > 0$, then

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\rho_2-\varepsilon}}\right).$$

Lemma 2 ([3]) *Suppose f is a nonconstant meromorphic function. Then the value a such that $\Theta(a,f) > 0$ are at most countable many and*

$$\sum_a \Theta(a,f) \leqslant 2.$$

Lemma 3 ([10]) *Let f be a meromorphic function of finite order, and let a be a small function of f. If $\sum\limits_{a\neq\infty} \delta(a,f) = 1$ and $\delta(\infty,f) = 1$, then f is of regular growth and $\rho(f)$ is a positive integer.*

Lemma 4 ([11]) *Let f be a nonconstant meromorphic function, and let a_i $(i = 1,2,3)$ be three distinct small functions of f. Then for any $0 < \varepsilon < 1$, we have*

$$2T(r,f) \leqslant \overline{N}(r,f) + \sum_{i=1}^{3} \overline{N}\left(r, \frac{1}{f-a_i}\right) + \varepsilon T(r,f) + S(r,f).$$

Lemma 5 ([3]) *Let f be a transcendental meromorphic function of finite order. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f).$$

Lemma 6 ([3]) *Let f be a meromorphic function with a positive order. If f has two distinct Borel exceptional values a_1 and a_2, then $\delta(a_1,f) = \delta(a_2,f) = 1$.*

Remark 2 Lemma 6 is also valid for $\rho(f) = 0$.

Lemma 7 ([12]) *Let f be a nonconstant meromorphic function of finite order. Then we have*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r),$$

and for each $\varepsilon > 0$, we have

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = O(r^{\rho(f)-1+\varepsilon}).$$

Lemma 8 ([13]) *Let f and g be two distinct meromorphic functions satisfying*

$$N(r,f) + N\left(r, \frac{1}{f}\right) = S(r,f),$$

$$N(r,g) + N\left(r, \frac{1}{g}\right) = S(r,g).$$

If f and g share 1 IM almost, then $f \equiv g$ or $fg \equiv 1$.

PROOF OF Theorem 1

Since f and $L(f)$ share a, b, ∞ CM, we can find two meromorphic functions H_1 and H_2 such that

$$\frac{L(f)-a}{f-a} = H_1, \quad \frac{L(f)-b}{f-b} = H_2, \tag{4}$$

where $\delta(0,H_1)=\delta(\infty,H_1)=1$ and $\delta(0,H_2)=\delta(\infty,H_2)=1$.

Obviously, by Lemma 3, we have $\rho(H_1)=k_1$ and $\rho(H_2)=k_2$, where k_1 and k_2 are positive integers.

By Lemma 3 and the definition of the order and the lower order of f, there exists a positive number ε_0 such that

$$r^{k_1-\varepsilon_0} \leqslant T(r,H_1) \leqslant r^{k_1+\varepsilon_0}, \tag{5}$$

$$T(r,H_2) \leqslant r^{k_2+\varepsilon_0}. \tag{6}$$

Next we consider the following two cases.

Case 1: $H_1 \equiv H_2$. From (4), we obtain the result of Theorem 1.

Case 2: $H_1 \not\equiv H_2$. By (4), we get

$$f=\frac{a(H_1-1)+b(1-H_2)}{H_1-H_2}. \tag{7}$$

Case 2.1: $k_1=k_2=k$.

From (5)–(7), we get

$$\begin{aligned} T(r,f) &= T\left(r,\frac{a(H_1-1)+b(1-H_2)}{H_1-H_2}\right)\\ &\leqslant 2T(r,H_1)+2T(r,H_2)+S(r,f)\\ &\leqslant 2r^{k+\varepsilon_0}+2r^{k+\varepsilon_0}+S(r,f)\\ &= 4r^{k+\varepsilon_0}+S(r,f). \end{aligned} \tag{8}$$

By (4) and Lemma 1, we obtain

$$\begin{aligned} T(r,H_1) &= T\left(r,\frac{L(f)-a}{f-a}\right)\\ &= m\left(r,\frac{L(f)-a}{f-a}\right)+N\left(r,\frac{L(f)-a}{f-a}\right)\\ &\leqslant m\left(r,\frac{1}{f-a}\right)+S(r,f)\\ &\leqslant T(r,f)+S(r,f). \end{aligned} \tag{9}$$

From (5), (8) and (9), we have

$$r^{k-\varepsilon_0} \leqslant T(r,f) \leqslant 4r^{k+\varepsilon_0}.$$

Obviously, $\rho(f)$ is an integer, a contradiction.

Case 2.2: $k_1 \neq k_2$. Without loss of generality, we assume that $k_1>k_2$.

By Lemma 3, we obtain $T(r,H_2)=S(r,H_1)$.

From (5) and (7), we get

$$\begin{aligned} T(r,f) &= T\left(r,\frac{a(H_1-1)+b(1-H_2)}{H_1-H_2}\right)\\ &\leqslant 2T(r,H_1)+S(r,f)\\ &\leqslant 2r^{k_1+\varepsilon_0}+S(r,f). \end{aligned} \tag{10}$$

Combing with (5), (9) and (10), we have

$$r^{k_1-\varepsilon_0} \leqslant T(r,f) \leqslant 2r^{k_1+\varepsilon_0}.$$

Hence, $\rho(f)$ is an integer, a contradiction.

This completes the proof of Theorem 1.

PROOF OF Theorem 2

Firstly, we consider the case that f is a nonconstant rational function. Obviously, $a, m_1, m_2, \dots m_n$ are constants. By

$$E(0,f) \subset E(0,L(f)),$$
$$E(\infty,f) \supset E(\infty,L(f)),$$

we get

$$\frac{L(f)}{f}=h, \tag{11}$$

where h is an entire function.

From (11), we have

$$\begin{aligned} \lim_{z\to\infty} h(z) &= \lim_{z\to\infty}\frac{\sum_{i=1}^{n} m_i(z)f(z+c_i)}{f(z)}\\ &= m_1+m_2+\cdots+m_n. \end{aligned}$$

Let $A=m_1+m_2+\cdots+m_n$. So we have $L(f)\equiv Af$.

Next we consider two cases.

Case 1: $A=0$. So we have $L(f)\equiv 0$. Since a is a nonzero constant, f and $L(f)$ share a IM, so f can be written as $f=a+\frac{1}{P}$, where P is a polynomial with $\deg(P)=p_1\geqslant 1$. Hence, we have

$$T(r,f)=p_1\log r+O(1),$$

and

$$\overline{N}(r,f)\geqslant \log r,$$

a contradiction.

Case 2: $A\neq 0$. We consider the following two subcases.

Case 2.1: $A=1$. It follows that $f\equiv L(f)$.

Case 2.2: $A\neq 1$.

Since f and $L(f)$ share a IM, we have $f\neq a$ and $L(f)\neq a$. It follows that $f\neq \frac{a}{A}$, a contradiction.

Therefore, we deduce $f\equiv L(f)$ in this case.

Next, we consider the case that f is a transcendental meromorphic function.

Since h is an entire function and by Lemma 1, we have

$$T(r,h)=m\left(r,\frac{\sum_{i=1}^{n} m_i(z)f(z+c_i)}{f(z)}\right)=S(r,f).$$

From (11) and Nevanlinna's first fundamental theorem, we have

$$\begin{aligned} T(r,L(f)) &\leqslant T(r,h)+T(r,f)\\ &\leqslant T(r,f)+S(r,f),\\ T(r,f) &\leqslant T(r,L(f))+T\left(r,\frac{1}{h}\right)\\ &\leqslant T(r,L(f))+S(r,f). \end{aligned}$$

Thus, we obtain

$$S(r,f)=S(r,L(f)). \tag{12}$$

If $h\equiv 1$, then by (11), we obtain the result of Theorem 2.

If $h\not\equiv 1$, then by f and $L(f)$ share a IM, we get

$$\overline{N}\left(r,\frac{1}{f-a}\right)=\overline{N}\left(r,\frac{1}{L(f)-a}\right)\leqslant N\left(r,\frac{1}{h-1}\right)\leqslant T(r,h)+S(r,f)=S(r,f). \tag{13}$$

It follows that

$$\overline{N}\left(r,\frac{1}{f-\frac{a}{h}}\right)=\overline{N}\left(r,\frac{1}{L(f)-a}\right)=S(r,f). \tag{14}$$

From (13), (14) and Nevanlinna's second fundamental theorem, we get

$$T(r,f)\leqslant \overline{N}(r,f)+\overline{N}\left(r,\tfrac{1}{f-a}\right)+\overline{N}\left(r,\tfrac{1}{f-\frac{a}{h}}\right)+S(r,f)\leqslant S(r,f), \tag{15}$$

a contradiction.

Therefore, we have $f\equiv L(f)$. This completes the proof of Theorem 2.

PROOF OF Corollary 1

Under the assumptions of Corollary 1, f is transcendental. Since

$$E(0,f^m)\subset E(0,H(f)),\quad E(\infty,f^m)\supset E(\infty,H(f)),$$

we get

$$\frac{H(f)}{f^m}=q, \tag{16}$$

where q is an entire function. By Lemma 1, we have

$$T(r,q)=m(r,q)+N(r,q)=m\left(r,\frac{H(f)}{f^m}\right)=S(r,f).$$

From (16) and Nevanlinna's first fundamental theorem, we get

$$T(r,H(f))\leqslant T(r,q)+T(r,f^m)\leqslant T(r,f^m)+S(r,f),$$
$$T(r,f^m)\leqslant T(r,H(f))+T\left(r,\tfrac{1}{q}\right)\leqslant T(r,H(f))+S(r,f).$$

Thus, we have

$$S(r,f^m)=S(r,H(f)). \tag{17}$$

If $q\equiv 1$, then by (16), we obtain the result of Corollary 1. If $q\not\equiv 1$, then by f^m and $H(f)$ share a IM, we get

$$\overline{N}\left(r,\frac{1}{f^m-a}\right)=\overline{N}\left(r,\frac{1}{H(f)-a}\right)\leqslant N\left(r,\frac{1}{q-1}\right)\leqslant T(r,q)+S(r,f)=S(r,f).$$

It follows that

$$\Theta(a,f^m)=1,\quad \Theta(a,H(f))=1. \tag{18}$$

So we have

$$\Theta\left(\tfrac{a}{q},f^m\right)=1. \tag{19}$$

Since $m\geqslant 2$, we get

$$\begin{aligned}\Theta(\infty,f^m)&=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,f^m)}{T(r,f^m)}\\&=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,f)}{mT(r,f)}\\&\geqslant 1-\overline{\lim_{r\to\infty}}\frac{T(r,f)}{mT(r,f)}\\&=1-\tfrac{1}{m}>0.\end{aligned} \tag{20}$$

Combing with (18)–(20) and Lemma 2, we have

$$\Theta(a,f^m)+\Theta\left(\tfrac{a}{q},f^m\right)+\Theta(\infty,f^m)=3-\tfrac{1}{m}>2,$$

a contradiction.

Therefore, we have $f^m\equiv H(f)$. This completes the proof of Corollary 1.

PROOF OF Theorem 3

Set $G=\frac{L(f)}{f}$. If $G\equiv 1$, then $f\equiv L(f)$. In the following, we assume $G\not\equiv 1$.

From $\delta(\infty,f)=1$, we have $\delta(\infty,L(f))=1$. Next we consider two cases.

Case 1: One of a_1, a_2, and a_3 is infinity. Without loss of generality, we assume that $a_3\equiv\infty$.

By $\overline{E}(a_j,f)\subseteq\overline{E}(a_j,L(f))$ (for $j=1,2,3$) and Lemma 4, for any $0<\varepsilon<1$, we have

$$\begin{aligned}&2T(r,f)\\&\leqslant\sum_{j=1}^{2}\overline{N}\left(r,\tfrac{1}{f-a_j}\right)+\overline{N}(r,f)+\overline{N}\left(r,\tfrac{1}{f}\right)+\varepsilon T(r,f)+S(r,f)\\&\leqslant\overline{N}\left(r,\tfrac{1}{f-L(f)}\right)+\overline{N}\left(r,\tfrac{1}{f}\right)+\varepsilon T(r,f)+S(r,f)\\&\leqslant T(r,f-L(f))+\overline{N}\left(r,\tfrac{1}{f}\right)+\varepsilon T(r,f)+S(r,f)\\&=m(r,f-L(f))+\overline{N}\left(r,\tfrac{1}{f}\right)+\varepsilon T(r,f)+S(r,f)\\&\leqslant m(r,f)+\overline{N}\left(r,\tfrac{1}{f}\right)+\varepsilon T(r,f)+S(r,f)\\&\leqslant T(r,f)+\overline{N}\left(r,\tfrac{1}{f}\right)+\varepsilon T(r,f)+S(r,f).\end{aligned}$$

So we obtain

$$(1-\varepsilon)T(r,f)\leqslant\overline{N}\left(r,\tfrac{1}{f}\right)+S(r,f).$$

Hence, we have $\delta(0,f)=0$, a contradiction.

Case 2: $a_j\not\equiv\infty$, $(j=1,2,3)$.

By $\overline{E}(a_j,f) \subseteq \overline{E}(a_j,L(f))$ (for $j = 1,2,3$), $\delta(\infty,f)=1$ and Lemma 4, we get

$$\begin{aligned}2T(r,f) &\leqslant \sum_{j=1}^{3}\overline{N}\left(r,\tfrac{1}{f-a_j}\right)+\overline{N}\left(r,\tfrac{1}{f}\right)+\varepsilon T(r,f)+S(r,f)\\ &\leqslant N\left(r,\tfrac{1}{f-L(f)}\right)+N\left(r,\tfrac{1}{f}\right)+\varepsilon T(r,f)+S(r,f)\\ &\leqslant T(r,f-L(f))+N\left(r,\tfrac{1}{f}\right)+\varepsilon T(r,f)+S(r,f)\\ &= m(r,f-L(f))+N\left(r,\tfrac{1}{f}\right)+\varepsilon T(r,f)+S(r,f)\\ &\leqslant m(r,f)+N\left(r,\tfrac{1}{f}\right)+\varepsilon T(r,f)+S(r,f)\\ &\leqslant T(r,f)+N\left(r,\tfrac{1}{f}\right)+\varepsilon T(r,f)+S(r,f).\end{aligned}$$

So we have

$$(1-\varepsilon)T(r,f)\leqslant N\left(r,\tfrac{1}{f}\right)+S(r,f).$$

Hence, we have $\delta(0,f)=0$, a contradiction.

Therefore, we have $f\equiv L(f)$. This completes the proof of Theorem 3.

PROOF OF Theorem 4

Set

$$\frac{W(f)-a}{f-a}=\varphi, \tag{21}$$

where φ is a meromorphic function. Since f and $W(f)$ share a,∞ CM, we have

$$N(r,\varphi)=S(r,f), \qquad N\left(r,\tfrac{1}{\varphi}\right)=S(r,f).$$

It follows from (21) that

$$\frac{1}{a-W(\alpha)-(a-\alpha)\varphi}\left(\frac{W(f-\alpha)}{f-\alpha}-\varphi\right)=\frac{1}{f-\alpha}. \tag{22}$$

By Lemma 1, Lemma 5 and Nevanlinna's first fundamental theorem, we have

$$\begin{aligned}T(r,\varphi)&=m(r,\varphi)+N(r,\varphi)\\ &=m\left(r,\frac{W(f)-a}{f-a}\right)+S(r,f)\\ &\leqslant m\left(r,\frac{\sum_{j\in J}A_j(z)f^{(k_j)}(z+a_j)-a}{f-a}\right)+S(r,f)\\ &\leqslant m\left(r,\frac{\sum_{j\in J}A_j(z)\left[f^{(k_j)}(z+a_j)-a^{(k_j)}(z+a_j)\right]}{f-a}\right)\\ &\quad+m\left(r,\frac{\sum_{j\in J}A_j(z)a^{(k_j)}(z+a_j)-a}{f-a}\right)+S(r,f)\\ &\leqslant \sum_{j\in J}m(r,A_j(z))+m\left(r,\tfrac{1}{f-a}\right)+S(r,f)\\ &\quad+\sum_{j\in J}m\left(r,\frac{f^{(k_j)}(z+a_j)-a^{(k_j)}(z+a_j)}{f-a}\right)\\ &\leqslant m\left(r,\tfrac{1}{f-a}\right)+S(r,f)\\ &\leqslant T(r,f)+S(r,f).\end{aligned}$$

It follows

$$S(r,\varphi)=S(r,f). \tag{23}$$

Since α is a Nevanlinna exceptional small function of f, we deduce that

$$m\left(r,\tfrac{1}{f-\alpha}\right)\geqslant \gamma T(r,f),$$

for sufficiently large r, where γ is some positive constant. Then, by (22), we have

$$\begin{aligned}T(r,f)&\leqslant \frac{1}{\gamma}m\left(r,\frac{1}{f-\alpha}\right)\\ &\leqslant \frac{1}{\gamma}\left[m\left(r,\frac{1}{a-W(\alpha)-(a-\alpha)\varphi}\right)+m(r,\varphi)\right]+S(r,f)\\ &\leqslant \frac{2}{\gamma}T(r,\varphi)+S(r,f).\end{aligned}$$

It follows

$$S(r,f)=S(r,\varphi). \tag{24}$$

By (23), (24), $a\not\equiv W(\alpha)$ and Nevanlinna's second fundamental theorem, we have

$$\begin{aligned}&T(r,\varphi)\\ &\leqslant \overline{N}(r,\varphi)+\overline{N}\left(r,\frac{1}{\varphi}\right)+\overline{N}\left(r,\frac{1}{\varphi-\frac{a-W(\alpha)}{a-\alpha}}\right)+S(r,f)\\ &\leqslant \overline{N}\left(r,\frac{a-\alpha}{(a-\alpha)\varphi-a+W(\alpha)}\right)+S(r,f)\\ &\leqslant \overline{N}(r,a-\alpha)+\overline{N}\left(r,\frac{1}{(a-\alpha)\varphi-a+W(\alpha)}\right)\\ &\leqslant T(r,\varphi)+S(r,f).\end{aligned}$$

Thus, we have

$$N\left(r,\frac{1}{a-W(\alpha)-(a-\alpha)\varphi}\right)=T(r,\varphi)+S(r,f).$$

It follows

$$m\left(r,\frac{1}{a-W(\alpha)-(a-\alpha)\varphi}\right)=S(r,f). \tag{25}$$

By (25), we have

$$\begin{aligned}&m\left(r,\frac{\varphi}{a-W(\alpha)-(a-\alpha)\varphi}\right)\\ &=m\left(r,\frac{1}{\alpha-a}+\frac{W(\alpha)-a}{\alpha-a}\cdot\frac{1}{a-W(\alpha)-(a-\alpha)\varphi}\right)\\ &\leqslant m\left(r,\frac{1}{\alpha-a}\right)+m\left(r,\frac{W(\alpha)-a}{\alpha-a}\right)\\ &\quad+m\left(r,\frac{1}{a-W(\alpha)-(a-\alpha)\varphi}\right)+S(r,f)\\ &=S(r,f).\end{aligned} \tag{26}$$

It follows from (22), (25), (26), Lemma 1 and Lemma 5 that

$$\begin{aligned} m\left(r,\frac{1}{f-\alpha}\right) &= m\left(r,\frac{1}{a-W(\alpha)-(a-\alpha)\varphi}\left(\frac{W(f-\alpha)}{f-\alpha}-\varphi\right)\right) \\ &\leqslant m\left(r,\frac{1}{a-W(\alpha)-(a-\alpha)\varphi}\cdot\frac{W(f-\alpha)}{f-\alpha}\right) \\ &\quad + m\left(r,\frac{\varphi}{a-W(\alpha)-(a-\alpha)\varphi}\right)+S(r,f) \\ &\leqslant m\left(r,\frac{1}{a-W(\alpha)-(a-\alpha)\varphi}\right)+S(r,f) \\ &= S(r,f), \end{aligned} \tag{27}$$

which contradicts with α is a Nevanlinna exceptional small function of f. Hence, φ is a constant. That is,

$$\frac{W(f)-a}{f-a}=\tau.$$

Obviously, $\tau=\varphi\neq 0$.

This completes the proof of Theorem 4.

PROOF OF Theorem 5

Set

$$F=\frac{f-\alpha}{a-\alpha},\ G=\frac{W(f)-W(\alpha)}{a-W(\alpha)}. \tag{28}$$

Obviously, we have

$$T(r,F)=T(r,f)+S(r,f), \tag{29}$$

$$T(r,G)=T(r,W(f))+S(r,f), \tag{30}$$

$$N\left(r,\tfrac{1}{F}\right)=N\left(r,\tfrac{1}{f-\alpha}\right)+S(r,f), \tag{31}$$

$$N\left(r,\tfrac{1}{G}\right)=N\left(r,\tfrac{1}{W(f)-W(\alpha)}\right)+S(r,f). \tag{32}$$

Set

$$\overline{\lim_{r\to\infty}}\frac{\log^+ N\left(r,\frac{1}{f-\alpha}\right)}{\log r}=\lambda_1.$$

Since α is a Borel exceptional function of f, we get

$$N\left(r,\tfrac{1}{f-\alpha}\right)\leqslant r^{\frac{\lambda_1+\rho(f)}{2}}. \tag{33}$$

Set $\varepsilon=\frac{1}{2}$. By Lemma 7, then we have

$$S(r,f)=O\left(r^{M_1}\right), \tag{34}$$

where $M_1=\max\left\{\frac{\lambda_1+\rho(f)}{2},\rho(f)-\frac{1}{2}\right\}$. From (31), (33) and (34), we obtain

$$N\left(r,\tfrac{1}{F}\right)\leqslant r^{\frac{\lambda_1+\rho(f)}{2}}+O\left(r^{M_1}\right)\leqslant O\left(r^{M_1}\right).$$

It follows that

$$\overline{\lim_{r\to\infty}}\frac{\log^+ N\left(r,\frac{1}{F}\right)}{\log r}\leqslant M_1<\rho(f)=\rho(F). \tag{35}$$

Thus, 0 is a Borel exceptional value of F. Similarly, we deduce that ∞ is also a Borel exceptional value of F. By Lemma 5, we have

$$\begin{aligned} m\left(r,\frac{1}{f-\alpha}\right) &\leqslant m\left(r,\frac{W(f)-W(\alpha)}{f-\alpha}\right)+m\left(r,\frac{1}{W(f)-W(\alpha)}\right) \\ &\leqslant m\left(r,\frac{1}{W(f)-W(\alpha)}\right)+S(r,f). \end{aligned}$$

Combing with Nevanlinna's first fundamental theorem, we get

$$\begin{aligned} N\left(r,\frac{1}{W(f)-W(\alpha)}\right) &\leqslant N\left(r,\frac{1}{f-\alpha}\right)+T(r,W(f))-T(r,f)+S(r,f) \\ &\leqslant N\left(r,\frac{1}{f-\alpha}\right)+N(r,W(f))-N(r,f)+S(r,f) \\ &= N\left(r,\frac{1}{f-\alpha}\right)+O(N(r,f))+S(r,f). \end{aligned} \tag{36}$$

Set

$$\overline{\lim_{r\to\infty}}\frac{\log^+ N(r,f)}{\log r}=\lambda_2.$$

Since ∞ is a Borel exceptional value of f, we get

$$N(r,f)\leqslant r^{\frac{\lambda_2+\rho(f)}{2}}. \tag{37}$$

From (33), (34), (36) and (37), we obtain

$$N\left(r,\frac{1}{W(f)-W(\alpha)}\right)\leqslant r^{\frac{\lambda_1+\rho(f)}{2}}+O\left(r^{\frac{\lambda_2+\rho(f)}{2}}\right)+O\left(r^{M_1}\right)\leqslant O\left(r^{M_2}\right),$$

where $M_2=\max\left\{\frac{\lambda_1+\rho(f)}{2},\frac{\lambda_2+\rho(f)}{2},\rho(f)-\frac{1}{2}\right\}$. It follows that

$$\overline{\lim_{r\to\infty}}\frac{\log^+ N\left(r,\frac{1}{W(f)-W(\alpha)}\right)}{\log r}\leqslant M_2<\rho(f)=\rho(W(f)). \tag{38}$$

Hence, $W(\alpha)$ is a Borel exceptional function of $W(f)$. Thus, we deduce that both 0 and ∞ are Borel exceptional values of G.

Since f and $W(f)$ share a IM, we know that F and G share 1 IM almost.

From Lemma 6 and Lemma 8, we get $F\equiv G$ or $FG\equiv 1$.

If $F\equiv G$, then we obtain the result of Theorem 5.

If $FG\equiv 1$, then we have

$$\frac{f-\alpha}{a-\alpha}\cdot\frac{W(f)-W(\alpha)}{a-W(\alpha)}\equiv 1. \tag{39}$$

It follows that

$$\frac{W(f)-W(\alpha)}{f-\alpha}\cdot\frac{1}{(a-\alpha)(a-W(\alpha))}\equiv\frac{1}{(f-\alpha)^2}.$$

Thus, we get

$$m\left(r,\frac{1}{(f-\alpha)^2}\right)=S(r,f).$$

Hence, we have

$$m\left(r,\frac{1}{f-\alpha}\right)=S(r,f). \tag{40}$$

From (39), we have

$$(f-\alpha)(W(f)-W(\alpha))\equiv(a-\alpha)(a-W(\alpha)).$$

It follows that

$$N\left(r,\frac{1}{f-\alpha}\right)=S(r,f). \tag{41}$$

By (40) and (41), we obtain $T(r,f)=S(r,f)$, a contradiction.

This completes the proof of Theorem 5.

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