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Singular value inequalities on 2 × 2 block accretive partial transpose matrices

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ABSTRACT: A 2 × 2 block matrix $\begin{pmatrix} A & X \\ Y^* & B \end{pmatrix}$ is accretive partial transpose (APT) if both $\begin{pmatrix} A & X \\ Y^* & B \end{pmatrix}$ and $\begin{pmatrix} A & Y^* \\ X & B \end{pmatrix}$ are accretive. This article presents some singular value inequalities related to this class of matrices. Our results complement the presented inequality in [*Oper Matrices* **9** (2015):917–924].

KEYWORDS: accretive partial transpose matrices, positive semidefinite matrices, singular value inequalities

MSC2020: 47A63 15A45

INTRODUCTION

The space of $m \times n$ complex matrices is denoted by $\mathbb{M}_{m\times n}$. If $m = n$, we write $\mathbb{M}_n$ instead of $\mathbb{M}_{n\times n}$. If the matrix $A \in \mathbb{M}_n$ is positive semidefinite (resp., definite), then we write $A \geqslant 0$ (resp., $A > 0$). We denote by $\mathbb{M}_m(\mathbb{M}_n)$ the set of block matrices of order m with each block in $\mathbb{M}_n$. We say that the matrix $A \in \mathbb{M}_n$ is accretive if its real part $\mathrm{Re} A := \frac{A+A^*}{2}$ is positive definite, where A^* means the conjugate transpose of A. Clearly, the accretive matrices is a larger class of matrices than positive definite matrices. Accretive matrices have been the subject of a number of recent papers [1, 2]. For any complex matrix $A \geqslant 0$, there exists a unique matrix $B \geqslant 0$ such that $B^2 = A$ [3] and we denote $A^{1/2} = B$. If all eigenvalues of A are real, then they are arranged nonincreasingly $\lambda_1(A) \geqslant \cdots \geqslant \lambda_n(A)$; the singular values of $A \in \mathbb{M}_n$, denoted by $s_j(X)$, are similarly arranged. Note that the singular values of A are the eigenvalues of $|A|$, where $|A| = (A^*A)^{\frac{1}{2}}$, i.e., $s_j(A) = \lambda_j(|A|)$, $j = 1, \ldots, n$. The geometric mean of two positive definite matrices $A, C \in \mathbb{M}_n$ is defined by

$$A\sharp C := A^{\frac{1}{2}}\left(A^{-\frac{1}{2}} C A^{-\frac{1}{2}}\right)^{\frac{1}{2}} A^{\frac{1}{2}}. \qquad (1)$$

It is known that the notion of geometric mean could be extended to cover all positive semidefinite matrices; see [4]. Recently, Drury [5] defined the geometric mean of two accretive matrices via the following formula

$$A\sharp C = \left(\frac{2}{\pi}\int_0^\infty (tA + t^{-1}C)^{-1}\frac{\mathrm{d}t}{t}\right)^{-1},$$

and proved the relationship (1) holds for two accretive matrices $A, C \in \mathbb{M}_n$. The readers can consult [5] for more properties.

For the 2 × 2 block matrix

$$M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$$

with each block in $\mathbb{M}_n$, its partial transpose is defined by

$$M^\tau := \begin{pmatrix} A & B^* \\ B & C \end{pmatrix}.$$

A matrix M is called partial positive transpose (PPT) if M and M^τ are positive semidefinite; see [6, 7]. We extend the notion to accretive matrices. If

$$M = \begin{pmatrix} A & X \\ Y^* & C \end{pmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$$

and

$$M^\tau := \begin{pmatrix} A & Y^* \\ X & C \end{pmatrix}$$

are both accretive, then we say that M is accretive partial transpose (i.e., APT); see [1]. Clearly, the class of APT matrices includes the class of PPT matrices.

Lin [6] obtained a singular value inequality involving the off-diagonal block of a PPT matrix and the geometric mean of its diagonal blocks.

Theorem 1 ([6]) *Let* $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ *be PPT. Then*

$$\prod_{j=1}^k s_j(B) \leqslant \prod_{j=1}^k s_j(A\sharp C), \quad k = 1, \ldots, n. \qquad (2)$$

Fu et al [8] present an alternative proof of the above singular value inequality.

Under the same condition as in Theorem 1, a stronger level inequality

$$s_j(B) \leqslant s_j(A\sharp C), \quad k = 1, \ldots, n,$$

is not true. Even the weaker singular value inequality

$$s_j(B) \leqslant s_j\left(\frac{A+C}{2}\right), \quad k=1,\dots,n,$$

also fails; see a counter-example in [6].

In this paper, we will present a singular value inequality relation between the off-diagonal block of an APT matrix and the geometric mean of its diagonal blocks which includes the case of PPT matrices. This complements the result in Theorem 1.

SINGULAR VALUE INEQUALITIES

Now we present our main results.

Theorem 2 *Let* $\begin{pmatrix} A & X \\ Y^* & C \end{pmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ *be APT. Then*

$$s_j\left(\frac{X+Y}{2}\right) \leqslant s_{\left[\frac{j+1}{2}\right]}(\mathrm{Re}\, A \sharp \mathrm{Re}\, C), \quad j=1,\dots,n,$$

where $[a]$ *is the greatest integer less than or equal to* a.

Proof: Since $\begin{pmatrix} A & X \\ Y^* & C \end{pmatrix}$ and $\begin{pmatrix} A & Y^* \\ X & C \end{pmatrix}$ are accretive,

$$\mathrm{Re}\begin{pmatrix} A & X \\ Y^* & C \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\, A & \frac{X+Y}{2} \\ \frac{X^*+Y^*}{2} & \mathrm{Re}\, C \end{pmatrix}$$

and

$$\mathrm{Re}\begin{pmatrix} A & Y^* \\ X & C \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\, A & \frac{X^*+Y^*}{2} \\ \frac{X+Y}{2} & \mathrm{Re}\, C \end{pmatrix}$$

are positive definite. This means that

$$\begin{pmatrix} \mathrm{Re}\, A & \frac{X+Y}{2} \\ \frac{X^*+Y^*}{2} & \mathrm{Re}\, C \end{pmatrix}$$

is PPT.

With the help of unitary similarity transformations, we have

$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}\begin{pmatrix} \mathrm{Re}\, A & \frac{X+Y}{2} \\ \frac{X^*+Y^*}{2} & \mathrm{Re}\, C \end{pmatrix}\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\, C & -\frac{X^*+Y^*}{2} \\ -\frac{X+Y}{2} & \mathrm{Re}\, A \end{pmatrix} \geqslant 0,$$

and

$$\begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}\begin{pmatrix} \mathrm{Re}\, A & \frac{X^*+Y^*}{2} \\ \frac{X+Y}{2} & \mathrm{Re}\, C \end{pmatrix}\begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\, A & -\frac{X^*+Y^*}{2} \\ -\frac{X+Y}{2} & \mathrm{Re}\, C \end{pmatrix} \geqslant 0.$$

By [9],

$$\begin{pmatrix} \mathrm{Re}\, A \sharp \mathrm{Re}\, C & -\frac{X^*+Y^*}{2} \\ -\frac{X+Y}{2} & \mathrm{Re}\, A \sharp \mathrm{Re}\, C \end{pmatrix} \geqslant 0,$$

which is equivalent to

$$\begin{pmatrix} \mathrm{Re}\, A \sharp \mathrm{Re}\, C & 0 \\ 0 & \mathrm{Re}\, A \sharp \mathrm{Re}\, C \end{pmatrix} \geqslant \begin{pmatrix} 0 & \frac{X^*+Y^*}{2} \\ \frac{X+Y}{2} & 0 \end{pmatrix}.$$

Suppose that the singular values of $\frac{X+Y}{2}$ are arranged nonincreasingly $s_1\left(\frac{X+Y}{2}\right) \geqslant s_2\left(\frac{X+Y}{2}\right) \geqslant \cdots \geqslant s_n\left(\frac{X+Y}{2}\right)$. Thus, by [10], the eigenvalues of

$$\begin{pmatrix} 0 & \frac{X^*+Y^*}{2} \\ \frac{X+Y}{2} & 0 \end{pmatrix}$$

are

$$s_1\left(\tfrac{X+Y}{2}\right) \geqslant \cdots \geqslant s_n\left(\tfrac{X+Y}{2}\right) \geqslant -s_n\left(\tfrac{X+Y}{2}\right) \geqslant \cdots \geqslant -s_1\left(\tfrac{X+Y}{2}\right).$$

Using Weyl's monotonicity principle [10], we have

$$s_{\left[\frac{j+1}{2}\right]}(\mathrm{Re}\, A \sharp \mathrm{Re}\, C) \geqslant s_j\left(\tfrac{X+Y}{2}\right), \quad j=1,\dots,n.$$

□

By Theorem 2, the following result becomes immediate.

Remark 1 Inspired by the proof methods of [11, Theorem 3.2] and [12, Theorem 2.3], we give the above proof of our proposed results.

Corollary 1 *Let* $\begin{pmatrix} A & X \\ X^* & C \end{pmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ *be PPT. Then*

$$s_j(X) \leqslant s_{\left[\frac{j+1}{2}\right]}(A \sharp C), \quad j=1,\dots,n,$$

where $[a]$ *is the greatest integer less than or equal to* a.

Remark 2 Obviously, our result complements Lin's inequality (2).

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