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On existence of meromorphic solutions for certain q-difference equation

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ABSTRACT: We consider the existence of transcendental meromorphic solutions of q-difference equation

$$\sum_{j=1}^{n} c_j(z) f(q^j z) = \frac{P(z, f(z))}{Q(z, f(z))},$$

where $P(z, f(z))$ and $Q(z, f(z))$ are polynomials in f having rational coefficients and no common roots, $c_j(z)$ are rational functions, $q \in \mathbb{C}$ and $0 < |q| \leqslant 1$. We obtain that such equation has no transcendental meromorphic solutions for the case $m = \deg_f P - \deg_f Q \geqslant 2$.

KEYWORDS: q-difference equation, difference equation, existence of transcendental meromorphic solution

MSC2020: 30D35 39B32

INTRODUCTION AND RESULTS

A function $f(z)$ is called meromorphic if it is analytic in the complex plane $\mathbb{C}$ except at isolated poles. In what follows, we use standard notations in the Nevanlinna's value distribution theory, see [1, 2]. Let $f(z)$ be a meromorphic function. We also use notations $\sigma(f)$, $\mu(f)$ for the order and the lower order, respectively.

Recently, there are some papers focusing on the existence and the growth of meromorphic solutions of q-difference equations, see [3–6].

Zhang and Korhonen [7] studied the existence of zero-order transcendental meromorphic solutions of the certain q-difference equation, and showed the following theorem.

Theorem 1 ([7]) *Let $q_1, \dots, q_n \in \mathbb{C}\backslash\{0\}$, and let $a_0(z), \dots, a_p(z)$, $b_0(z), \dots, b_d(z)$ be rational functions. If the q-difference equation*

$$\sum_{j=1}^{n} f(q_j z) = \frac{P(z, f(z))}{Q(z, f(z))} = \frac{a_0(z) + a_1(z)f(z) + \cdots + a_p(z)f(z)^p}{b_0(z) + b_1(z)f(z) + \cdots + b_d(z)f(z)^d}, \quad (1)$$

where $P(z, f(z))$ and $Q(z, f(z))$ do not have any common factors in $f(z)$, admits a transcendental meromorphic solution of zero order, then $\max\{p, d\} \leqslant n$.

Zheng and Chen [8] considered the growth problem for transcendental meromorphic solutions of complex q-difference equation, and obtained the following result.

Theorem 2 ([8]) *Suppose that f is a transcendental meromorphic solution of equation*

$$\sum_{j=1}^{n} c_j(z) f(q^j z) = R(z, f(z)) = \frac{P(z, f(z))}{Q(z, f(z))}, \quad (2)$$

where $q \in \mathbb{C}$, $|q| > 1$, the coefficients $c_j(z)$ are rational functions and P, Q are relatively prime polynomials in f over the field of rational functions satisfying $p = \deg_f P$, $d = \deg_f Q$, $m = p - d \geqslant 2$. If f has infinitely many poles, then for sufficiently large r, $n(r, f) \geqslant K m^{\log r / n \log |q|}$ holds for some constant $K > 0$. Thus, the lower order of f, which has infinitely many poles, satisfies $\mu(f) \geqslant \frac{\log m}{n \log |q|}$.

In Theorem 2, condition $|q| > 1$ is necessary. It is natural to ask if $0 < |q| \leqslant 1$, what do we get? In the following, we will answer the above question, and obtain Theorem 3 as show below.

Theorem 3 *Let $c_j(z), j = 1, \dots, n$, $a_i(z), i = 0, 1, \dots, p$ and $b_k(z), k = 0, 1, \dots, d$ be rational functions with $a_p(z) b_d(z) \not\equiv 0$. Consider q-difference equation*

$$\sum_{j=1}^{n} c_j(z) f(q^j z) = \frac{P(z, f(z))}{Q(z, f(z))} = \frac{a_0(z) + a_1(z)f(z) + \cdots + a_p(z)f(z)^p}{b_0(z) + b_1(z)f(z) + \cdots + b_d(z)f(z)^d}, \quad (3)$$

where $P(z, f(z))$ and $Q(z, f(z))$ do not have any common factors in $f(z)$, $q \in \mathbb{C}$ and $m = p - d \geqslant 2$. If $0 < |q| \leqslant 1$, then equation (3) has no transcendental meromorphic solution.

From Theorem 1 and Theorem 3, we can get the following Corollary 1.

Corollary 1 *Suppose that the q-difference equation (1) satisfies the hypothesis of Theorem 1. If $p-d \geqslant 2$ and $0<|q_j|\leqslant 1$ $(j=1,\dots,n)$, then equation (1) does not possess transcendental meromorphic solution with finitely many poles.*

Remark 1 ([9]) We shall also use the observation that

$$\begin{aligned} M(r,f(qz)) &= M(|q|r,f),\\ N(r,f(qz)) &= N(|q|r,f)+O(1),\\ \text{and}\quad T(r,f(qz)) &= T(|q|r,f)+O(1)\end{aligned}$$

hold for any meromorphic function f and any non-zero constant q.

PROOF OF Theorem 3

Without loss of generality, suppose that the coefficients $c_j(z)$, $a_i(z)$ $(i=0,1,\dots,p)$ and $b_k(z)$ $(k=0,1,\dots,d)$ in (3) are polynomials.

On the contrary, suppose that equation (3) has a transcendental meromorphic solution f. Our conclusion holds for the cases.

Case 1: Suppose that f, the solution of (3), is transcendental entire.

Denote $p_j=\deg c_j$, $l_k=\deg b_k$, $t=\deg a_p$. Note that $M(r,f(qz))=M(|q|r,f)$ for z satisfying $|z|=r$. Set $h=1+\max\{p_1,\dots,p_n\}$ and $v=1+\max\{l_0,l_1,\dots,l_d\}$. It follows that

$$\begin{aligned} M\left(r,\frac{P(z,f(z))}{Q(z,f(z))}\right) &= M\left(r,\sum_{j=1}^{n}c_j(z)f(q^jz)\right)\\ &\leqslant nr^hM\left(r,f(z)\right), \end{aligned} \tag{4}$$

when r is large enough and $0<|q|\leqslant 1$. Furthermore

$$\begin{aligned} &\left|\sum_{i=0}^{p}a_i(z)f(z)^i\right|\\ &\quad\geqslant |a_p(z)f(z)^p|-(|a_{p-1}(z)f(z)^{p-1}|+\dots+|a_0(z)|)\\ &\quad\geqslant \frac{1}{2}|a_p(z)f(z)^p| = \frac{1}{2}r^t|f(z)|^p(1+o(1)), \end{aligned}$$

when r is sufficiently large. And

$$\begin{aligned} \left|\sum_{k=0}^{d}b_k(z)f(z)^k\right| &\leqslant \sum_{k=0}^{d}|b_k(z)f(z)^k|\\ &\leqslant \sum_{k=0}^{d}r^v|f(z)|^d=(d+1)r^v|f(z)|^d, \end{aligned}$$

when r is large enough. Hence

$$\begin{aligned} \left|\frac{P(z,f(z))}{Q(z,f(z))}\right| &= \left|\frac{\sum_{i=0}^{p}a_i(z)f(z)^i}{\sum_{k=0}^{d}b_k(z)f(z)^k}\right|\\ &\geqslant \frac{|a_p(z)f(z)^p|-(|a_{p-1}(z)f(z)^{p-1}|+\dots+|a_0(z)|)}{|b_d(z)f(z)^d|+\dots+|b_1(z)f(z)|+|b_0(z)|}\\ &\geqslant \frac{\frac{1}{2}r^t|f(z)|^p(1+o(1))}{(d+1)r^v|f(z)|^d}\\ &= \frac{1}{2(d+1)}r^{(t-v)}|f(z)|^{(p-d)}(1+o(1)), \end{aligned}$$

when r is sufficiently large. Thus

$$M\left(r,\frac{P(z,f(z))}{Q(z,f(z))}\right)\geqslant \frac{r^{(t-v)}M(r,f(z))^m}{2(d+1)}, \tag{5}$$

when r is sufficiently large. We have by (4) and (5) that

$$\log M(r,f(z))\geqslant m\log M(r,f(z))+g(r), \tag{6}$$

where $|g(r)|<K\log r$ for some $K>0$, when r is large enough, and (6) is a contradiction since $m\geqslant 2$.

Case 2: Suppose that f, the solution of (3), is transcendental meromorphic with finitely many poles. Then there exists a polynomial $H(z)$ such that $F(z)=H(z)f(z)$ is transcendental entire. Substituting $f(z)=F(z)/H(z)$ into (3) and multiplying away the denominators, we will obtain an equation similar to (3). Applying the same reasoning above to $F(z)$, we obtain that for sufficiently large r

$$\log M(r,f)=\log M(r,F)+O(1)\geqslant m\log M(r,F)+g(r).$$

It is a contradiction since $m\geqslant 2$.

Case 3: Suppose that f, the solution of (3), is meromorphic with infinitely many poles. Since $a_i(z)$ $(i=0,1,\dots,p)$, $b_k(z)$ $(k=0,1,\dots,d)$ and $c_j(z)$ are polynomials, there are two constants $R>0$ and $M>0$ such that all nonzero zeros of $a_i(z)$ $(i=0,1,\dots,p)$, $b_k(z)$ $(k=0,1,\dots,d)$ and $c_j(z)$ are in $D_1=\{z: M\leqslant |z|\leqslant R\}$. Set $D=\{z:|z|>R\}$.

Since $f(z)$ has infinitely many poles, there exists a pole $z_0(\in D)$ of $f(z)$ having multiplicity $\tau\geqslant 1$. Then the right-hand side of (3) has a pole of multiplicity $m\tau$ at z_0. Thus, there exists at least one index $j_1\in\{1,2,\dots,n\}$ such that $q^{j_1}z_0$ is a pole of $f(z)$ of multiplicity $\tau_1=m\tau$.

We need to discuss the following two subcases.

Subcase 1: $|q|=1$. Replacing z by $q^{j_1}z_0$ in (3), we have

$$\begin{aligned} &\sum_{j=1}^{n}c_j(q^{j_1}z_0)f(q^{j+j_1}z_0)\\ &\qquad = \frac{a_0(q^{j_1}z_0)+\dots+a_p(q^{j_1}z_0)f^p(q^{j_1}z_0)}{b_0(q^{j_1}z_0)+\dots+b_d(q^{j_1}z_0)f^d(q^{j_1}z_0)}. \end{aligned} \tag{7}$$

Since $|q^{j_1}z_0| = |z_0|$, the coefficients of (3) cannot have a zero at $q^{j_1}z_0$, thus the right side of (7) has a pole of multiplicity $m\tau_1$ at $q^{j_1}z_0$. Hence, there exists at least one index $j_2 \in \{1, 2, \ldots, n\}$ such that $q^{j_1+j_2}z_0$ is a pole of $f(z)$ of multiplicity $\tau_2 = m\tau_1 = m^2\tau$.

We proceed to follow the step above. Since the coefficients of (3) have no zeros in D and f has infinitely many poles again, we may construct poles $\xi_l = q^{j_1+\cdots+j_l}z_0$ $(j_1, \ldots, j_l \in \{1, 2, \ldots, n\})$ of $f(z)$ of multiplicity τ_l for all $l \in \mathbb{N}$, satisfying $\tau_l = m^l\tau \to \infty$ as $l \to \infty$, and $|\xi_l| = |z_0|$ since $|q| = 1$. Thus, $f(z)$ is not a meromorphic function. It is a contradiction.

Subcase 2: $0 < |q| < 1$. Set $\deg a_p = A$ $(\geqslant 0)$. Since $z_0 \in D$, we know that $q^{j_1}z_0$ has two possibilities:

(a): If $q^{j_1}z_0 \in D_1$, this process will be terminated and we have to choose another pole z_0 of $f(z)$ in the way we did above.

(b): If $q^{j_1}z_0 \notin D_1$, then $q^{j_1}z_0$ is a pole of $f(z)$ of multiplicity $\tau_1 = m\tau$, since the right-hand side of (3) has a pole of multiplicity $m\tau$ at z_0.

If $q^{j_1}z_0 \notin D \cup D_1$, that is $0 < |q^{j_1}z_0| < M$, then we choose pole z_0 of $f(z)$ and substitute $q^{j_1}z_0$ for z in (3).

If $q^{j_1}z_0 \in D$, that is $|q^{j_1}z_0| > R$, then we substitute $q^{j_1}z_0$ for z in (3) to obtain (7). Similarly as above, there exists at least one index $j_2 \in \{1, 2, \ldots, n\}$ such that $q^{j_1+j_2}z_0$ is a pole of $f(z)$ of multiplicity $\tau_2 = m\tau_1 = m^2\tau$.

We proceed to follow the steps (a) and (b) as above. Since there are infinitely many poles of $f(z)$ in D, we will find a pole $z_0(\in D)$ of $f(z)$ such that $q^{j_1+\cdots+j_{n_1}}z_0(\in D)$ is a pole of $f(z)$ of multiplicity $\tau_{n_1} = m^{n_1}\tau$. And z_0 satisfies $q^{j_1+\cdots+j_{n_1}+j_{n_1+1}}z_0 \in D_1$. By (3) and $m = p - d \geqslant 2$, we conclude that $q^{j_1+\cdots+j_{n_1}+j_{n_1+1}}z_0$ is a pole of $f(z)$ of multiplicity $\tau_{(n_1+1)} = m\tau_{n_1} = m^{n_1+1}\tau$.

Substitute $\hat{z} := q^{j_1+\cdots+j_{n_1+1}}z_0$ for z in (3) to obtain

$$\sum_{j=1}^{n} c_j(\hat{z})f(q^j\hat{z}) = \frac{a_0(\hat{z}) + \cdots + a_p(\hat{z})f^p(\hat{z})}{b_0(\hat{z}) + \cdots + b_d(\hat{z})f^d(\hat{z})}. \quad (8)$$

We see that the right-hand side of (8) has a pole of multiplicity at least $p\tau_{(n_1+1)} - A - d\tau_{(n_1+1)} = m\tau_{(n_1+1)} - A$ at $q^{j_1+\cdots+j_{n_1+1}}z_0$. Without loss of generality, suppose that the right-hand side of (8) has a pole of multiplicity $m\tau_{(n_1+1)} - A$ at $q^{j_1+\cdots+j_{n_1+1}}z_0$.

By $m \geqslant 2$, when $n_1 > \max\left\{\frac{\log A - \log(m^2-1)\tau}{\log m}, 1\right\}$, we have $m\tau_{(n_1+1)} - A = m^{n_1+2}\tau - A > m^{n_1}\tau$. Thus $m\tau_{(n_1+1)} - A > \tau_{n_1}$.

We proceed to follow the step as above. We will find that $q^{j_1+\cdots+j_{n_1}+\cdots+j_{n_1+n_2}}z_0$ is a pole of $f(z)$ of multiplicity $\tau_{(n_1+n_2)} = m^{n_1+n_2}\tau - A(m^{n_2-2} + \cdots + m + 1)$ such that $0 < \left|q^{j_1+\cdots+j_{n_1+n_2}}z_0\right| < M$, that is $q^{j_1+\cdots+j_{n_1+n_2}}z_0 \notin D \cup D_1$.

Set $s := \tau_{(n_1+n_2)} = m^{n_1+n_2}\tau - A(m^{n_2-2} + \cdots + m + 1)$. Then

$$s = m^{n_1+n_2}\tau - A\frac{m^{n_2-1} - 1}{m - 1}.$$

That is

$$s = \frac{m^{n_2-1}}{m-1}\left[(m-1)m^{n_1+1}\tau - A\right] + \frac{A}{m-1}.$$

When $n_2 \geqslant 2$ and $n_1 > \max\left\{\frac{\log(A+1) - \log(m-1)\tau}{\log m} - 1, 1\right\}$, we have $(m-1)m^{n_1+1}\tau > A+1$, that is $(m-1)m^{n_1+1}\tau - A > 1$. Hence $s \geqslant 1$.

Set $z_1 = q^{j_1+\cdots+j_{n_1+n_2}}z_0(0 < |q^{j_1+\cdots+j_{n_1+n_2}}z_0| < M)$. Then z_1 is a pole of $f(z)$ of multiplicity $s \geqslant 1$. Specially, when $n_1 = 1$ and $n_2 = 0$, then $z_1 = q^{j_1}z_0$ is a pole of $f(z)$ of multiplicity $s = \tau_1 = m\tau$.

Using the same reasoning as Subcase 1, we conclude that $\zeta_\nu = q^{j_1+\cdots+j_\nu}z_1(\notin D \cup D_1)$ is a pole of $f(z)$ of multiplicity $k_\nu = m^\nu s$. Thus, there is a sequence $\{\zeta_\nu, \nu = 1, 2, \ldots\}$ which are the poles of $f(z)$. Since $0 < |q| < 1$, we have $\zeta_\nu \to 0$ as $\nu \to \infty$. Thus, $f(z)$ is not a meromorphic function. It is a contradiction.

Thus, Theorem 3 is proved.

VALUE DISTRIBUTION OF MEROMORPHIC SOLUTION OF DIFFERENCE EQUATION

Recently, there are also papers focusing on complex difference equations, see [10–13]. Ablowitz et al [14] looked at a difference equation of the type

$$f(z+1) + f(z-1) = R(z, f),$$

where R is rational in both of its arguments, and showed the following theorem.

Theorem 4 ([14]) *If the second-order difference equation*

$$f(z+1) + f(z-1) = \frac{a_0(z) + a_1(z)f(z) + \cdots + a_p(z)f(z)^p}{b_0(z) + b_1(z)f(z) + \cdots + b_d(z)f(z)^d}, \quad (9)$$

where a_i and b_i are polynomials, admits a non-rational meromorphic solution of finite order, then $\max\{p, d\} \leqslant 2$.

In Theorem 4, we see that if equation (9) admits a transcendental meromorphic solution of finite order , then $\max\{p, d\} \leqslant 2$. A natural question is: what is the result when $p - d \geqslant 2$ in (9)? Corresponding to this question, we get Theorem 5.

Theorem 5 *Let $a_0(z), \ldots, a_p(z)$, $b_0(z), \ldots, b_d(z)$ be rational functions with $a_p(z)b_d(z) \not\equiv 0$. Suppose that f is a transcendental meromorphic solution of equation*

$$f(z+1) + f(z-1) = \frac{P(z, f(z))}{Q(z, f(z))} = \frac{a_0(z) + a_1(z)f(z) + \cdots + a_p(z)f(z)^p}{b_0(z) + b_1(z)f(z) + \cdots + b_d(z)f(z)^d}, \quad (10)$$

where $P(z, f(z))$ and $Q(z, f(z))$ are relatively prime polynomials in f. Let $m = p - d \geqslant 2$.

(i) *If f is entire or has finitely many poles, then there exist constants $K > 0$ and $r_0 > 0$ such that*

$$\log M(r, f) \geqslant Km^r$$

holds for all $r \geqslant r_0$.

(ii) *If f has infinitely many poles, then there exist constants $K > 0$ and $r_0 > 0$ such that*

$$n(r, f) \geqslant Km^r$$

holds for all $r \geqslant r_0$.

From Theorem 4 and Theorem 5, we can get the following Corollary 2.

Corollary 2 *Suppose that the second-order difference equation (9) satisfies the hypothesis of Theorem 4. If equation (9) admits a non-rational meromorphic solution of finite order, then* $\max\{p, d\} \leqslant 2$ *and* $p - d \leqslant 1$.

In fact, many authors studied special forms of equation (9) when $\max\{p, d\} \leqslant 2$ and $p-d \leqslant 1$. Especially, they mainly considered three types of equations as show below.

$$f(z+1)+f(z-1) = \frac{az+b}{f(z)} + c, \tag{11}$$

$$f(z+1)+f(z-1) = \frac{az+b}{f(z)} + \frac{c}{f^2(z)}, \tag{12}$$

$$f(z+1)+f(z-1) = \frac{(az+b)f(z)+c}{1-f^2(z)}, \tag{13}$$

where a, b and c are constants. These equations are now known as the Painlevé equations. (11)–(13) are difference Painlevé equations I and II. Some results about transcendental meromorphic solutions of finite order to equations (11)–(13), can be found in [14–16].

From this, we see that the equation (10) is an important class of difference equations. It will play an important role for research of difference Painlevé equations I and II.

Remark 2 By Theorem 5, we obtain that meromorphic solutions of (10) are infinite order when $p-d \geqslant 2$. Under the conditions of $\max\{p, d\} \leqslant 2$ and $p - d \leqslant 1$, equation (9) may have meromorphic solution of infinite order, which can be seen by the following example.

Example 1 The difference equation

$$f(z+1)+f(z-1) = 2f(z)$$

has a solution $f(z) = \exp\{e^{2\pi i z}\}$, where $\sigma(f) = \infty$.

PROOF OF Theorem 5

Without loss of generality, suppose that $a_i(z)$ ($i = 0, 1, \dots, p$) and $b_v(z)$ ($v = 0, 1, \dots, d$) are polynomials.

(i): Suppose that f, the solution of (10), is transcendental entire. Denote $l_v = \deg b_v$, $t = \deg a_p$. The maximum modulus principle yields

$$M(r+1, f(z)) \geqslant M(r, f(z \pm 1))$$

for z satisfying $|z| = r$. Choosing $h = 1 + \max\{l_0, l_1, \dots, l_d\}$, it follows that

$$M\left(r, \frac{P(z, f(z))}{Q(z, f(z))}\right) = M(r, f(z+1)+f(z-1)) \leqslant CM(r+1, f(z)), \tag{14}$$

when r is large enough, where C is a positive constant. Using the same methods as the proof of Theorem 3, we have

$$M\left(r, \frac{P(z, f(z))}{Q(z, f(z))}\right) \geqslant \frac{r^{(t-h)}|f(z)|^{p-d}}{2(d+1)} = \frac{r^{(t-h)}M(r, f(z))^m}{2(d+1)}, \tag{15}$$

when r is sufficiently large. We have by (14) and (15) that

$$\log M(r+1, f(z)) \geqslant m \log M(r, f(z)) + g(r), \tag{16}$$

where $|g(r)| < K \log r$ for some $K > 0$ and r is large enough. Iterating (16), we have

$$\log M(r+j, f(z)) \geqslant m^j \log M(r, f(z)) + E_j(r), \tag{17}$$

where

$$|E_j(r)| = \left|m^{j-1}g(r)+m^{j-2}g(r+1)+\cdots+g(r+(j-1))\right| \leqslant Km^{j-1}\sum_{k=0}^{j-1}\frac{\log(r+k)}{m^k} \leqslant Km^{j-1}\sum_{k=0}^{\infty}\frac{\log(r+k)}{m^k}.$$

Since $\log(r+k) \leqslant (\log r)(\log k)$ for sufficiently large r and k, we have

$$\sum_{k=0}^{\infty}\frac{\log(r+k)}{m^k} \leqslant \sum_{k=0}^{\infty}\frac{(\log r)(\log k)}{m^k} = \log r\sum_{k=0}^{\infty}\frac{\log k}{m^k}.$$

Obviously, the series $I = \sum_{k=0}^{\infty}\frac{\log k}{m^k}$ is convergent. Hence

$$|E_j(r)| \leqslant K' m^j \log r. \tag{18}$$

Since, by the hypothesis, f is transcendental entire, we get the inequality $\log M(r, f) \geqslant 2K' \log r$ for sufficiently large r. Thus, (17) and (18) imply

$$\log M(r+j, f(z)) \geqslant K' m^j \log r, \tag{19}$$

which holds for r sufficiently large, say $r \geqslant r_0$. By choosing $r \in [r_0, r_0+1)$ arbitrarily and letting $j \to \infty$

for each choice of r, and set $s=r+j$, then $j=s-r\geqslant s-(r_0+1)$. We have by (19) that

$$\log M(s,f(z))=\log M(r+j,f(z)) \geqslant K'm^{s-r_0-1}\log r_0=K''m^s$$

holds for all $s\geqslant s_0=r_0+1$, where $K''=K'm^{-(r_0+1)}\log r_0$. We have proved the assertion in the case of f being entire.

Suppose now that f, the solution of (10), is meromorphic with finitely many poles. Then there exists a polynomial $H(z)$ such that $F(z)=H(z)f(z)$ is entire. Substituting $f(z)=F(z)/H(z)$ into (10) and again multiplying away the denominators, we will obtain an equation similar to (10). Applying the same reasoning above to $F(z)$, we obtain that for sufficiently large r, $\log M(r,f)=\log M(r,F)+O(1)\geqslant(K''-\varepsilon)m^r=K'''m^r$, where $K'''(>0)$ is some constant.

Thus, part (i) is proved.

(ii): Suppose that $f(z)$ is a meromorphic function with infinitely many poles. Since $a_i(z)$ $(i=0,1,\dots,p)$, $b_v(z)$ $(v=0,1,\dots,d)$ are polynomials, there is a constant $M>0$ such that all zeros of $a_i(z)$ $(i=0,1,\dots,p)$ and $b_v(z)$ $(v=0,1,\dots,d)$ are in $D=\{z:|\mathrm{Re}(z)|<M,|\mathrm{Im}(z)|<M\}$. Set

$$D_1=\{z:\mathrm{Re}(z)>M\};\quad D_2=\{z:\mathrm{Re}(z)<-M\};$$
$$D_3=\{z:\mathrm{Im}(z)>M\};\quad D_4=\{z:\mathrm{Im}(z)<-M\}.$$

Since $f(z)$ has infinitely many poles, there exists at least one of D_s $(s=1,2,3,4)$ such that $f(z)$ has infinitely many poles in it. Suppose that z_0 is in one of D_s $(s=1,2,3,4)$ such that D_s has infinitely many poles of $f(z)$, and z_0 is a pole of $f(z)$ having multiplicity $k_0\geqslant 1$. Then the right-hand side of (10) has a pole of multiplicity mk_0 at z_0. Thus, there is $l_1\in\{1,-1\}$ such that z_0+l_1 is a pole of $f(z)$ of multiplicity $k_1=mk_0$.

Our conclusion holds for the following cases.

Case 1: $l_1=1$. Then z_0+1 is a pole of $f(z)$ of multiplicity k_1.

Suppose that $f(z)$ has infinitely many poles in D_1 and $z_0\in D_1$. Then $z_0+1\in D_1$ since $z_0\in D_1$. Substitute z_0+1 for z in (10) to obtain

$$f(z_0+2)+f(z_0) = \frac{a_0(z_0+1)+\cdots+a_p(z_0+1)f^p(z_0+1)}{b_0(z_0+1)+\cdots+b_d(z_0+1)f^d(z_0+1)}. \quad (20)$$

By (20) and $m=p-d\geqslant 2$, we conclude that z_0+2 is a pole of $f(z)$ of multiplicity $k_2=mk_1=m^2k_0$. Obviously $z_0+2\in D_1$.

Similarly, $z_0+n\in D_1$ is a pole of $f(z)$ of multiplicity $k_n=mk_{n-1}=m^nk_0$. Thus, there is a sequence $\{z_0+j\in D_1,\ j=1,2,\dots\}$ which are the poles of $f(z)$. Since $m^jk_0\to\infty$, as $j\to\infty$, and since $f(z)$ does not have essential singularities in the finite plane, we must have $|z_0+j|\to\infty$, as $j\to\infty$. It is clear that, for j large enough, say $j>j_0$,

$$m^jk_0\leqslant k_0(1+m+\cdots+m^j)\leqslant n(|z_0+j|,f) \leqslant n(|z_0|+j,f)\leqslant n(t+j,f),$$

where $t\in[|z_0|,|z_0|+1]$ can be chosen arbitrarily. Letting $j\to\infty$ for each choice of t, and set $r=t+j$, then $j=r-t\geqslant r-(|z_0|+1)$. Thus, the above inequality implies

$$n(r,f)\geqslant m^jk_0\geqslant k_0m^{r-(|z_0|+1)}=Km^r,$$

which holds for all $r\geqslant r_0:=j_0+1+|z_0|$, where $K=k_0m^{-(|z_0|+1)}$. The fact that r_0 and K both depend on $|z_0|$ is not a problem, since z_0 is fixed.

Suppose that $f(z)$ has infinitely many poles in D_3 (or D_4). Then we may use the same method as above.

Suppose that $f(z)$ has infinitely many poles in D_2 and $z_0\in D_2$. Set $\deg a_p=A(\geqslant 0)$. Since $z_0\in D_2$, we know that z_0+1 has two possibilities:

(a): If $z_0+1\notin D_2$, this process will be terminated and we have to choose another pole z_0 of $f(z)$ in the way we did above.

(b): If $z_0+1\in D_2$, then z_0+1 is a pole of $f(z)$ of multiplicity $k_1=mk_0$, since the right-hand side of (10) has a pole of multiplicity mk_0 at z_0.

Substitute z_0+1 for z in (10) to obtain (20). And we conclude that z_0+2 is a pole of $f(z)$ of multiplicity $k_2=mk_1$.

We proceed to follow the steps (a) and (b) as above. Since there are infinitely many poles of $f(z)$ in D_2, we will find a pole $z_0(\in D_2)$ of $f(z)$ such that $z_0+n_1(\in D_2)$ is a pole of $f(z)$ of multiplicity $k_{n_1}=mk_{(n_1-1)}=m^{n_1}k_0$. And z_0 satisfies $z_0+n_1+1\notin D_2$, that is $\mathrm{Re}(z_0+n_1+1)\geqslant -M$. By (10) and $m=p-d\geqslant 2$, we conclude that z_0+n_1+1 is a pole of $f(z)$ of multiplicity $k_{(n_1+1)}=m^{n_1+1}k_0$.

Substitute z_0+n_1+1 for z in (10) to obtain

$$f(z_0+n_1+2)+f(z_0+n_1) =\frac{a_0(z_0+n_1+1)+\cdots+a_p(z_0+n_1+1)f^p(z_0+n_1+1)}{b_0(z_0+n_1+1)+\cdots+b_d(z_0+n_1+1)f^d(z_0+n_1+1)}. \quad (21)$$

We see that the right-hand side of (21) has a pole of multiplicity at least $pk_{(n_1+1)}-A-dk_{(n_1+1)}=mk_{(n_1+1)}-A$ at z_0+n_1+1. Without loss of generality, suppose that the right-hand side of (21) has a pole of multiplicity $mk_{(n_1+1)}-A$ at z_0+n_1+1.

In the left-hand side of (21), $f(z-1)$ has a pole of multiplicity $k_{n_1}=m^{n_1}k_0$ at z_0+n_1+1. By $m\geqslant 2$, when $n_1>\max\left\{\frac{\log(A+1)-\log[(m^2-1)k_0]}{\log m},1\right\}$, we have $mk_{(n_1+1)}-A=m^{n_1+2}k_0-A>m^{n_1}k_0+1$.

Hence, by (21), we conclude that z_0+n_1+2 is a pole of $f(z)$ of multiplicity $k_{(n_1+2)}=mk_{(n_1+1)}-A=m^{n_1+2}k_0-A$.

We proceed to follow the step as above. We will find $z_0+n_1+n_2$ is a pole of $f(z)$ of multiplicity $k_{(n_1+n_2)}=$

$m^{n_1+n_2}k_0 - A[m^{n_2-2}+\cdots+m+1]$ such that $\mathrm{Re}(z_0+n_1+n_2) > M$, that is $z_0+n_1+n_2 \in D_1$.

Set $k := k_{(n_1+n_2)} = m^{n_1+n_2}k_0 - A[m^{n_2-2}+\cdots+m+1]$. Then

$$k = \frac{m^{n_2-1}}{m-1}\left[(m-1)m^{n_1+1}k_0 - A\right] + \frac{A}{m-1}.$$

When $n_2 \geqslant 2$ and $n_1 > \max\left\{\frac{\log(A+1)-\log(m-1)k_0}{\log m} - 1, 1\right\}$, we have $(m-1)m^{n_1+1}k_0 > A+1$, that is $(m-1)m^{n_1+1}k_0 - A > 1$. Hence $k \geqslant 1$.

Set $z_1 := z_0 + n_1 + n_2 (\in D_1)$. Then z_1 is a pole of $f(z)$ of multiplicity at least $k \geqslant 1$. Specially, when $n_1 = 1$ and $n_2 = 0$, then $z_1 = z_0 + 1$ is a pole of $f(z)$ of multiplicity $k = k_1 = mk_0$.

Applying the same reasoning that $f(z)$ has infinitely many poles in D_1, we obtain that

$$n(r, f) \geqslant Km^r$$

holds for all $r \geqslant r_0$. The fact that r_0 and K both depend on $|z_1|$ is not a problem, since $z_1 \in D_1$ is fixed by $z_0 \in D_2$.

Case 2: $l_1 = -1$. Then $z_0 - 1$ is a pole of $f(z)$ of multiplicity $k_1 = mk_0$.

Suppose that $f(z)$ has infinitely many poles in D_2 and $z_0 \in D_2$. Then $z_0 - 1 \in D_2$ since $z_0 \in D_2$. Substitute $z_0 - 1$ for z in (10) to obtain

$$f(z_0) + f(z_0-2) = \frac{a_0(z_0-1) + \cdots + a_p(z_0-1)f^p(z_0-1)}{b_0(z_0-1) + \cdots + b_d(z_0-1)f^d(z_0-1)}. \quad (22)$$

By (22) and $m = p - d \geqslant 2$, we conclude that $z_0 - 2$ is a pole of $f(z)$ of multiplicity $k_2 = m^2k_0$. Obviously $z_0 - 2 \in D_2$.

Similarly, $z_0 - n\ (\in D_2)$ is a pole of $f(z)$ of multiplicity $k_n = m^n k_0$. Thus, there is a sequence $\{z_0 - j \in D_2, j = 1, 2, \ldots\}$ which are the poles of $f(z)$ of multiplicity $k_j = m^j k_0$. Since $k_j = m^j k_0 \to \infty$, as $j \to \infty$, and since f does not have essential singularities in the finite plane, we must have $|z_0 - j| \to \infty$, as $j \to \infty$. It is clear that, for j large enough, say $j > j_0$,

$$\begin{aligned} m^j k_0 &\leqslant k_0(1 + m + \cdots + m^j) \\ &\leqslant n(|z_0 - j|, f) \leqslant n(|z_0| + j, f) \leqslant n(t + j, f), \end{aligned}$$

where $t \in [|z_0|, |z_0| + 1]$ can be chosen arbitrarily. By the same method as Case 1, we have

$$n(r, f) \geqslant Km^r.$$

Suppose that $f(z)$ has infinitely many poles in D_3 (or D_4). Then we may use the same method as above.

Suppose that $f(z)$ has infinitely many poles in D_1 and $z_0 \in D_1$. Set $\deg a_p = A (\geqslant 0)$. Since $z_0 \in D_1$, we know that $z_0 - 1$ has two possibilities:

(a): If $z_0 - 1 \notin D_1$, this process will be terminated and we have to choose another pole z_0 of $f(z)$ in the way we did above.

(b): If $z_0 - 1 \in D_1$, then $z_0 - 1$ is a pole of $f(z)$ of multiplicity $k_1 = mk_0$, since the right-hand side of (10) has a pole of multiplicity mk_0 at z_0.

Substitute $z_0 - 1$ for z in (10) to obtain (22). And we conclude that $z_0 - 2$ is a pole of $f(z)$ of multiplicity $k_2 = mk_1$.

We proceed to follow the steps (a) and (b) as above. Since there are infinitely many poles of $f(z)$ in D_1, we will find a pole $z_0 (\in D_1)$ of $f(z)$ such that $z_0 - n_1 (\in D_1)$ is a pole of $f(z)$ of multiplicity $k_{n_1} = m^{n_1}k_0$. And z_0 satisfies $z_0 - n_1 - 1 \notin D_1$, that is $\mathrm{Re}(z_0 - n_1 - 1) \leqslant M$. By (10) and $m = p - d \geqslant 2$, we conclude that $z_0 - n_1 - 1$ is a pole of $f(z)$ of multiplicity $k_{(n_1+1)} = m^{n_1+1}k_0$.

Substitute $z_0 - n_1 - 1$ for z in (10) to obtain

$$f(z_0-n_1) + f(z_0-n_1-2) = \frac{a_0(z_0-n_1-1) + \cdots + a_p(z_0-n_1-1)f^p(z_0-n_1-1)}{b_0(z_0-n_1-1) + \cdots + b_d(z_0-n_1-1)f^d(z_0-n_1-1)}. \quad (23)$$

We see that the right-hand side of (23) has a pole of multiplicity at least $pk_{(n_1+1)} - A - dk_{(n_1+1)} = mk_{(n_1+1)} - A$ at $z_0 - n_1 - 1$. Without loss of generality, suppose that the right-hand side of (23) has a pole of multiplicity $mk_{(n_1+1)} - A$ at $z_0 - n_1 - 1$.

We proceed to follow the step as above. We will find $z_0 - n_1 - n_2$ is a pole of $f(z)$ of multiplicity $k_{(n_1+n_2)} = m^{n_1+n_2}k_0 - A[m^{n_2-2} + \cdots + m + 1]$ such that $\mathrm{Re}(z_0 - n_1 - n_2) < -M$, that is $z_0 - n_1 - n_2 \in D_2$.

Using the same reasoning as Case 1, we obtain that

$$n(r, f) \geqslant Km^r$$

holds for all $r \geqslant r_0$.

Thus, Theorem 5 is proved.

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