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On a convergence rate of generalized truncated hypersingular integrals associated to generalized Flett potentials

Simten Bayrakci[a], Sinem Sezer Evcan[b,*], Melih Eryiğit[a], Selim Çobanoğlu[c]

[a] Department of Mathematics, Akdeniz University, Antalya 07058 Turkey
[b] Faculty of Education, Akdeniz University, Antalya 07058 Turkey
[c] Finike Republic Vocational and Technical Anatolian High School, Antalya 07940 Turkey

*Corresponding author, e-mail: sinemsezer@akdeniz.edu.tr

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ABSTRACT: In this paper, a family of truncated hypersingular integrals depending on a parameter ε and generated by the generalized Poisson semigroup is introduced. Then the convergence rate of these families of truncated hypersingular integrals, which converge to $L_{p,\nu}$-function φ as $\varepsilon \to 0$, is obtained.

KEYWORDS: Flett potentials, truncated hypersingular integrals, rate of convergence, generalized Poisson semigroup, generalized translation

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INTRODUCTION

In the Fourier harmonic analysis, the clasical Flett potentials are defined in terms of the Fourier transform F by

$$F(\mathscr{F}^{\alpha} f)(x) = (1+|x|)^{-\alpha} F(f)(x), \quad x \in \mathbb{R}^n,\ \alpha > 0,$$

where the equality is understood in the sense of distribution theory [1]. This potentials are interpreted as a negative fractional powers of the operator $(E+(-\Delta)^{\frac{1}{2}})$. Here E is the identity operator and Δ is the Laplacian.

Finding inversion formulas for potentials such as Riesz, Bessel, Flett, parabolic potentials, etc., and examining their approximation properties have been interesting topics in harmonic analysis.

The hypersingular integral technique, a very powerful tool for inversion of potentials, was introduced and studied by [2–6] and others.

In [7](see also [2, pp 217–222]), Rubin introduced some families of "truncated" hypersingular integrals generated by the classical Gauss-Weierstrass semigroup, and obtained new inversion formulas for classical Riesz and Bessel potentials, reducing the multivariate problem to the univariate problem. Recently, using this technique the explicit inversion formulas for the classical Riesz, Bessel, Flett potentilas and their approximation properties were studied by [8–13].

In the framework of the Fourier-Bessel harmonic analysis associated with the Laplace-Bessel differential operator, the similar studies have been investigated by [14–16].

The aim of this article is to define a new "truncated" hypersingular operator family, associated with the generalized singular Bessel differential operator defined by

$$\Delta_\nu = \sum_{k=1}^{n} \left(\frac{\partial^2}{\partial x_k^2} + \frac{2\nu_k}{x_k} \frac{\partial}{\partial x_k} \right), \qquad \nu_1, \dots, \nu_n > 0, \quad (1)$$

for the generalized Flett potentials and obtain the "rate of convergence" of this family to $\varphi(x_0)$ as ε tends to zero for the function $\varphi \in L_{p,\nu}$ with the some smoothness property at $x_0 \in \mathbb{R}^n_+$. The generalized Flett potentials, associated with Δ_ν, are formally defined in terms of Fourier-Bessel transform F_ν by

$$F_\nu(\mathscr{F}^{\alpha}_\nu f)(x) = (1+|x|)^{-\alpha} F_\nu(f)(x),\ x \in \mathbb{R}^n_+,\ \alpha > 0. \quad (2)$$

These potentials are interpreted as negative fractional powers of the operator $(E + (-\Delta_\nu)^{1/2})$.

In this article, we firstly introduce "a new family of the generalized truncated hypersingular integral operators $\mathfrak{D}^{\alpha}_{\varepsilon} f, (\varepsilon > 0)$" generated by generalized Poisson semigroup associated with Δ_ν. Secondly, using this we obtain some relationship between the "order of smoothness" of function φ and for the generalized Flett potential $\mathscr{F}^{\alpha}_\nu \varphi$, the "rate of convergence" of the families $\mathfrak{D}^{\alpha}_{\varepsilon} \mathscr{F}^{\alpha}_\nu \varphi(x_0)$ to $\varphi(x_0)$ as ε tends to zero, where $\varphi \in L_{p,\nu}$ has some smoothness properties at $x_0 \in \mathbb{R}^n_+$.

PRELIMINARIES

Let $\mathbb{R}^n_+ = \{x \mid x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n,\ x_i > 0,\ i = 1, \dots, n\}$ and

$$L_{p,\nu} \equiv L_{p,\nu}(\mathbb{R}^n_+) = \left\{ f : \|f\|_{p,\nu} = \left(\int_{\mathbb{R}^n_+} |f(x)|^p x^{2\nu}\, dx \right)^{1/p} < \infty \right\}$$

where $1 \leqslant p < \infty$, $\nu = (\nu_1, \dots, \nu_n)$, $\nu_i > 0$ for $i = 1, \dots, n$ and $x^{2\nu}\, dx = x_1^{2\nu_1} \cdots x_n^{2\nu_n}\, dx_1\, dx_2 \cdots dx_n$. In the case $p=\infty$ we identify $L_{\infty,\nu}$ with $C_0=C_0(\mathbb{R}^n_+)$, the corresponding space of continuous functions vanishing at infinity.

Fourier-Bessel transform F_ν of a function f are defined by

$$(F_\nu f)(x)=\int_{\mathbb{R}^n_+} f(y)\Big(\prod_{k=1}^{n} j_{\nu_k-\frac{1}{2}}(x_k y_k)\Big)y^{2\nu}\,\mathrm{d}y$$

where $j_\lambda(\tau)=2^\lambda\Gamma(\lambda+1)J_\lambda(\tau)\tau^{-\lambda}$; $(\tau>0,\lambda>-\frac{1}{2})$, $J_\lambda(\tau)$ is the Bessel function of the first kind (see for detail [17]).

For $x\in\mathbb{R}^n_+$, $y\in\mathbb{R}^n_+$ the generalized translation operator of a function $f:\mathbb{R}^n_+\to\mathbb{C}$ is defined by [15],

$$T^y f(x)=\pi^{-n/2}\prod_{k=1}^{n}\Gamma\Big(\nu_k+\frac{1}{2}\Big)\Gamma^{-1}(\nu_k)\times \int_0^\pi\cdots\int_0^\pi\prod_{k=1}^{n}\sin^{2\nu_k-1}\alpha_k f\Big(\sqrt{x_1^2-2x_1y_1\cos\alpha_1+y_1^2},\ldots,\sqrt{x_n^2-2x_ny_n\cos\alpha_n+y_n^2}\Big)\,\mathrm{d}\alpha_1\cdots\mathrm{d}\alpha_n.$$

It is note that T^y is closely connected with the Bessel differential operator

$$B_t=\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\frac{2\nu}{t}\frac{\mathrm{d}}{\mathrm{d}t},\quad 0<t<\infty.$$

Here, we deal with the Bessel translation with in $x=(x_1,x_2,\ldots,x_n)$, which is defined by

$$B^s f(t)=\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu)\Gamma(\frac{1}{2})}\int_0^\pi f\big(\sqrt{t^2-2ts\cos\theta+s^2}\big)\sin^{2\nu-1}\theta\,\mathrm{d}\theta.$$

It is well known that for $1\leqslant p\leqslant\infty$ (see [18])

$$\|T^y f\|_{p,\nu}\leqslant\|f\|_{p,\nu},\tag{3}$$

$$\|T^y f-f\|_{p,\nu}\to 0\quad\text{as}\quad |y|\to 0.\tag{4}$$

The generalized Poisson kernel is defined as

$$p_\nu(y,t)=\sqrt{c(n,\nu)}\frac{1}{\sqrt{\pi}}2^{\nu_1+\cdots+\nu_n}\Gamma(\tfrac{n+1}{2}+\nu_1+\cdots+\nu_n)\times\frac{t}{(|y|^2+t^2)^{\frac{n+1}{2}+\nu_1+\cdots+\nu_n}},\tag{5}$$

where $c(n,\nu)=\big(\prod_{k=1}^{n}2^{2\nu_k}\Gamma^2(\nu_k+\frac{1}{2})\big)^{-1}$. Note also that (see [19])

$$\|p_\nu(\cdot,t)\|_{1,\nu}=1,\quad\forall t>0.\tag{6}$$

Given a function f, we introduce the generalized ν-maximal function

$$(M_\nu f)(x)=\sup_{r>0}\frac{1}{r^{n+2\nu_1+\cdots+2\nu_n}\omega(n,\nu)}\int_{B_r^+}|T^y f(x)|y^{2\nu}\mathrm{d}y,\tag{7}$$

where $B_r^+=\{y\in\mathbb{R}^n_+:|y|\leqslant r\}$ and $\omega(n,\nu)=\int_{B_1^+}y^{2\nu}\,\mathrm{d}y$.

Note that for $f\in L_{p,\nu}$ (see [15, 20])

$$\|M_\nu f\|_{p,\nu}\leqslant c\|f\|_{p,\nu},\quad 1<p\leqslant\infty.\tag{8}$$

We define the generalized Poisson integral (semigroup) $\mathscr{P}_t f$, $(t>0)$ generated by the generalized translation operator as follows.

$$(\mathscr{P}_t f)(x)=\int_{\mathbb{R}^n_+}p_\nu(y,t)(T^y f(x))y^{2\nu}\,\mathrm{d}y.\tag{9}$$

The following lemma gives some properties of the generalized Poisson integral $\mathscr{P}_t f$, which will be used later.

Lemma 1 ([19]) *Let $f\in L_{p,\nu}$ and $\mathscr{P}_t f$ is defined as in (9). Then,*

(i)

$$\|\mathscr{P}_t f\|_{p,\nu}\leqslant\|f\|_{p,\nu},\quad 1\leqslant p\leqslant\infty,\ \forall t>0;\tag{10}$$

(ii)

$$\sup_{x\in\mathbb{R}^n_+}|\mathscr{P}_t f(x)|\leqslant c\,t^{-(n+2\nu_1+\cdots+2\nu_n)/p}\|f\|_{p,\nu},\tag{11}$$

where $1\leqslant p<\infty$ and c is independent of t;

(iii)

$$\sup_{t>0}|\mathscr{P}_t f(x)|\leqslant(M_\nu f)(x);\tag{12}$$

(iv)

$$\mathscr{P}_t(\mathscr{P}_s f)(x)=(\mathscr{P}_{t+s}f)(x),\quad t,s>0;\tag{13}$$

(v)

$$\lim_{t\to 0^+}\mathscr{P}_t f(x)=f(x),\tag{14}$$

where the limit is in the $L_{p,\nu}$ $(1\leqslant p<\infty)$ sense or pointwise for almost all $x\in\mathbb{R}^n_+$. If $f\in C_0$, then the convergence is uniform on $\mathbb{R}^n_+$.

Using the generalized Poisson integral $\mathscr{P}_t f$, we define modified generalized Poisson integral as

$$(\mathscr{P}_t^m f)(x,t)=\mathrm{e}^{-t}(\mathscr{P}_t f)(x),\quad 0<t<\infty.\tag{15}$$

For $t=0$ we set

$$(\mathscr{P}_0 f)(x)=(\mathscr{P}_0^m f)(x,0)=f(x).$$

The generalized Flett potentials $\mathscr{F}_\nu^\alpha$ initially defined by (2), can be represented as an integral operator (see [16, p 120]),

$$(\mathscr{F}_\nu^\alpha f)(x)=\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\,\mathrm{e}^{-t}(\mathscr{P}_t f)(x)\,\mathrm{d}t,\tag{16}$$

and by making use of the notion $\mathscr{P}_t^m f$ defined as (15), Eq. (16) can be write as follows.

$$(\mathscr{F}_\nu^\alpha f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}(\mathscr{P}_t^m f)(x,t)\,\mathrm{d}t. \quad (17)$$

Eq. (17) shows that there is a relation between the generalized Flett potentials and the modified generalized Poisson integral.

In addition, it is not difficult to show that

$$\|\mathscr{F}_\nu^\alpha f\|_{p,\nu} \leqslant \|f\|_{p,\nu}, \quad 1\leqslant p\leqslant\infty, \forall\alpha>0. \quad (18)$$

The finite difference with order $l\in\mathbb{N}$ and step $\tau\in\mathbb{R}$ of the function $g(t)$, $(t\in\mathbb{R})$ is defined by

$$\Delta_\tau^l[g](t) = \sum_{k=0}^{l}\binom{l}{k}(-1)^k g(t+k\tau). \quad (19)$$

By making use of this finite difference and modified generalized Poisson semigroup $(\mathscr{P}_t^m f)$, we can define the following “Balakrishnan-Rubin type truncated integral” (see [2, p 220]).

Definition 1 Let $f\in L_{p,\nu}$, $1\leqslant p<\infty$, $\alpha>0$ and $l>\alpha$, $(l\in\mathbb{N})$. Then the construction

$$(\mathfrak{D}_\varepsilon^\alpha f)(x) = \frac{1}{\aleph_l(\alpha)}\int_\varepsilon^\infty \Delta_\tau^l\left[(\mathscr{P}_\cdot^m f)(x)\right](0)\frac{\mathrm{d}\tau}{\tau^{1+\alpha}}$$
$$= \frac{1}{\aleph_l(\alpha)}\int_\varepsilon^\infty\Big[\sum_{k=0}^{l}\binom{l}{k}(-1)^k\,\mathrm{e}^{-k\tau}(\mathscr{P}_{k\tau}f)(x)\Big]\frac{\mathrm{d}\tau}{\tau^{1+\alpha}} \quad (20)$$

will be called a “generalized truncated integrals” with parameter $\varepsilon>0$. Here the normalized coefficient $\aleph_l(\alpha)$ is defined by

$$\aleph_l(\alpha) = \int_0^\infty (1-\mathrm{e}^{-t})^l t^{-1-\alpha}\,\mathrm{d}t.$$

The following lemma, gives us that there is a close connection between the construction (20) and the generalized Flett potential $(\mathscr{F}_\nu^\alpha f)$.

Lemma 2 ([15]) *Let $\varphi\in L_{p,\nu}$, $(1\leqslant p<\infty)$ and $0<\alpha<\frac{n+2(\nu_1+\cdots+\nu_n)}{p}$. Then for any $\varepsilon>0$ and for a.e. $x\in\mathbb{R}_+^n$*

$$(\mathfrak{D}_\varepsilon^\alpha\mathscr{F}_\nu^\alpha\varphi)(x) = \int_0^\infty \mathscr{K}_\alpha^{(l)}(\eta)(\mathscr{P}_{\varepsilon\eta}^m\varphi)(x,\varepsilon\eta)\,\mathrm{d}\eta, \quad (21)$$

where the function $\mathscr{K}_\alpha^{(l)}(\eta)$ is defined by

$$\mathscr{K}_\alpha^{(l)}(\eta) = \frac{1}{\Gamma(1+\alpha)\aleph_l(\alpha)}\frac{1}{\eta}\sum_{k=0}^{l}\binom{l}{k}(-1)^k(\eta-k)_+^\alpha,$$

$$(\eta-k)_+^\alpha = \begin{cases}(\eta-k)^\alpha, & \eta>k,\\ 0, & \eta\leqslant k.\end{cases}$$

The next lemma gives some properties of the function $\mathscr{K}_\alpha^{(l)}(\eta)$ that will be used later.

Lemma 3 ([2, 6])
(i) $\mathscr{K}_\alpha^{(l)}(\eta)\in L_1(0,\infty)$ *and* $\int_0^\infty \mathscr{K}_\alpha^{(l)}(\eta)\,\mathrm{d}\eta = 1$*;*
(ii) $\mathscr{K}_\alpha^{(l)}(\eta) = \begin{cases}O(\eta^{\alpha-1}), & \eta\to 0^+\\ O(\eta^{\alpha-l-1}), & \eta\to\infty.\end{cases}$

Definition 2 Let $\rho\in(0,1)$ be a fixed parameter and the function $\mu(r)$, $(0\leqslant r\leqslant\rho)$ be continuous and strictly increasing on $[0,\rho]$ and $\mu(0)=0$. We say that a function $\varphi\in L_{1,\nu}^{\mathrm{loc}}(\mathbb{R}_+^n)$ has μ-smoothness property at a point $x_0\in\mathbb{R}_+^n$ if

$$(\mathscr{S}_\mu\varphi)(x_0) \equiv \sup_{0<r\leqslant\rho}\frac{1}{r^{n+2(\nu_1+\cdots+\nu_n)}\mu(r)}$$
$$\times\int_{B_r^+}|T^x\varphi(x_0)-\varphi(x_0)|x^{2\nu}\,\mathrm{d}x<\infty. \quad (22)$$

From now on, we will assume $\mu(t)\geqslant st$, $(0\leqslant t\leqslant\rho)$, for some $s>0$ and $\mu(t)=\mu(\rho)$ for $\rho\leqslant t<\infty$.

Lemma 4 ([8, 21, 22]) *Let a function $\varphi\in L_{p,\nu}$, $(1\leqslant p<\infty)$ has the μ-smoothness property at a point $x_0\in\mathbb{R}_+^n$, the function $\psi(r)$, $(0\leqslant r<\rho)$ be decreasing, nonnegative and continuously differentiable on $[0,\rho]$. Then*

$$\int_{B_r^+}|T^x\varphi(x_0)-\varphi(x_0)|\psi(|x|)x^{2\nu}\,\mathrm{d}x$$
$$\leqslant(\mathscr{S}_\mu\varphi)(x_0)\Big[\rho^{n+2(\nu_1+\cdots+\nu_n)}\mu(\rho)\psi(\rho)$$
$$+\int_0^\rho r^{n+2(\nu_1+\cdots+\nu_n)}\mu(r)(-\psi'(r))\,\mathrm{d}r\Big]. \quad (23)$$

Proof: Let us take $k(x)=|T^x\varphi(x_0)-\varphi(x_0)|$ and $x=r\theta$, where $r=|x|$. Then

$$J\equiv\int_{B_r^+}k(x)\psi(|x|)x^{2\nu}\,\mathrm{d}x$$
$$=\int_0^\rho r^{n-1+2(\nu_1+\cdots+\nu_n)}\psi(r)\Big(\int_{|\theta|=1}k(r\theta)\,\mathrm{d}\sigma(\theta)\Big)\mathrm{d}r$$

If we call

$$\begin{cases}u(r)=\displaystyle\int_{|\theta|=1}k(r\theta)\,\mathrm{d}\sigma(\theta),\\ \Omega(r)=\displaystyle\int_0^r u(t)t^{n-1+2(\nu_1+\cdots+\nu_n)}\,\mathrm{d}t,\end{cases} \quad (24)$$

then we get

$$\begin{aligned}J &\equiv \int_0^\rho \psi(r)u(r)r^{n-1+2(\nu_1+\cdots+\nu_n)}\,\mathrm{d}r = \int_0^\rho \psi(r)\,\mathrm{d}\Omega(r)\\ &= \psi(r)\Omega(r)\Big|_0^\rho - \int_0^\rho \Omega(r)\psi'(r)\,\mathrm{d}r\\ &= \psi(\rho)\Omega(\rho) + \int_0^\rho \Omega(r)(-\psi'(r))\,\mathrm{d}r.\end{aligned}$$

It follows from (22) that

$$\begin{aligned}\Omega(r) &= \int_0^r u(t)t^{n-1+2(\nu_1+\cdots+\nu_n)}\,\mathrm{d}t = \int_{|x|\leqslant r} k(x)x^{2\nu}\,\mathrm{d}x\\ &= \int_{|x|\leqslant r} |T^x\varphi(x_0)-\varphi(x_0)|x^{2\nu}\,\mathrm{d}x\\ &\leqslant r^{n+2(\nu_1+\cdots+\nu_n)}\mu(r)(\mathscr{S}_\mu\varphi)(x_0).\end{aligned}$$

Hence,

$$J \leqslant (\mathscr{S}_\mu\varphi)(x_0)\Big[\rho^{n+2(\nu_1+\cdots+\nu_n)}\mu(\rho)\psi(\rho) + \int_0^\rho r^{n+2(\nu_1+\cdots+\nu_n)}\mu(r)(-\psi'(r))\,\mathrm{d}r\Big].$$

□

Lemma 5 *Let $p_\nu(x,\varepsilon)$ be the generalized Poisson kernel defined as in (5), i.e. for $x\in\mathbb{R}^n_+$,*

$$p_\nu(x,\varepsilon) = \sqrt{c(n,\nu)}\frac{1}{\sqrt{\pi}}2^{\nu_1+\cdots+\nu_n}\Gamma\big(\tfrac{n+1}{2}+\nu_1+\cdots+\nu_n\big) \times \frac{\varepsilon}{(|x|^2+\varepsilon^2)^{\frac{n+1}{2}+\nu_1+\cdots+\nu_n}}.$$

Then there exist $c>0$ such that

$$\int_{|x|\leqslant r} |T^x\varphi(x_0)-\varphi(x_0)|p_\nu(x,\varepsilon)x^{2\nu}\,\mathrm{d}x \leqslant c(\mathscr{S}_\mu\varphi)(x_0)\Big[\varepsilon + \int_0^\infty \mu(\varepsilon t)\frac{\mathrm{d}t}{1+t^2}\Big]. \quad (25)$$

Proof: Let $a_{n,\nu} = \sqrt{c(n,\nu)}\frac{1}{\sqrt{\pi}}2^{\nu_1+\cdots+\nu_n}\Gamma(\frac{n+1}{2}+\nu_1+\cdots+\nu_n)$. If we take $\psi(|x|) = p_\nu(x,\varepsilon)$ in (23).

$$\begin{aligned}&\int_{|x|\leqslant\rho} |T^x\varphi(x_0)-\varphi(x_0)|p_\nu(x,\varepsilon)x^{2\nu}\,\mathrm{d}x\\ &\leqslant (\mathscr{S}_\mu\varphi)(x_0)\Big[\rho^{n+2(\nu_1+\cdots+\nu_n)}\mu(\rho)\frac{a_{n,\nu}\varepsilon}{(\rho^2+\varepsilon^2)^{\frac{n+1}{2}+\nu_1+\cdots+\nu_n}}\\ &\quad + \int_0^\rho r^{n+2(\nu_1+\cdots+\nu_n)}\mu(r)\Big(-\frac{a_{n,\nu}\varepsilon}{(r^2+\varepsilon^2)^{\frac{n+1}{2}+\nu_1+\cdots+\nu_n}}\Big)'\,\mathrm{d}r\Big].\end{aligned}$$

Since

$$\begin{aligned}-\psi'(r) &= \Big(-\frac{a_{n,\nu}\,\varepsilon}{(r^2+\varepsilon^2)^{\frac{n+1}{2}+\nu_1+\cdots+\nu_n}}\Big)'\\ &= c_1\frac{\varepsilon r}{(r^2+\varepsilon^2)^{\frac{n+3}{2}+\nu_1+\cdots+\nu_n}},\end{aligned}$$

$c_1 = 2a_{n,\nu}(n+1)+\nu_1+\cdots+\nu_n$ and

$$\rho^{n+2(\nu_1+\cdots+\nu_n)}\mu(\rho)\frac{a_{n,\nu}\varepsilon}{(\rho^2+\varepsilon^2)^{\frac{n+1}{2}+\nu_1+\cdots+\nu_n}} \leqslant c_2\varepsilon,$$

$c_2 = a_{n,\nu}\frac{\mu(\rho)}{\rho}$, for $\rho<1$ and $c=\max\{c_1,c_2\}$ we have

$$\begin{aligned}&\int_{|x|\leqslant\rho} |T^x\varphi(x_0)-\varphi(x_0)|p_\nu(x,\varepsilon)x^{2\nu}\,\mathrm{d}x\\ &\leqslant c(S_\mu\varphi)(x_0)\Big[\varepsilon + \int_0^\rho \varepsilon\frac{r^{n+1+2(\nu_1+\cdots+\nu_n)}}{(r^2+\varepsilon^2)^{\frac{n+3}{2}+\nu_1+\cdots+\nu_n}}\mu(r)\,\mathrm{d}r\Big]\\ &\cdots(r=\varepsilon t,\quad \mathrm{d}r=\varepsilon\,\mathrm{d}t)\cdots\\ &= c(S_\mu\varphi)(x_0)\Big[\varepsilon + \int_0^{\frac{\rho}{\varepsilon}} \frac{\varepsilon(\varepsilon t)^{n+1+2(\nu_1+\cdots+\nu_n)}}{(\varepsilon^2t^2+\varepsilon^2)^{\frac{n+3}{2}+\nu_1+\cdots+\nu_n}}\mu(\varepsilon t)\varepsilon\mathrm{d}t\Big]\\ &= c(S_\mu\varphi)(x_0)\Big[\varepsilon + \int_0^{\frac{\rho}{\varepsilon}} \frac{t^{n+1+2(\nu_1+\cdots+\nu_n)}}{(1+t^2)^{\frac{n+3}{2}+\nu_1+\cdots+\nu_n}}\mu(\varepsilon t)\,\mathrm{d}t\Big]\\ &\leqslant c(S_\mu\varphi)(x_0)\Big[\varepsilon + \int_0^\infty \mu(\varepsilon t)\frac{t^{n+1+2(\nu_1+\cdots+\nu_n)}}{(1+t^2)^{\frac{n+3}{2}+\nu_1+\cdots+\nu_n}}\,\mathrm{d}t\Big]\\ &\leqslant c(S_\mu\varphi)(x_0)\Big[\varepsilon + \int_0^\infty \frac{\mu(\varepsilon t)}{1+t^2}\,\mathrm{d}t\Big],\end{aligned}$$

using the following statement we obtained last inequality:

$$\begin{aligned}\frac{t^{n+1+2(\nu_1+\cdots+\nu_n)}}{(1+t^2)^{\frac{n+3}{2}+\nu_1+\cdots+\nu_n}} &= \frac{(t^2)^{\frac{n+1+2(\nu_1+\cdots+\nu_n)}{2}}}{(1+t^2)^{\frac{n+3}{2}+\nu_1+\cdots+\nu_n}}\\ &\leqslant \frac{(1+t^2)^{\frac{n+1+2(\nu_1+\cdots+\nu_n)}{2}}}{(1+t^2)^{\frac{n+3}{2}+\nu_1+\cdots+\nu_n}}\\ &= \frac{1}{1+t^2}.\end{aligned}$$

□

Corollary 1 *Let the function $\mu(r)$, $(0\leqslant r\leqslant\rho<1)$ be continuous on $[0,\rho]$, positive on $(0,\rho]$ and $\mu(0)=0$. Let also, $\mu(t)\geqslant st$, $0\leqslant t\leqslant\rho$ for some $s>0$ and $\mu(t)=\mu(\rho)$ for $\rho\leqslant t<\infty$. If there exists a locally bounded function $\omega(t)>0$ such that for $0<\varepsilon<\rho$, $0<t<\infty$,*

$$\mu(\varepsilon t)\leqslant\mu(\varepsilon)\omega(t) \quad \text{and} \quad \int_0^\infty \frac{\omega(t)}{1+t^2}\,\mathrm{d}t<\infty, \quad (26)$$

then there exist $A>0$ not depending on $\varepsilon\in(0,\rho)$, such that

$$\int_{|x|\leqslant\rho} |T^x\varphi(x_0)-\varphi(x_0)|p_\nu(x,\varepsilon)x^{2\nu}\,\mathrm{d}x \leqslant A\mu(\varepsilon). \quad (27)$$

Proof: By taking into account (26) in (25) and using the condition

$$\mu(\varepsilon) \geqslant s\varepsilon, \quad 0 \leqslant \varepsilon \leqslant \rho,$$

we obtain desired inequality (27). □

Now, we will give the examples of the functions that satisfy all conditions of Corollary 1.

Example 1 Let $0 < \gamma < 1$ and $0 < \beta < \infty$. Then the function

$$\mu(r) = \begin{cases} 0, & r = 0, \\ r^{\gamma}|\ln r|^{\beta}, & 0 < r < \rho, \\ \rho^{\gamma}|\ln \rho|^{\beta}, & r \geqslant \rho. \end{cases}$$

satisfies all conditions of Corollary 1 for

$$\omega(t) = t^{\gamma}(1 + \frac{|\ln t|}{|\ln \rho|})^{\beta}.$$

Indeed, for $0 < \varepsilon < \rho$ and $0 < t < \infty$ we have

$$\mu(\varepsilon t) = \begin{cases} 0, & t = 0, \\ \varepsilon^{\gamma} t^{\gamma}|\ln \varepsilon + \ln t|^{\beta}, & 0 < t < \frac{\rho}{\varepsilon}, \\ \rho^{\gamma}|\ln \rho|^{\beta}, & t > \frac{\rho}{\varepsilon} \end{cases}$$
$$\leqslant \begin{cases} 0, & t = 0, \\ \mu(\varepsilon) t^{\gamma}(1 + \frac{|\ln t|}{|\ln \varepsilon|})^{\beta}, & 0 < t < \frac{\rho}{\varepsilon}, \\ \rho^{\gamma}|\ln \rho|^{\beta}, & t > \frac{\rho}{\varepsilon} \end{cases}$$
$$\leqslant \mu(\varepsilon)\omega(t), \quad 0 < \varepsilon < \rho,\ t > 0,$$

where $\omega(t) = t^{\gamma}(1 + |\ln t|/|\ln \rho|)^{\beta}$.

Example 2 Let $0 < \gamma < 1$. Then the function

$$\mu(r) = \begin{cases} r^{\gamma}, & 0 \leqslant r \leqslant \rho < 1, \\ \rho^{\gamma}, & r \geqslant \rho \end{cases}$$

satisfies all conditions of Corollary 1 with $\omega(t) = t^{\gamma}$.

MAIN RESULTS

Theorem 1 *Let the function $\mu(r), (0 < r < \infty)$ satisfy all conditions of Corollary 1. Further, let the function $\varphi \in L_{p,\nu}$, $(1 \leqslant p < \infty)$ has the μ-smoothness property (22) at a point $x_0 \in \mathbb{R}^n_+$. If the operator $\mathfrak{D}^{\alpha}_{\varepsilon}$ is defined by (20) and the parameter $l \in \mathbb{N}$ satisfies the condition $l > \alpha/2 + 1$, then*

$$|(\mathfrak{D}^{\alpha}_{\varepsilon}\mathscr{F}^{\alpha}_{\nu}\varphi)(x_0) - \varphi(x_0)| = O(\mu(\varepsilon)) \text{ as } \varepsilon \to 0^+, \quad (28)$$

where $\mathscr{F}^{\alpha}_{\nu}\varphi$, $(\alpha > 0)$ is the generalized Flett potentials associated with the Bessel differential operator Δ_{ν}.

Proof: By making use of Lemma 2 and Lemma 3(i) we have

$$|(\mathfrak{D}^{\alpha}_{\varepsilon}\mathscr{F}^{\alpha}_{\nu}\varphi)(x_0) - \varphi(x_0)|$$
$$= \left| \int_0^{\infty} \mathscr{K}^{(l)}_{\alpha}(\eta)(\mathscr{P}^{m}_{\varepsilon\eta}\varphi)(x_0, \varepsilon\eta)\,\mathrm{d}\eta - \int_0^{\infty} \mathscr{K}^{(l)}_{\alpha}(\eta)\varphi(x_0)\,\mathrm{d}\eta \right|$$
$$\leqslant \int_0^{\infty} \left|\mathscr{K}^{(l)}_{\alpha}(\eta)\right| |(\mathscr{P}^{m}_{\varepsilon\eta}\varphi)(x_0, \varepsilon\eta) - \varphi(x_0)|\,\mathrm{d}\eta. \quad (29)$$

Let us estimate $k_0 = |(\mathscr{P}^{m}_{\varepsilon\eta}\varphi)(x_0, \varepsilon\eta) - \varphi(x_0)|$. We have

$$k_0 \leqslant (1 - \mathrm{e}^{-\varepsilon\eta})|(\mathscr{P}_{\varepsilon\eta}\varphi)(x_0)| + |(\mathscr{P}_{\varepsilon\eta}\varphi)(x_0) - \varphi(x_0)|$$
$$= k_1 + k_2.$$

Since

$$(1 - \mathrm{e}^{-\varepsilon\eta}) \leqslant \varepsilon\eta \quad \text{and} \quad \sup_{\tau > 0}|(\mathscr{P}_{\tau}\varphi)(x_0)| \leqslant (M_{\nu}\varphi)(x_0),$$

we have $k_1 \leqslant k_3\varepsilon\eta$, where $(M_{\nu}\varphi)$ is the generalized ν-maximal function.

Now let us estimate k_2. Using (6), we can write the following.

$$k_2 = |(\mathscr{P}_{\varepsilon\eta}\varphi)(x_0) - \varphi(x_0)|$$
$$= \left| \int_{\mathbb{R}^n_+} p_{\nu}(y, \varepsilon\eta)\,[T^y\varphi(x_0) - \varphi(x_0)]\,y^{2\nu}\,\mathrm{d}y \right|$$
$$\leqslant \int_{|y| \leqslant \rho} p_{\nu}(y, \varepsilon\eta)|T^y\varphi(x_0) - \varphi(x_0)|y^{2\nu}\,\mathrm{d}y$$
$$+ \int_{|y| > \rho} p_{\nu}(y, \varepsilon\eta)|T^y\varphi(x_0) - \varphi(x_0)|y^{2\nu}\,\mathrm{d}y$$
$$\equiv i_1 + i_2. \quad (30)$$

Using (27) we have

$$i_1 \leqslant A\mu(\varepsilon\eta)$$

where A does not depend on ε and η. Now let's estimate i_2. If we use Hölder's inequality, we get

$$i_2 \leqslant |\varphi(x_0)| \int_{|y| > \rho} p_{\nu}(y, \varepsilon\eta)y^{2\nu}\,\mathrm{d}y$$
$$+ \|\varphi\|_{p,\nu}\left(\int_{|y| > \rho} |p_{\nu}(y, \varepsilon\eta)|^{p'} y^{2\nu}\,\mathrm{d}y \right)^{1/p'} \equiv j_1 + j_2.$$

From (5), we have

$$j_1 = |\varphi(x_0)| \int_{|y| > \rho} \frac{a_{n,\nu}\varepsilon\eta}{((\varepsilon\eta)^2 + |y|^2)^{\frac{n+1}{2} + \nu_1 + \cdots + \nu_n}} y^{2\nu}\,\mathrm{d}y$$
$$= c\varepsilon\eta \int_{|y| > \rho} \frac{1}{((\varepsilon\eta)^2 + |y|^2)^{\frac{n+1}{2} + \nu_1 + \cdots + \nu_n}} y^{2\nu}\,\mathrm{d}y$$
$$(y = r\theta,\ \rho < r < \infty,\ \theta \in S^{n-1},\ \mathrm{d}y = r^{n-1}\,\mathrm{d}r\,\mathrm{d}\sigma(\theta))$$
$$= c_2\varepsilon\eta \int_{\rho}^{\infty} \frac{r^{n-1}}{((\varepsilon\eta)^2 + r^2)^{\frac{n+1}{2} + \nu_1 + \cdots + \nu_n}} r^{2(\nu_1 + \cdots + \nu_n)}\,\mathrm{d}r$$
$$\leqslant c_2\varepsilon\eta \int_{\rho}^{\infty} \frac{r^{n-1}}{r^{n+1}}\,\mathrm{d}r = c_3\varepsilon\eta,$$

where $c_3 = c_3(\rho, \eta)$ is independent of ε and η.

Similarly,

$$\begin{aligned} j_2 &= \|\varphi\|_{p,\nu}\bigg(\int_{|y|>\rho} |p_\nu(y,\varepsilon\eta)|^{p'} y^{2\nu}\,\mathrm{d}y\bigg)^{1/p'} \\ &= \|\varphi\|_{p,\nu}\bigg(\int_{|y|>\rho} \frac{(a_{n,\nu}\varepsilon\eta)^{p'}}{((\varepsilon\eta)^2+|y|^2)^{(\frac{n+1}{2}+\nu_1+\cdots+\nu_n)p'}} y^{2\nu}\mathrm{d}y\bigg)^{1/p'} \\ &= c_4\varepsilon\eta\bigg(\int_{|y|>\rho} \frac{1}{((\varepsilon\eta)^2+|y|^2)^{(\frac{n+1}{2}+\nu_1+\cdots+\nu_n)p'}} y^{2\nu}\mathrm{d}y\bigg)^{1/p'} \end{aligned}$$

$(y = r\theta,\ \rho < r < \infty,\ \theta \in S^{n-1},\ \mathrm{d}y = r^{n-1}\,\mathrm{d}r\,\mathrm{d}\sigma(\theta))$

$$\begin{aligned} &= c_4\varepsilon\eta\bigg(\int_\rho^\infty \frac{r^{2(\nu_1+\cdots+\nu_n)}r^{n-1}}{((\varepsilon\eta)^2+r^2)^{(\frac{n+1}{2}+\nu_1+\cdots+\nu_n)p'}}\,\mathrm{d}r\bigg)^{1/p'} \\ &\leqslant c_4\varepsilon\eta\bigg(\int_\rho^\infty \frac{r^{n-1+2(\nu_1+\cdots+\nu_n)}}{r^{(n+1+2(\nu_1+\cdots+\nu_n))p'}}\,\mathrm{d}r\bigg)^{1/p'} \leqslant c_5\varepsilon\eta, \end{aligned}$$

where c_5 is independent of ε and η.

As a result, we get the following estimate for (30).

$$\begin{aligned} &|(\mathscr{P}_{\varepsilon\eta}\varphi)(x_0)-\varphi(x_0)| \\ &\qquad\leqslant \int_{\mathbb{R}^n_+} p_\nu(y,\varepsilon\eta)|T^y\varphi(x_0)-\varphi(x_0)|y^{2\nu}\,\mathrm{d}y \\ &\qquad\leqslant c_6(\mu(\varepsilon\eta)+\varepsilon\eta). \end{aligned} \tag{31}$$

We use the last estimation (31) in (29) using the conditions $\mu(\varepsilon\eta) \leqslant \mu(\varepsilon)\omega(\eta)$ and $\mu(\varepsilon) \geqslant s\varepsilon$ for $\varepsilon \in (0,\rho)$,

$$\begin{aligned} &|(\mathfrak{D}^\alpha_\varepsilon\mathscr{F}^\alpha_\nu\varphi)(x_0)-\varphi(x_0)| \\ &\qquad\leqslant c_7\int_0^\infty |\mathscr{K}^{(l)}_\alpha(\eta)|(\mu(\varepsilon\eta)+\varepsilon\eta)\,\mathrm{d}\eta \\ &\qquad\leqslant c_7\int_0^\infty |\mathscr{K}^{(l)}_\alpha(\eta)|(\mu(\varepsilon)\omega(\eta)+\varepsilon\eta)\,\mathrm{d}\eta \\ &\qquad\leqslant c_8\mu(\varepsilon)\int_0^\infty |\mathscr{K}^{(l)}_\alpha(\eta)|(\omega(\eta)+\eta)\,\mathrm{d}\eta. \end{aligned} \tag{32}$$

By making use of Lemma 3(ii) and the formula (26) we can write

$$\begin{aligned} &\int_0^\infty |\mathscr{K}^{(l)}_\alpha(\eta)|\omega(\eta)\,\mathrm{d}\eta \\ &\qquad= \int_0^1 |\mathscr{K}^{(l)}_\alpha(\eta)|\omega(\eta)\,\mathrm{d}\eta + \int_1^\infty |\mathscr{K}^{(l)}_\alpha(\eta)|\omega(\eta)\,\mathrm{d}\eta \\ &\qquad\leqslant c_9 + \int_1^\infty \frac{\omega(\eta)}{1+\eta^2}(1+\eta^2)|\mathscr{K}^{(l)}_\alpha(\eta)|\,\mathrm{d}\eta \\ &\qquad\leqslant c_9 + c_{10}\int_1^\infty \frac{\omega(\eta)}{1+\eta^2}\,\mathrm{d}\eta \equiv c_{11} < \infty, \end{aligned}$$

when $\mathscr{K}^{(l)}_\alpha(\eta) = O(\eta^{\alpha-l-1})$ as $\eta \to \infty$ and $l > \alpha+1$.

We also have

$$\begin{aligned} &\int_0^\infty |\mathscr{K}^{(l)}_\alpha(\eta)|\eta\,\mathrm{d}\eta \\ &\qquad= \int_0^1 |\mathscr{K}^{(l)}_\alpha(\eta)|\eta\,\mathrm{d}\eta + \int_1^\infty |\mathscr{K}^{(l)}_\alpha(\eta)|\eta\,\mathrm{d}\eta \\ &\qquad\leqslant c_{12} + \int_1^\infty |\mathscr{K}^{(l)}_\alpha(\eta)|\eta\,\mathrm{d}\eta \leqslant c_{13}. \end{aligned}$$

Finally, using the summation of these estimations in (32) we obtain

$$|(\mathfrak{D}^\alpha_\varepsilon\mathscr{F}^\alpha_\nu\varphi)(x_0)-\varphi(x_0)| \leqslant c\mu(\varepsilon)$$

as $\varepsilon \to 0^+$, where c being independent of ε. The proof is completed. □

Corollary 2 *Let $\mu(t) = t^\gamma$, $0 < \gamma < 1$, $t \in [0,\rho)$ and $x_0 \in \mathbb{R}^n_+$ be a μ-smoothness point of $\varphi \in L_{p,\nu}$. Then*

$$|(\mathfrak{D}^\alpha_\varepsilon\mathscr{F}^\alpha_\nu\varphi)(x_0)-\varphi(x_0)| = O(\varepsilon^\gamma)$$

as $\varepsilon \to 0^+$.

REFERENCES

1. Flett TM (1971) Temperatures, Bessel potentials and Lipschitz spaces. *Proc Lond Math Soc* **s3-22**, 385–451.
2. Rubin B (1996) *Fractional Integrals and Potentials*, Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman, Harlow.
3. Stein E (1971) *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ.
4. Lizorkin PI (1964) The functions of Hirshman type and relations between the spaces $B^r_p(E_n)$ and $L^r_p(\mathbb{R}^n)$. *Math Sat* **63**, 505–535. [in Russian]
5. Wheeden RL (1968) On hypersingular integrals and Lebesgue spaces of differentiable functions. *Trans Amer Math Soc* **134**, 421–435.
6. Samko SG, Kilbas AA, Marichev OI (1993) *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, London.
7. Rubin BS (1986) A method of characterization and inversion of Bessel and Riesz potentials. *Russian Math (Iz VUZ)* **30**, 78–89. [in Russian]
8. Aliev IA, Eryigit M (2013) On a rate of convergence of truncated hypersingular integrals associated to Riesz and Bessel potentials. *J Math Anal Appl* **406**, 352–359.
9. Aliev IA, Cobanoglu S (2014) The rate of convergence of truncated hypersingular integrals generated by the Poisson and metaharmonic semigroups. *Integral Transform Spec Funct* **25**, 943–954.
10. Eryigit M, Cobanoglu S (2017) On the rate of L_p-convergence of Balakrishnan-Rubin-type hypersingular integrals associated to Gauss-Weierstrass semigroup. *Turk J Math* **41**, 1376–1384.
11. Eryigit M, Sezer Evcan S, Cobanoglu S (2020) The convergence rate of truncated hypersingular integrals generated by the modified Poisson semigroup. *J Inequal Appl* **2020**, 201.

12. Sezer Evcan S, Eryigit M, Cobanoglu S (2020) A Balakrishnan-Rubin type hypersingular integral operator and inversion of Flett potentials. *Hacet J Math Stat* **49**, 1405–1413.
13. Lyakhov LN, Shishkina EL (2009) Inversion of general Riesz B-potentials with homogeneous characteristic in weight classes of functions. *Dokl Math* **79**, 377–381.
14. Gadjiev AD, Aral A, Aliev IA (2007) On behavior of the Riesz and generalized Riesz potentials as order tends to zero. *Math Inequal Appl* **10**, 875–888.
15. Aliev IA, Bayrakci S (1998) On inversion of B-elliptic potentials by the method of Balakrishnan-Rubin. *Fract Calc Appl Anal* **1**, 365–384.
16. Sezer S (2009) An inversion formula for the generalized Flett potentials. *Int J Math Comput* **2**, 115–125.
17. Levitan BM (1951) Expansion in Fourier series and integrals with Bessel functions. *Uspekhi Mat Nauk* **6**, 102–143. [in Russian]
18. Lofstorm J, Peetre J (1969) Approximation theorems connected with generalized translation. *Math Ann* **181**, 255–268.
19. Uyhan S (2004) On Poisson semigroup generated by the generalized B-Translation. In: *Dynamical Systems and Applications*, Antalya, Turkey, pp 679–686.
20. Guliev VS (1998) The Sobolev theorem for the B-Riesz potentials. *Dokl Math* **57**, 72–73.
21. Aliev IA (1999) On the Bochner-Riesz summability and restoration of μ-smooth functions by means of their Fourier transforms. *Fract Calc Appl Anal* **2**, 265–277.
22. Sezer S, Aliev IA (2011) On the Gauss Weierstrass summability of multiple trigonometric series at μ-smoothness points. *Acta Math Sin (Engl Ser)* **27**, 741–746.