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The periodicity on a transcendental entire function with its differential-difference polynomials

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ABSTRACT: According to a conjecture by C. C. Yang [Houston J Math **45** (2019):431–437], if $\omega(z)\omega^{(k)}(z)$ is a periodic function, where $\omega(z)$ is a transcendental entire function and k is a positive integer, then $\omega(z)$ is also a periodic function. We consider the related questions, which can be viewed as differential-difference versions of Yang's conjecture. We discuss the periodicity of a transcendental entire function $\omega(z)$ when differential-difference polynomials in $\omega(z)$ are periodic.

KEYWORDS: entire functions, periodicity, differential-difference equations, hyper-order

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INTRODUCTION AND MAIN RESULTS

Periodicity is important and easy to recognise property for meromorphic functions. Rényi and Rényi [1] have proved that if ω is a nonconstant entire function and $P(z)$ is a polynomial with $\deg(P(z)) \geqslant 3$, then the entire function $\omega(P(z))$ cannot be a periodic function. If $\deg(P(z)) = 2$, then there exists a transcendental entire function ω such that $\omega(P(z))$ is periodic.

Titchmarsh [2, p. 267] considered the real transcendental entire solutions of the differential equation

$$\omega(z)\omega^{(k)}(z) = p(z)\sin^2 z,$$

where $p(z)$ is a non-zero polynomial and obtained the following theorem.

Theorem A *The differential equation $\omega(z)\omega''(z) = -\sin^2 z$ has non real entire function of finite order other than $\omega(z) = \pm\sin z$.*

Li et al [3] generalized Theorem A, and obtained the following theorem.

Theorem B *If $\omega(z)$ is an entire function satisfying $\omega(z)\omega''(z) = p(z)\sin^2 z$, where $p(z)$ is a non-zero polynomial with real coefficients and real zeros, then $p(z)$ must be a non-zero constant p, and $\omega(z) = a\sin z$, where a is a constant satisfying $a^2 = -p$.*

They also raised an interesting question on the periodicity of transcendental entire functions, also mentioned in [4]. We formulate the question as follows.

Yang's Conjecture *Let $\omega(z)$ be a transcendental entire function and k be a positive integer. If $\omega(z)\omega^{(k)}(z)$ is a periodic function, then $\omega(z)$ is also a periodic function.*

Wang and Hu [4] showed that Yang's conjecture holds for $k = 1$, while Liu and Yu [5] proved that Yang's conjecture also holds for an arbitrary k if $\omega(z)$ has a non-zero Picard exceptional value.

Some results on the periodicity of transcendental meromoprhic functions can be found in [3–8]. In this article, we use the basic notations of Nevanlinna theory [9, 10]. In the following, we will use $\sigma(\omega)$ to denote the order of $\omega(z)$, and $\lambda(\omega)$ and $\lambda(1/\omega)$ to denote, respectively, the exponent of convergence of zeros and poles of $\omega(z)$.

More recently, Lü and Zhang [8] regarded Yang's conjecture, and they obtained the following theorems.

Theorem C *Let $\omega(z)$ be a transcendental entire function of hyper-order strictly less than 1, and n, k be positive integers. Suppose that $\omega(z)$ has a finite Borel exceptional value l, and $\omega^n(z)\omega^{(k)}(z)$ is a periodic function, then $\omega(z)$ is also a periodic function.*

Theorem D *Let $\omega(z)$ be a transcendental entire function of hyper-order strictly less than 1, and $n\,(\geqslant 2)$, $k\,(\geqslant 1)$ be integers. If $\omega^n(z) + b_1(\omega(z))' + \cdots + b_m(\omega(z))^{(m)}$ is a periodic function, where $b_1, \ldots, b_m$ are constants, then $\omega(z)$ is also a periodic function.*

A natural question would arise: what will happen if we replace the derivative of $\omega(z)$ with $\triangle_c\omega = \omega(z+c) - \omega(z)$, where c is a non-zero constant. We obtain the following results.

Theorem 1 *Let $\omega(z)$ be a transcendental entire function with $\rho_2(\omega) < 1$, and n, k be positive integers. Suppose that $\omega(z)$ has a finite non-zero Borel exceptional value l, and $\omega^n(z)(\omega(z+c) - \omega(z))^{(k)}$ is a periodic function with period c, then $\omega(z)$ is also a periodic function.*

Theorem 2 *Let $\omega(z)$ be a transcendental entire function with $\rho_2(\omega)<1$, and $n,m\,(\geqslant 1)$ be integers.*

(i) *If $n=2$ or $n\geqslant 4$ and $\omega^n(z)+b_1(\omega(z+c)-\omega(z))'+\cdots+b_m(\omega(z+c)-\omega(z))^{(m)}$ is a periodic function with period c, where $b_1,\dots,b_m$ are constants, then $\omega(z)$ is also a periodic function.*

(ii) *If $\omega^3(z)+b_1(\omega(z+c)-\omega(z))'+\cdots+b_m(\omega(z+c)-\omega(z))^{(m)}$ is a periodic function with period c, then $(\omega(z)-\delta\omega(z+c))(\omega(z)-\delta^2\omega(z+c))$ is a periodic function, where $\delta\,(\neq 1)$ is a cube-root of the unity.*

Remark 1 Theorem 2 is not true for $n=1$. We know $\omega(z)=z\,\mathrm{e}^{-z}$ is not a periodic function, but

$$\omega(z)+\mathrm{e}[\omega(z+1)-\omega(z)]'+\mathrm{e}[\omega(z+1)-\omega(z)]'' + \frac{\mathrm{e}}{1-\mathrm{e}}[\omega(z+1)-\omega(z)]'''=\mathrm{e}^{-z}\,\frac{1-\mathrm{e}-\mathrm{e}^2}{1-\mathrm{e}}$$

is a periodic function.

We give two examples to illustrate the preceding theorems.

Example 1 $(\mathrm{e}^z+1)^n(\mathrm{e}^{z+c}+1-\mathrm{e}^z-1)^{(k)}=(\mathrm{e}^z+1)^n(\mathrm{e}^c-1)\mathrm{e}^z$ is a periodic function, here e^z+1 is a also periodic function.

Example 2 $(\mathrm{e}^z)^n+b_1(\mathrm{e}^{z+c}-\mathrm{e}^z)'+\cdots+b_m(\mathrm{e}^{z+c}-\mathrm{e}^z)^{(m)}$ is a periodic function, here e^z is a also periodic function.

PRELIMINARY LEMMAS

Lemma 1 ([10]) *Suppose that $\omega_j\,(j=1,2,\dots,n)$ $(n\geqslant 3)$ are meromorphic functions which are not constants except for ω_n. Furthermore, let*

$$\sum_{j=1}^{n}\omega_j=1.$$

If $\omega_n\not\equiv 0$ and

$$\sum_{j=1}^{n}N(r,\tfrac{1}{\omega_j})+(n-1)\sum_{j=1}^{n}\overline{N}(r,\omega_j)<(l+o(1))T(r,\omega_k),$$

where $r\in I$, I is a set whose linear measure is infinite, $k\in\{1,2,\dots,n-1\}$ and $l<1$, then $\omega_n\equiv 1$.

Lemma 2 ([11]) *Let ω be a non-constant meromorphic function with $\rho_2(\omega)<1$ and let c be a non-zero complex number. Then*

$$m\left(r,\frac{\omega(z+c)}{\omega(z)}\right)=S(r,\omega),$$

outside of a possible exceptional set with finite logarithmic measure.

Lemma 3 ([12]) *Let ω be a non-constant meromorphic function with $\rho(\omega)<\infty$ and let c be a non-zero complex number. Then for each ε, we have*

$$m\left(r,\frac{\omega(z+c)}{\omega(z)}\right)=O(r^{\rho(\omega)-1+\varepsilon}),$$

outside of a possible exceptional set with finite logarithmic measure.

By applying Lemma 2 and the logarithmic derivative Lemma, we can obtain the following result.

Lemma 4 *Let ω be a non-constant meromorphic function with $\rho_2(\omega)<1$ and let c be a non-zero complex number and k be a positive integer. Then*

$$m\left(r,\frac{\omega^{(k)}(z+c)}{\omega(z)}\right)=S(r,\omega),$$

outside of a possible exceptional set with finite logarithmic measure.

Lemma 5 ([10], Lemma 5.1) *Let ω denote a non-constant periodic function. Then $\sigma(\omega)\geqslant 1$.*

PROOF OF Theorem 1

Suppose $\omega(z)$ has a finite non-zero Borel exceptional value l. Then by the Hadamard factorization theorem, it follows that

$$\omega(z)-l=U(z)\,\mathrm{e}^{V(z)}, \tag{1}$$

where $U(z)$ is canonical product ($U(z)$ may be a polynomial) formed by zeros of ω, $V(z)$ is non-constant entire function such that $\sigma(U)=\lambda(U)=\lambda(\omega-l)<\sigma(\omega-l)=\sigma(\omega)=\sigma(\mathrm{e}^{V(z)})$. Assume that $(\omega(z))^n(\triangle_c\omega)^{(k)}$ is a periodic function with period c. Thus

$$(\omega(z))^n(\triangle_c\omega)^{(k)}=(\omega(z+c))^n(\omega(z+2c)-\omega(z+c))^{(k)}. \tag{2}$$

Together (1) with (2), we have

$$\begin{aligned}(U(z)\,\mathrm{e}^{V(z)}+l)^n(\mathrm{e}^{V(z+c)}G_1(z+c)-\mathrm{e}^{V(z)}G_1(z))\\ =(U(z+c)\,\mathrm{e}^{V(z+c)}+l)^n(\mathrm{e}^{V(z+2c)}G_1(z+2c)\\ -\mathrm{e}^{V(z+c)}G_1(z+c)),\end{aligned} \tag{3}$$

where $G_1(z)=U^{(k)}(z)+kU^{(k-1)}(z)V'(z)+B_2(z)U^{(k-2)}(z)V''(z)+\cdots+B_k(z)U(z)$, when $B_j\,(j=2,\dots,k)$ are polynomials formed by $V(z)$ and its derivatives. By the expression of $G_1(z)$, we have

$$\sigma(G_1(z))\leqslant\max\{\sigma(U(z)),\sigma(V(z))\}<\sigma(\omega(z)).$$

Eq. (3) implies that

$$\begin{aligned}&\big(U(z)^n\,\mathrm{e}^{nV(z)}+C_n^1 lU(z)^{n-1}\,\mathrm{e}^{(n-1)V(z)}+\cdots+l^n\big)\\ &\big(\mathrm{e}^{V(z+c)}G_1(z+c)-\mathrm{e}^{V(z)}G_1(z)\big)\\ &=\big(U(z+c)^n\mathrm{e}^{nV(z+c)}+C_n^1 lU(z+c)^{n-1}\mathrm{e}^{(n-1)V(z+c)}+\cdots+l^n\big)\\ &(\mathrm{e}^{V(z+2c)}G_1(z+2c)-\mathrm{e}^{V(z+c)}G_1(z+c)).\end{aligned} \tag{4}$$

Obviously,

$$l^n(e^{V(z+c)}G_1(z+c)-e^{V(z)}G_1(z))\not\equiv 0.$$

Otherwise, if

$$l^n(e^{V(z+c)}G_1(z+c)-e^{V(z)}G_1(z))\equiv 0,$$

then

$$\frac{e^{V(z+c)}}{e^{V(z)}}\equiv\frac{G_1(z)}{G_1(z+c)},$$

and Lemma 3 implies that

$$m\left(r,\tfrac{e^{V(z+c)}}{e^{V(z)}}\right)=O\left(r^{\sigma(\omega(z))-1+\varepsilon}\right),$$
$$m\left(r,\tfrac{G_1(z)}{G_1(z+c)}\right)=O\left(r^{\sigma(G_1(z)-1+\varepsilon)}\right),$$

a contradiction with $\sigma(\omega(z))>\sigma(G_1(z))$. Hence

$$l^n(e^{V(z+c)}G_1(z+c)-e^{V(z)}G_1(z))\not\equiv 0.$$

Dividing both sides of (4) by $l^n(e^{V(z+c)}G_1(z+c)-e^{V(z)}G_1(z))$,

$$\begin{aligned}&\frac{e^{V(z+2c)}G_1(z+2c)-e^{V(z+c)}G_1(z+c)}{l^n(e^{V(z+c)}G_1(z+c)-e^{V(z)}G_1(z))}U(z+c)^n e^{nV(z+c)}\\&+C_n^1\frac{e^{V(z+2c)}G_1(z+2c)-e^{V(z+c)}G_1(z+c)}{l^{n-1}(e^{V(z+c)}G_1(z+c)-e^{V(z)}G_1(z))}U(z+c)^{n-1}\\&e^{(n-1)V(z+c)}+\cdots+\frac{e^{V(z+2c)}G_1(z+2c)-e^{V(z+c)}G_1(z+c)}{e^{V(z+c)}G_1(z+c)-e^{V(z)}G_1(z)}\\&-\frac{U(z)^n e^{nV(z)}}{l^n}-\frac{C_n^1U(z)^{n-1}e^{(n-1)V(z)}}{l^{n-1}}-\cdots\\&-\frac{C_n^{n-1}U(z)e^{V(z)}}{l}=1.\end{aligned}\tag{5}$$

By Lemma 1 and (5), we have

$$\frac{e^{V(z+2c)}G_1(z+2c)-e^{V(z+c)}G_1(z+c)}{e^{V(z+c)}G_1(z+c)-e^{V(z)}G_1(z)}\equiv 1.$$

That is

$$\begin{aligned}&e^{V(z+2c)}G_1(z+2c)-e^{V(z+c)}G_1(z+c)\\&=e^{V(z+c)}G_1(z+c)-e^{V(z)}G_1(z).\end{aligned}\tag{6}$$

By (3) and (6), we have

$$(U(z)e^{V(z)}+l)^n=(U(z+c)e^{V(z+c)}+l)^n.\tag{7}$$

Eqs. (1) and (7) imply that

$$(\omega(z))^n=(\omega(z+c))^n.$$

By this, we know ω is a periodic function with period c or nc. Hence Theorem 1 holds.

PROOF OF Theorem 2

Since $\omega^n(z)+b_1(\omega(z+c)-\omega(z))'+\cdots+b_m(\omega(z+c)-\omega(z))^{(m)}$ is a periodic function with period c, then we have

$$\begin{aligned}&\omega^n(z)+b_1(\omega(z+c)-\omega(z))'+\cdots+b_m(\omega(z+c)-\omega(z))^{(m)}\\&=\omega^n(z+c)+b_1(\omega(z+2c)-\omega(z+c))'\\&+\cdots+b_m(\omega(z+2c)-\omega(z+c))^{(m)}.\end{aligned}\tag{8}$$

We next consider the following three cases separately. **Case 1**: If $n=2$, then (8) can be written as follows:

$$\begin{aligned}&(\omega(z)-\omega(z+c))(\omega(z)+\omega(z+c))\\&=b_1(\omega(z+2c)-\omega(z+c)-(\omega(z+c)-\omega(z)))'+\cdots\\&+b_m(\omega(z+2c)-\omega(z+c)-(\omega(z+c)-\omega(z)))^{(m)}.\end{aligned}\tag{9}$$

If $\omega(z)-\omega(z+c)\equiv 0$, then ω is a periodic function with period c.

We next consider the case that $\omega(z)-\omega(z+c)\not\equiv 0$. Dividing both sides of (9) by $\omega(z)-\omega(z+c)$, then we have

$$\begin{aligned}&\omega(z)+\omega(z+c)\\&=b_1\left(\frac{(\omega(z+2c)-\omega(z+c))'}{\omega(z)-\omega(z+c)}-\frac{(\omega(z+c)-\omega(z))'}{\omega(z)-\omega(z+c)}\right)+\cdots\\&+b_m\left(\frac{(\omega(z+2c)-\omega(z+c))^{(m)}}{\omega(z)-\omega(z+c)}-\frac{(\omega(z+c)-\omega(z))^{(m)}}{\omega(z)-\omega(z+c)}\right)\\&=-b_1\left(\frac{\eta'(z+c)}{\eta(z)}-\frac{\eta'(z)}{\eta(z)}\right)-\cdots\\&-b_m\left(\frac{\eta^{(m)}(z+c)}{\eta(z)}-\frac{\eta^{(m)}(z)}{\eta(z)}\right),\end{aligned}\tag{10}$$

where

$$\eta(z)=\omega(z)-\omega(z+c).\tag{11}$$

Let

$$\begin{aligned}\chi(z)=&-b_1\left(\frac{\eta'(z+c)}{\eta(z)}-\frac{\eta'(z)}{\eta(z)}\right)-\cdots\\&-b_m\left(\frac{\eta^{(m)}(z+c)}{\eta(z)}-\frac{\eta^{(m)}(z)}{\eta(z)}\right).\end{aligned}\tag{12}$$

By Lemma 4, we have

$$\begin{aligned}&T(r,\chi(z))=m(r,\chi(z))\\&\leqslant m\left(r,\frac{\eta'(z+c)}{\eta(z)}\right)+m\left(r,\frac{\eta'(z)}{\eta(z)}\right)+\cdots\\&+m\left(r,\frac{\eta^{(m)}(z+c)}{\eta(z)}\right)+m\left(r,\frac{\eta^{(m)}(z)}{\eta(z)}\right)+O(1)\\&\leqslant S(r,\eta(z)).\end{aligned}\tag{13}$$

Together (10) with (12), we obtain

$$\omega(z)+\omega(z+c)=\chi(z).\tag{14}$$

Combining (11) and (14), we have

$$\omega(z)=\tfrac{1}{2}(\chi(z)+\eta(z)),$$

and

$$\omega(z+c)=\tfrac{1}{2}(\chi(z)-\eta(z))=\tfrac{1}{2}(\chi(z+c)+\eta(z+c)).$$

By this, hence we have

$$\eta(z)+\eta(z+c)=\chi(z)-\chi(z+c). \tag{15}$$

Eq. (15) implies that

$$\eta^{(j)}(z)+\eta^{(j)}(z+c)=\chi^{(j)}(z)-\chi^{(j)}(z+c). \tag{16}$$

Together (12) with (16), we have

$$\begin{aligned}\chi(z)\eta(z)&+\chi(z+c)\eta(z+c)\\ &=-b_1(\eta'(z+c)-\eta'(z)+\eta'(z+2c)-\eta'(z+c))-\cdots\\ &-b_m(\eta^{(m)}(z+c)-\eta^{(m)}(z)+\eta^{(m)}(z+2c)-\eta^{(m)}(z+c))\\ &=-b_1(\eta'(z+2c)+\eta'(z+c)-(\eta'(z+c)+\eta'(z)))-\cdots\\ &-b_m(\eta^{(m)}(z+c)+\eta^{(m)}(z+2c)-(\eta^{(m)}(z+c)+\eta^{(m)}(z)))\\ &=-b_1(\chi'(z+c)-\chi'(z+2c)-(\chi'(z)-\chi'(z+c)))-\cdots\\ -b_m(\chi^{(m)}&(z+c)-\chi^{(m)}(z+2c)-(\chi^{(m)}(z)-\chi^{(m)}(z+c))).\end{aligned} \tag{17}$$

By (17) and (15), we have

$$\begin{aligned}\chi(z)\eta(z)&+\chi(z+c)(\chi(z)-\chi(z+c)-\eta(z))\\ &=-b_1(\chi'(z+c)-\chi'(z+2c)-(\chi'(z)-\chi'(z+c)))-\cdots\\ -b_m(\chi^{(m)}&(z+c)-\chi^{(m)}(z+2c)-(\chi^{(m)}(z)-\chi^{(m)}(z+c))).\end{aligned} \tag{18}$$

Next we show that $\chi(z)=\chi(z+c)$. If $\chi(z)\not\equiv\chi(z+c)$, then by (18), we have

$$\begin{aligned}\eta(z)=&\frac{-b_1(\chi'(z+c)-\chi'(z+2c)-(\chi'(z)-\chi'(z+c)))}{\chi(z)-\chi(z+c)}+\cdots\\ &+\frac{-b_m(\chi^{(m)}(z+c)-\chi^{(m)}(z+2c)-(\chi^{(m)}(z)-\chi^{(m)}(z+c)))}{\chi(z)-\chi(z+c)}\\ &-\chi(z+c).\end{aligned} \tag{19}$$

By (13), (19) and Lemma 4, we have

$$T(r,\eta(z))\leqslant S(r,\eta(z)),$$

a contradiction. Hence, we have $\chi(z)=\chi(z+c)$. Together with (15), we have $\eta(z)=-\eta(z+c)$. So we know $\omega(z)$ is a periodic function with period $2c$.

Case 2: $n=3$. Rewriting (8) as follows

$$\begin{aligned}(\omega(z)-\omega(z+c))&(\omega(z)-\delta\omega(z+c))(\omega(z)-\delta^2\omega(z+c))\\ &=b_1(\omega(z+2c)-\omega(z+c)-(\omega(z+c)-\omega(z)))'+\cdots\\ &+b_m(\omega(z+2c)-\omega(z+c)-(\omega(z+c)-\omega(z)))^{(m)},\end{aligned} \tag{20}$$

where $\delta(\neq 1)$ is a cube-root of the unity. If $\omega(z)\equiv\omega(z+c)$, then $(\omega(z)-\delta\omega(z+c))(\omega(z)-\delta^2\omega(z+c)$ is a periodic function with period c. If $\omega(z)\not\equiv\omega(z+c)$, we can write (20) as follows.

$$\begin{aligned}(\omega(z)&-\delta\omega(z+c))(\omega(z)-\delta^2\omega(z+c))\\ &=\frac{b_1(\omega(z+2c)-\omega(z+c)-(\omega(z+c)-\omega(z)))'}{\omega(z)-\omega(z+c)}+\cdots\\ &+\frac{b_m(\omega(z+2c)-\omega(z+c)-(\omega(z+c)-\omega(z)))^{(m)}}{\omega(z)-\omega(z+c)}.\end{aligned} \tag{21}$$

If $\chi(z)\equiv 0$, (12), (13) and (21) imply that $\omega(z)-\delta\omega(z+c)\equiv 0$ or $\omega(z)-\delta^2\omega(z+c)\equiv 0$, hence $\omega(z)$ is a periodic function with period $3c$. If $\chi(z)\not\equiv 0$, by the Hadamard factorization theorem, we have

$$\omega(z)-\delta\omega(z+c)=P_1(z)\,\mathrm{e}^{Q(z)}, \tag{22}$$

and

$$\omega(z)-\delta^2\omega(z+c)=P_2(z)\,\mathrm{e}^{-Q(z)}, \tag{23}$$

where $Q(z)$ is a non-constant entire function with $\sigma(Q)<1$, $T(r,P_i)=S(r,\omega(z))$, $i=1,2$. Eqs. (22) and (23) imply that

$$\omega(z)=\frac{\delta P_1(z)\,\mathrm{e}^{Q(z)}-P_2(z)\,\mathrm{e}^{-Q(z)}}{\delta-1}, \tag{24}$$

$$\begin{aligned}\omega(z+c)&=\frac{P_1(z)\,\mathrm{e}^{Q(z)}-P_2(z)\,\mathrm{e}^{-Q(z)}}{\delta(\delta-1)}\\ &=\frac{\delta P_1(z+c)\,\mathrm{e}^{Q(z+c)}-P_2(z+c)\,\mathrm{e}^{-Q(z+c)}}{\delta-1}.\end{aligned} \tag{25}$$

Eq. (25) implies that

$$\begin{aligned}\delta^2P_1(z+c)\,\mathrm{e}^{Q(z+c)}-\delta P_2(z+c)\,\mathrm{e}^{-Q(z+c)}&-P_1(z)\,\mathrm{e}^{Q(z)}\\ &+P_2(z)\,\mathrm{e}^{-Q(z)}=0.\end{aligned} \tag{26}$$

That is

$$\begin{aligned}-\delta^2\frac{P_1(z+c)}{P_2(z)}\,\mathrm{e}^{Q(z+c)+Q(z)}&+\delta\frac{P_2(z+c)}{P_2(z)}\,\mathrm{e}^{-Q(z+c)+Q(z)}\\ &+\frac{P_1(z)}{P_2(z)}\,\mathrm{e}^{2Q(z)}=1.\end{aligned} \tag{27}$$

We assume that $Q(z)+Q(z+c)$ is not a constant. Otherwise, if $Q(z)+Q(z+c)$ is a constant, then $Q'(z)$ is a periodic function with periodic $2c$, Lemma 5 implies that $\sigma(Q(z))=\sigma(Q'(z))\geqslant 1$, a contradiction. So $Q(z)+Q(z+c)$ is not a constant. By Lemma 1 and (27), we have

$$\delta\frac{P_2(z+c)}{P_2(z)}\,\mathrm{e}^{-Q(z+c)+Q(z)}\equiv 1. \tag{28}$$

On the other hand, dividing (26) by $P_1(z)\,\mathrm{e}^{Q(z)}$, we have

$$\begin{aligned}\delta^2\frac{P_1(z+c)}{P_1(z)}\,\mathrm{e}^{Q(z+c)-Q(z)}&-\delta\frac{P_2(z+c)}{P_1(z)}\,\mathrm{e}^{-Q(z)-Q(z+c)}\\ &+\frac{P_2(z)}{P_1(z)}\,\mathrm{e}^{-2Q(z)}=1.\end{aligned} \tag{29}$$

By Lemma 1 and (29), we have

$$\delta^2 \frac{P_1(z+c)}{P_1(z)} e^{Q(z+c)-Q(z)} = 1. \quad (30)$$

Eqs. (28) and (30) imply that

$$\delta^3 P_1(z+c) P_2(z+c) = P_1(z) P_2(z).$$

By this, (22) and (23), we have $(\omega(z) - \delta\omega(z+c))$ $(\omega(z) - \delta^2\omega(z+c))$.

If $n \geqslant 4$, then we can write (8) as follows.

$$\begin{aligned}&(\omega(z)-\omega(z+c))(\omega^{n-1}(z)+\omega^{n-2}(z)\omega(z+c)+\cdots\\&+\omega^{n-1}(z+c))\\&=b_1(\omega(z+2c)-\omega(z+c)-(\omega(z+c)-\omega(z)))'+\cdots\\&+b_m(\omega(z+2c)-\omega(z+c)-(\omega(z+c)-\omega(z)))^{(m)}. \quad (31)\end{aligned}$$

If $\omega(z)-\omega(z+c) \equiv 0$, then $\omega(z)$ is a periodic function with period c. If $\omega(z)-\omega(z+c) \not\equiv 0$, we can write (31) as follows.

$$\begin{aligned}&\omega^{n-1}(z)+\omega^{n-2}(z)\omega(z+c)+\cdots+\omega^{n-1}(z+c)\\&=\frac{b_1(\omega(z+2c)-\omega(z+c)-(\omega(z+c)-\omega(z)))'}{\omega(z)-\omega(z+c)}+\cdots\\&+\frac{b_m(\omega(z+2c)-\omega(z+c)-(\omega(z+c)-\omega(z)))^{(m)}}{\omega(z)-\omega(z+c)}. \quad (32)\end{aligned}$$

Set

$$\begin{aligned}\chi(z)=&\frac{b_1(\omega(z+2c)-\omega(z+c)-(\omega(z+c)-\omega(z)))'}{\omega(z)-\omega(z+c)}+\cdots\\&+\frac{b_m(\omega(z+2c)-\omega(z+c)-(\omega(z+c)-\omega(z)))^{(m)}}{\omega(z)-\omega(z+c)}. \quad (33)\end{aligned}$$

Let $L(z) = \frac{\omega(z+c)}{\omega(z)}$. If $\frac{\omega(z+c)}{\omega(z)} \equiv 1$, then $\omega(z)$ is a periodic function with periodic c. If $\frac{\omega(z+c)}{\omega(z)} \not\equiv 1$, and $L(z)$ is not a constant. Eq. (13) implies that

$$\eta(z) = (1 - \frac{\omega(z+c)}{\omega(z)})\omega(z). \quad (34)$$

Eqs. (32) and (33) imply that

$$\begin{aligned}\chi(z) = \omega^{n-1}(z)\bigg(&1 + \frac{\omega(z+c)}{\omega(z)} + \cdots\\&+ \frac{\omega^{n-2}(z+c)}{\omega^{n-2}(z)} + \frac{\omega^{n-1}(z+c)}{\omega^{n-1}(z)}\bigg). \quad (35)\end{aligned}$$

Together (34) with (35), we have

$$\frac{(1-L(z))^{n-1}}{L^{n-1}(z)+L^{n-2}(z)+\cdots+L(z)+1} = \frac{\eta^{n-1}(z)}{\chi(z)}. \quad (36)$$

By (36), we have

$$(n-1)T(r,\omega(z)) = (n-1)T(r,\eta(z)+S(r,\eta(z)),$$

$$\begin{aligned}N\left(r,\tfrac{1}{L^{n-1}(z)+L^{n-2}(z)+\cdots+L(z)+1}\right) &= N\left(r,\tfrac{1}{\chi(z)}\right)\\&\leqslant T(r,\eta(z)) = S(r,\eta(z)).\end{aligned}$$

Using the second main theorem of Nevanlinna theory, we obtain

$$\begin{aligned}&(n-2)T(r,L)\\&\leqslant N\left(r,\tfrac{1}{L-1}\right)+N\left(r,\tfrac{1}{L^{n-1}(z)+L^{n-2}(z)+\cdots+L(z)+1}\right)+S(r,L)\\&=N\left(r,\tfrac{1}{L-1}\right)+S(r,L)\leqslant T\left(r,\tfrac{1}{L-1}\right)+S(r,L),\end{aligned}$$

which is impossible for $n \geqslant 4$. Hence we obtain that L must be a constant and $L(z) \not\equiv 1$. By (34), we have

$$T(r,\eta(z)) = T(r,\omega(z)) + S(r,\omega(z)). \quad (37)$$

Eq. (35) implies that

$$(n-1)T(r,\omega(z)) = T(r,\chi(z)) + S(r,\omega(z)) = S(r,\omega(z)),$$

which is a contradiction.

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