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Stability for a general form of an alternative functional equation related to the Jensen's functional equation

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ABSTRACT: Given real numbers α,β,γ such that $(\alpha,\beta,\gamma)\neq(k,-2k,k)$ for all $k\in\mathbb{R}$ and $(\beta,\gamma)\notin\{(0,\alpha),(\alpha,\alpha),(\alpha+\gamma,\gamma)\}$, we investigate the stability of an alternative Jensen's functional equation of the form

$$f(xy^{-1})-2f(x)+f(xy)=0 \quad\text{or}\quad \alpha f(xy^{-1})+\beta f(x)+\gamma f(xy)=0,$$

where f is a mapping from an abelian group to a Banach space.

KEYWORDS: stability, alternative equation, Jensen's functional equation

MSC2020: 39B82 39B72

INTRODUCTION

The problem of alternative Cauchy functional equations has been studied by various authors (e.g., Kannappan et al [1], Ger [2] and Forti [3]). The Jensen's functional equation is a famous equation that is closely related to the Cauchy functional equation. Nakmahachalasint [4] first solved an alternative Jensen's functional equation of the form

$$f(x)\pm 2f(xy)+f(xy^2)=0 \tag{1}$$

on a semigroup. His research extended the work of Ng [5] and the work of Parnami et al [6] on the classical Jensen's functional equation

$$f(xy^{-1})-2f(x)+f(xy)=0 \tag{2}$$

on a group. The Hyers-Ulam stability (Hyers [7], Aoki [8], Bourgin [9], Rassias [10] and Gavruta [11]) of the alternative Jensen's functional equation (1) was proved by Nakmahachalasint [12].

Kitisin et al [13] establish a criterion for existence of the general solution to the alternative Jensen's functional equation of the form

$$\begin{aligned} &f(xy^{-1})-2f(x)+f(xy)=0 \quad\text{or}\\ &\alpha f(xy^{-1})+\beta f(x)+\gamma f(xy)=0, \end{aligned} \tag{3}$$

where f is a mapping from a group to a uniquely divisible abelian group, but its stability problem has not yet been investigated.

In this paper, we will prove the Hyers-Ulam stability of the alternative Jensen's functional equation (3) when α,β and γ are real numbers with

$$\begin{aligned} &(\alpha,\beta,\gamma)\neq(k,-2k,k) \text{ for all } k\in\mathbb{R} \quad\text{and}\\ &(\beta,\gamma)\notin\{(0,\alpha),(\alpha,\alpha),(\alpha+\gamma,\gamma)\} \end{aligned} \tag{4}$$

and f is a mapping from an abelian group $(G,\cdot)$ to a Banach space $(E,\|\cdot\|)$. In other words, we will prove that for every $\varepsilon\geqslant 0$, there exist $\delta_1,\delta_2\geqslant 0$ such that if a mapping $f:G\to E$ satisfies the inequalities

$$\begin{aligned} &\|f(xy^{-1})-2f(x)+f(xy)\|\leqslant\delta_1 \quad\text{or}\\ &\|\alpha f(xy^{-1})+\beta f(x)+\gamma f(xy)\|\leqslant\delta_2 \end{aligned} \tag{5}$$

for every $x,y\in G$, where α,β and γ are fixed real numbers with (4), then there exists a unique Jensen's mapping $J:G\to E$ with

$$\|f(x)-J(x)\|\leqslant\varepsilon$$

for all $x\in G$.

It should be noted that Kitisin et al [13] proved that if α,β and γ are integers satisfying (4), then the alternative Jensen's functional equation (3) is equivalent to Jensen's functional equation (2). On the other hand, when $(\beta,\gamma)\in\{(0,\alpha),(\alpha,\alpha),(\alpha+\gamma,\gamma)\}$, (3) is not necessarily equivalent to (2).

AUXILIARY LEMMAS

Let $(G,+)$ be a group and let E be a Banach space. Given real numbers α,β,γ as in (4) and a function $f:G\to E$. For every pair of $x,y\in G$, we will define

$$\mathscr{F}_y^{(\alpha,\beta,\gamma)}(x)=\|\alpha f(xy^{-1})+\beta f(x)+\gamma f(xy)\|$$

and

$$\mathscr{J}_y(x)=\|f(xy^{-1})-2f(x)+f(xy)\|.$$

For $\delta_1,\delta_2\geqslant 0$, we let

$$\mathscr{P}f_y^{(\alpha,\beta,\gamma)}(x)=\left(\mathscr{J}_y(x)\leqslant\delta_1 \text{ or } \mathscr{F}_y^{(\alpha,\beta,\gamma)}(x)\leqslant\delta_2\right)$$

and

$$\delta = \max\{\delta_1, \delta_2\}.$$

The set of solution to the statement $\mathscr{P} f_y^{(\alpha,\beta,\gamma)}(x)$ will be denoted by $\mathscr{A}_{(G,E)}^{(\alpha,\beta,\gamma)}$, i.e.,

$$\mathscr{A}_{(G,E)}^{(\alpha,\beta,\gamma)} = \{f : G \to E \mid \mathscr{P} f_y^{(\alpha,\beta,\gamma)}(x) \text{ for all } x, y \in G\}.$$

For each real number λ, we let

$$\mathscr{M}(\lambda) = \begin{cases} |\lambda|^{-1} & \text{if} \quad 0 < |\lambda| < 1; \\ |\lambda| & \text{if} \quad |\lambda| \geqslant 1; \\ 1 & \text{if} \quad \lambda = 0. \end{cases}$$

It should be noted that for every $\lambda \in \mathbb{R}$,
(i) $1 \leqslant \mathscr{M}(\lambda)$;
(ii) $|\lambda| \leqslant \mathscr{M}(\lambda)$;
(iii) $|\lambda|^{-1} \leqslant \mathscr{M}(\lambda)$ if $\lambda \neq 0$.
We denote $\Lambda = \{-3, -2, -1, 0, 1, 2, 3\}$ and

$$M = \max_{\sigma_1,\sigma_2,\sigma_3,\sigma_4 \in \Lambda} \{\mathscr{M}(\sigma_1\alpha + \sigma_2\beta + \sigma_3\gamma + \sigma_4)\}.$$

The above notations will be used extensively in the proofs below, and thus should be kept in mind.

First, we will give the bound of $\mathscr{J}_y(x)$ for a function $f \in \mathscr{A}_{(G,E)}^{(0,\beta,0)}$.

Lemma 1 *If $f \in \mathscr{A}_{(G,E)}^{(0,\beta,0)}$ and $x, y \in G$, then $\mathscr{J}_y(x) \leqslant 12M\delta$.*

Proof: Let $f \in \mathscr{A}_{(G,E)}^{(0,\beta,0)}$ and $x, y \in G$. By (4), we must have $\beta \neq 0$. Suppose $\mathscr{J}_y(x) > \delta_1$. Hence $\mathscr{F}_y^{(0,\beta,0)}(x) \leqslant \delta_2$ and we get $\|f(x)\| \leqslant M\delta$. Next, we will consider the alternatives in $\mathscr{P} f_y^{(0,\beta,0)}(xy^{-1})$ as follows.

Case (i). Assume that $\mathscr{J}_y(xy^{-1}) \leqslant \delta_1$. By $\|f(x)\| \leqslant M\delta$, we have

$$\|f(xy^{-2}) - 2f(xy^{-1})\| \leqslant 2M\delta. \tag{6}$$

By (6) and the alternatives in $\mathscr{P} f_y^{(0,\beta,0)}(xy^{-2})$, we have

$$\begin{gathered}\|f(xy^{-3}) - 3f(xy^{-1})\| \leqslant 5M\delta \text{ or} \\ \|f(xy^{-1})\| \leqslant 2M\delta.\end{gathered} \tag{7}$$

By (7) and the alternatives in $\mathscr{P} f_{y^2}^{(0,\beta,0)}(xy^{-1})$, we get

$$\begin{gathered}\|f(xy^{-1}) + f(xy)\| \leqslant 6M\delta \text{ or} \\ \|f(xy^{-1})\| \leqslant 2M\delta.\end{gathered} \tag{8}$$

If $\|f(xy^{-1}) + f(xy)\| \leqslant 6M\delta$, then by $\|f(xy^{-1}) + f(xy)\| \leqslant 6M\delta$ and $\|f(x)\| \leqslant M\delta$, we obtain $\mathscr{J}_y(x) \leqslant 8M\delta$. It remains to consider the case when $\|f(xy^{-1})\| \leqslant 2M\delta$. By the alternatives in $\mathscr{P} f_y^{(0,\beta,0)}(xy)$ and $\|f(x)\| \leqslant M\delta$, we have

$$\|2f(xy) - f(xy^2)\| \leqslant 2M\delta \text{ or } \|f(xy)\| \leqslant M\delta. \tag{9}$$

By the alternatives in $\mathscr{P} f_y^{(0,\beta,0)}(xy^2)$ and (9), we get

$$\begin{gathered}\|3f(xy) - f(xy^3)\| \leqslant 5M\delta \text{ or} \\ \|f(xy)\| \leqslant 2M\delta.\end{gathered} \tag{10}$$

By $\|f(xy^{-1})\| \leqslant 2M\delta$ and (10), the alternatives in $\mathscr{P} f_{y^2}^{(0,\beta,0)}(xy)$ gives

$$\|f(xy)\| \leqslant 8M\delta. \tag{11}$$

By $\|f(xy^{-1})\| \leqslant 2M\delta, \|f(x)\| \leqslant M\delta$ and (11), we get

$$\mathscr{J}_y(x) \leqslant 12M\delta. \tag{12}$$

Case (ii). Assume that $\mathscr{F}_y^{(0,\beta,0)}(xy^{-1}) \leqslant \delta_2$. We have $\|f(xy^{-1})\| \leqslant M\delta$. The proof is as in case (i) after referring the steps (9)–(12). □

Lemma 2 *Let $f \in \mathscr{A}_{(G,E)}^{(\alpha,\beta,\gamma)}$ with $\alpha \neq \gamma$ and $x, y \in G$. If $\mathscr{J}_y(x) > \delta_1$, then $\|f(xy^{-1}) - f(xy)\| \leqslant 2M\delta$.*

Proof: Assume that $\mathscr{J}_y(x) > \delta_1$. By the alternatives in $\mathscr{P} f_{y^{-1}}^{(\alpha,\beta,\gamma)}(x)$ and $\mathscr{P} f_y^{(\alpha,\beta,\gamma)}(x)$, we get $\mathscr{F}_{y^{-1}}^{(\alpha,\beta,\gamma)}(x) \leqslant \delta_2$ and $\mathscr{F}_y^{(\alpha,\beta,\gamma)}(x) \leqslant \delta_2$, respectively. Therefore,

$$\begin{aligned}&\|(\alpha-\gamma)(f(xy^{-1}) - f(xy))\| \\ &\leqslant \mathscr{F}_{y^{-1}}^{(\alpha,\beta,\gamma)}(x) + \mathscr{F}_y^{(\alpha,\beta,\gamma)}(x) \\ &\leqslant 2\delta.\end{aligned}$$

Since $\alpha \neq \gamma$, the proof is completed as desired. □

The above lemma states a necessary property for a function $f \in \mathscr{A}_{(G,E)}^{(\alpha,\beta,\gamma)}$ with $\alpha \neq \gamma$ in the case when $\mathscr{J}_y(x) > \delta_1$. Next, we will prove the bound of $\mathscr{J}_y(x)$ concerning the relation between $\mathscr{P} f_y^{(\alpha,\beta,\gamma)}(xy^{-1})$ and $\mathscr{P} f_y^{(\alpha,\beta,\gamma)}(x)$ with $\alpha \neq \gamma$ as in the following two lemmas.

Lemma 3 *Let $f \in \mathscr{A}_{(G,E)}^{(\alpha,\beta,\gamma)}$ with $\alpha \neq \gamma$ and $x, y \in G$. If $\mathscr{J}_y(xy^{-1}) > \delta_1$ and $\mathscr{J}_y(x) > \delta_1$, then $\mathscr{J}_y(x) \leqslant 34M^5\delta$.*

Proof: Assume that $\mathscr{J}_y(xy^{-1}) > \delta_1$ and $\mathscr{J}_y(x) > \delta_1$. By Lemma 2, we obtain that

$$\|f(xy^{-2}) - f(x)\| \leqslant 2M\delta \tag{13}$$

and

$$\|f(xy^{-1}) - f(xy)\| \leqslant 2M\delta. \tag{14}$$

From $\mathscr{J}_y(xy^{-1}) > \delta_1$ and $\mathscr{J}_y(x) > \delta_1$, the alternatives in $\mathscr{P} f_y^{(\alpha,\beta,\gamma)}(xy^{-1})$ and $\mathscr{P} f_y^{(\alpha,\beta,\gamma)}(x)$ gives $\mathscr{F}_y^{(\alpha,\beta,\gamma)}(xy^{-1}) \leqslant \delta_2$ and $\mathscr{F}_y^{(\alpha,\beta,\gamma)}(x) \leqslant \delta_2$, respectively. Eliminating $f(xy^{-2})$ from (13) and $\mathscr{F}_y^{(\alpha,\beta,\gamma)}(xy^{-1}) \leqslant \delta_2$, we get

$$\|\beta f(xy^{-1}) + (\alpha+\gamma)f(x)\| \leqslant 3M^2\delta. \tag{15}$$

By (14) and (15), we have

$$\|(\alpha+\gamma)f(x)+\beta f(xy)\| \leqslant 5M^2\delta. \tag{16}$$

Eliminating $f(xy^{-1})$ from (14) and $\mathscr{F}_y^{(\alpha,\beta,\gamma)}(x) \leqslant \delta_2$, we obtain

$$\|\beta f(x)+(\alpha+\gamma)f(xy)\| \leqslant 3M^2\delta. \tag{17}$$

We eliminate $f(xy)$ from (16) and (17) to get

$$\|(\beta-\alpha-\gamma)(\beta+\alpha+\gamma)f(x)\| \leqslant 8M^3\delta.$$

From $\beta \neq \alpha+\gamma$, we conclude that

$$\|(\beta+\alpha+\gamma)f(x)\| \leqslant 8M^4\delta. \tag{18}$$

First, we suppose $\beta \neq -\alpha-\gamma$. Hence (18) reduces to

$$\|f(x)\| \leqslant 8M^5\delta. \tag{19}$$

Eliminating $f(x)$ from (16) and (17), we conclude that

$$\|f(xy)\| \leqslant 8M^5\delta. \tag{20}$$

By (14), (19) and (20), we have

$$\mathscr{J}_y(x) \leqslant 34M^5\delta. \tag{21}$$

Next, we suppose $\beta = -\alpha-\gamma$. If $\alpha+\gamma = 0$, then $\beta = 0$ which contradicts $\beta \neq \alpha+\gamma$. Hence $\alpha+\gamma \neq 0$. Substituting $\beta = -\alpha-\gamma$ in (17), we get

$$\|(\alpha+\gamma)(f(x)-f(xy))\| \leqslant 3M^2\delta.$$

Thus we conclude that

$$\|f(x)-f(xy)\| \leqslant 3M^3\delta. \tag{22}$$

By (14) and (22), $\mathscr{J}_y(x) \leqslant 8M^3\delta \leqslant 34M^5\delta$. □

Lemma 4 *Let $f \in \mathscr{A}_{(G,E)}^{(\alpha,\beta,\gamma)}$ with $\alpha \neq \gamma$ and $x, y \in G$. If $\mathscr{J}_y(xy^{-1}) \leqslant \delta_1$ and $\mathscr{J}_y(x) > \delta_1$, then*

$$\mathscr{J}_y(x) \leqslant 56M^5\delta.$$

Proof: Assume that $\mathscr{J}_y(xy^{-1}) \leqslant \delta_1$ and $\mathscr{J}_y(x) > \delta_1$. By Lemma 2, we have

$$\|f(xy^{-1})-f(xy)\| \leqslant 2M\delta. \tag{23}$$

By $\mathscr{J}_y(x) > \delta_1$, the alternatives in $\mathscr{P}f_y^{(\alpha,\beta,\gamma)}(x)$ gives $\mathscr{F}_y^{(\alpha,\beta,\gamma)}(x) \leqslant \delta_2$. Eliminating $f(xy^{-1})$ from (23) and $\mathscr{F}_y^{(\alpha,\beta,\gamma)}(x) \leqslant \delta_2$, we obtain that

$$\|\beta f(x)+(\alpha+\gamma)f(xy)\| \leqslant 3M^2\delta. \tag{24}$$

We eliminate $f(xy^{-1})$ from (23) and $\mathscr{J}_y(xy^{-1}) \leqslant \delta_1$ to get

$$\|f(xy^{-2})+f(x)-2f(xy)\| \leqslant 5M\delta. \tag{25}$$

Next, we will consider the alternatives in $\mathscr{P}f_y^{(\alpha,\beta,\gamma)}(xy^{-2})$ as follows.

Case (i). Assume that $\mathscr{J}_y(xy^{-2}) > \delta_1$. By Lemma 2, we have

$$\|f(xy^{-3})-f(xy^{-1})\| \leqslant 2M\delta. \tag{26}$$

The alternatives in $\mathscr{P}f_y^{(\alpha,\beta,\gamma)}(xy^{-2})$ gives $\mathscr{F}_y^{(\alpha,\beta,\gamma)}(xy^{-2}) \leqslant \delta_2$. Eliminating $f(xy^{-3})$ from (26) and $\mathscr{F}_y^{(\alpha,\beta,\gamma)}(xy^{-2}) \leqslant \delta_2$, we get

$$\|\beta f(xy^{-2})+(\alpha+\gamma)f(xy^{-1})\| \leqslant 3M^2\delta. \tag{27}$$

By (23) and (27), we obtain

$$\|\beta f(xy^{-2})+(\alpha+\gamma)f(xy)\| \leqslant 5M^2\delta. \tag{28}$$

Eliminating $f(xy^{-2})$ from (25) and (28), we have

$$\|\beta f(x)-(\alpha+2\beta+\gamma)f(xy)\| \leqslant 10M^2\delta. \tag{29}$$

By (24) and (29), we obtain that

$$\|\beta(f(x)-f(xy))\| \leqslant 13M^2\delta. \tag{30}$$

If $\beta \neq 0$, then (30) reduces to

$$\|f(x)-f(xy)\| \leqslant 13M^3\delta. \tag{31}$$

By (23) and (31), we obtain that $\mathscr{J}_y(x) \leqslant 28M^3\delta$. If $\beta = 0$, then (4) gives $\alpha+\gamma \neq 0$. Thus (24) reduces to

$$\|f(xy)\| \leqslant 3M^3\delta. \tag{32}$$

By (25) and (32), we obtain that

$$\|f(xy^{-2})+f(x)\| \leqslant 11M^3\delta. \tag{33}$$

Next, we will consider two cases of $\mathscr{P}f_y^{(\alpha,0,\gamma)}(xy)$ as follows. If $\mathscr{J}_y(xy) \leqslant \delta_1$, then by (32), we get

$$\|f(x)+f(xy^2)\| \leqslant 7M^3\delta. \tag{34}$$

Eliminating $f(xy^{-2})$ and $f(xy^2)$ from (33), (34) and the alternatives in $\mathscr{P}f_{y^2}^{(\alpha,0,\gamma)}(x)$, we conclude that

$$\|f(x)\| \leqslant 19M^5\delta. \tag{35}$$

If $\mathscr{J}_y(xy) > \delta_1$, then we have $\mathscr{F}_y^{(\alpha,0,\gamma)}(xy) \leqslant \delta_2$. Since $\mathscr{J}_y(xy) > \delta_1$, Lemma 2 gives

$$\|f(x)-f(xy^2)\| \leqslant 2M\delta. \tag{36}$$

By $\mathscr{F}_y^{(\alpha,0,\gamma)}(xy) \leqslant \delta_2$ and (36), we get (35). By (23), (32) and (35), we obtain

$$\mathscr{J}_y(x) \leqslant 46M^5\delta. \tag{37}$$

Case (ii). Assume that $\mathscr{J}_y(xy^{-2}) \leqslant \delta_1$. Eliminating $f(xy^{-1})$ from (23) and $\mathscr{J}_y(xy^{-2}) \leqslant \delta_1$, we get

$$\|f(xy^{-3}) - 2f(xy^{-2}) + f(xy)\| \leqslant 3M\delta. \tag{38}$$

Eliminating $f(xy^{-2})$ from (25) and (38), we have

$$\|f(xy^{-3}) + 2f(x) - 3f(xy)\| \leqslant 13M\delta. \tag{39}$$

Eliminating $f(x)$ from (24) and (39), we obtain

$$\|\beta f(xy^{-3}) - (2\alpha + 3\beta + 2\gamma)f(xy)\| \leqslant 19M^2\delta. \tag{40}$$

Next, we will consider two cases of $\mathscr{P} f_{y^2}^{(\alpha,\beta,\gamma)}(xy^{-1})$ as follows. We first assume that $\mathscr{J}_{y^2}(xy^{-1}) \leqslant \delta_1$. Eliminating $f(xy^{-1})$ from (23) and $\mathscr{J}_{y^2}(xy^{-1}) \leqslant \delta_1$, we get

$$\|f(xy^{-3}) - f(xy)\| \leqslant 5M\delta. \tag{41}$$

By (40) and (41), we get

$$\|2(\beta + \alpha + \gamma)f(xy)\| \leqslant 24M^2\delta. \tag{42}$$

If $\beta \neq -\alpha - \gamma$, then (42) reduces to

$$\|f(xy)\| \leqslant 12M^3\delta. \tag{43}$$

By (24) and (43), we have

$$\|\beta f(x)\| \leqslant 15M^4\delta.$$

Suppose $\beta \neq 0$. We get

$$\|f(x)\| \leqslant 15M^5\delta. \tag{44}$$

By (23), (43) and (44), we obtain

$$\mathscr{J}_y(x) \leqslant 56M^5\delta. \tag{45}$$

Suppose $\beta = 0$. Repeating to the steps (32)–(36), we get (37). If $\beta = -\alpha - \gamma$, then $\alpha + \gamma \neq 0$. Thus (24) reduces to

$$\|f(x) - f(xy)\| \leqslant 3M^3\delta. \tag{46}$$

By (23) and (46), we conclude that (45). We next assume that $\mathscr{J}_{y^2}(xy^{-1}) > \delta_1$. Lemma 2 gives

$$\|f(xy^{-3}) - f(xy)\| \leqslant 2M\delta.$$

Repeating the steps (41)–(46), we obtain (45).

The desired results follows from the consideration of the above two cases. □

Next, we will prove the bound of $f(x)$ concerning the relation between $\mathscr{P} f_y^{(1,\beta,1)}(xy^{-1})$, $\mathscr{P} f_y^{(1,\beta,1)}(x)$ and $\mathscr{P} f_y^{(1,\beta,1)}(xy)$ as in the following two lemmas. It should be noted that $\beta \notin \{-2, 0, 1, 2\}$.

Lemma 5 *Let $f \in \mathscr{A}_{(G,E)}^{(1,\beta,1)}$ and let $x, y \in G$.*

(i) *If $\mathscr{J}_y(xy^{-1}) \leqslant \delta_1$, $\mathscr{F}_y^{(1,\beta,1)}(x) \leqslant \delta_2$ and $\mathscr{J}_y(xy) \leqslant \delta_1$, then $\|f(x)\| \leqslant 5M\delta$.*
(ii) *If $\mathscr{F}_y^{(1,\beta,1)}(xy^{-1}) \leqslant \delta_2$, $\mathscr{F}_y^{(1,\beta,1)}(x) \leqslant \delta_2$ and $\mathscr{F}_y^{(1,\beta,1)}(xy) \leqslant \delta_2$, then $\|f(x)\| \leqslant 4M^3\delta$.*

Proof: Assume that all assumptions in the lemma hold.

(i) We observe that

$$\begin{aligned}&\|f(xy^{-2}) + (2+2\beta)f(x) + f(xy^2)\| \\ &\leqslant \mathscr{J}_y(xy^{-1}) + 2\mathscr{F}_y^{(1,\beta,1)}(x) + \mathscr{J}_y(xy) \\ &\leqslant 4\delta.\end{aligned} \tag{47}$$

Consider the alternatives in $\mathscr{P} f_{y^2}^{(1,\beta,1)}(x)$. The inequality $\mathscr{J}_{y^2}(x) \leqslant \delta_1$ and (47) give

$$\|(4 + 2\beta)f(x)\| \leqslant 5\delta,$$

while the inequality $\mathscr{F}_{y^2}^{(1,\beta,1)}(x) \leqslant \delta_2$ and (47) also give

$$\|(2 + \beta)f(x)\| \leqslant 5\delta.$$

Hence $\|f(x)\| \leqslant 5M\delta$.

(ii) We observe that

$$\begin{aligned}&\|f(xy^{-2}) + (2-\beta^2)f(x) + f(xy^2)\| \\ &\leqslant \mathscr{F}_y^{(1,\beta,1)}(xy^{-1}) + |\beta|\mathscr{F}_y^{(1,\beta,1)}(x) \\ &\quad + \mathscr{F}_y^{(1,\beta,1)}(xy) \\ &\leqslant 3M\delta.\end{aligned} \tag{48}$$

Consider the alternatives in $\mathscr{P} f_{y^2}^{(1,\beta,1)}(x)$. The inequality $\mathscr{J}_{y^2}(x) \leqslant \delta_1$ and (48) give

$$\|(4 - \beta^2)f(x)\| \leqslant 4M\delta,$$

while the inequality $\mathscr{F}_{y^2}^{(1,\beta,1)}(x) \leqslant \delta_2$ and (48) also give

$$\|(2 - \beta - \beta^2)f(x)\| \leqslant 4M\delta.$$

Hence $\|f(x)\| \leqslant 4M^3\delta$.

□

Lemma 6 *Let $f \in \mathscr{A}_{(G,E)}^{(1,\beta,1)}$ and let $x, y \in G$. If $\mathscr{J}_y(xy^{-1}) \leqslant \delta_1$, $\mathscr{F}_y^{(1,\beta,1)}(x) \leqslant \delta_2$ and $\mathscr{F}_y^{(1,\beta,1)}(xy) \leqslant \delta_2$, then $\|f(x)\| \leqslant 46M^7\delta$.*

Proof: Assume that the assumption in the lemma holds. By $\mathscr{J}_y(xy^{-1}) \leqslant \delta_1$ and $\mathscr{F}_y^{(1,\beta,1)}(x) \leqslant \delta_2$, we get

$$\|f(xy^{-2}) + (1 + 2\beta)f(x) + 2f(xy)\| \leqslant 3\delta. \tag{49}$$

Next, we will consider two possible cases in $\mathscr{P} f_{y^2}^{(1,\beta,1)}(x)$ as follows.

Case (i). Assume that $\mathscr{J}_{y^2}(x) \leqslant \delta_1$. Using $\mathscr{F}_y^{(1,\beta,1)}(xy) \leqslant \delta_2$, $\mathscr{J}_{y^2}(x) \leqslant \delta_1$ and (49), we obtain

$$\|2f(x) + f(xy)\| \leqslant 5M\delta \tag{50}$$

and

$$\|(1-2\beta)f(x)+f(xy^2)\| \leqslant 6M^2\delta. \tag{51}$$

Eliminating $f(xy)$ from (50) and the alternatives in $\mathscr{P}f_y^{(1,\beta,1)}(xy^2)$, we have

$$\begin{aligned}&\|2f(x)+2f(xy^2)-f(xy^3)\| \leqslant 6M\delta \text{ or}\\&\|2f(x)-\beta f(xy^2)-f(xy^3)\| \leqslant 6M\delta.\end{aligned} \tag{52}$$

By (51) and (52), we obtain

$$\begin{aligned}&\|4\beta f(x)-f(xy^3)\| \leqslant 18M^2\delta \text{ or}\\&\|(2\beta^2-\beta-2)f(x)+f(xy^3)\| \leqslant 12M^3\delta.\end{aligned} \tag{53}$$

Consider the alternatives in $\mathscr{P}f_{y^2}^{(1,\beta,1)}(xy)$.

- If $\mathscr{J}_{y^2}(xy) \leqslant \delta_1$, then we use $\mathscr{J}_{y^2}(xy) \leqslant \delta_1$ and $\mathscr{F}_y^{(1,\beta,1)}(x) \leqslant \delta_2$ to get

  $$\|\beta f(x)+3f(xy)-f(xy^3)\| \leqslant 2\delta. \tag{54}$$

  By (53) and (54), we obtain

  $$\begin{aligned}&\|3\beta f(x)-3f(xy)\| \leqslant 20M^2\delta \text{ or}\\&\|(2\beta^2-2)f(x)+3f(xy)\| \leqslant 14M^3\delta.\end{aligned} \tag{55}$$

  Eliminating $f(xy)$ from (50) and (55), we have $\|f(x)\| \leqslant 15M^5\delta$.

- If $\mathscr{F}_{y^2}^{(1,\beta,1)}(xy) \leqslant \delta_2$, then we use $\mathscr{F}_{y^2}^{(1,\beta,1)}(xy) \leqslant \delta_2$ and $\mathscr{F}_y^{(1,\beta,1)}(x) \leqslant \delta_2$ to get

  $$\|\beta f(x)+(1-\beta)f(xy)-f(xy^3)\| \leqslant 2\delta. \tag{56}$$

  By (53) and (56), we obtain

  $$\begin{aligned}&\|3\beta f(x)+(\beta-1)f(xy)\| \leqslant 20M^2\delta \text{ or}\\&\|(2\beta^2-2)f(x)+(1-\beta)f(xy) \leqslant 14M^3\delta.\end{aligned} \tag{57}$$

  Eliminating $f(xy)$ from (50) and (57), we get $\|f(x)\| \leqslant 25M^5\delta$.

Case (ii). Assume that $\mathscr{F}_{y^2}^{(1,\beta,1)}(x) \leqslant \delta_2$. By $\mathscr{F}_y^{(1,\beta,1)}(xy) \leqslant \delta_2, \mathscr{F}_{y^2}^{(1,\beta,1)}(x) \leqslant \delta_2$ and (49), we obtain

$$\|f(x)+f(xy)\| \leqslant 5M\delta \tag{58}$$

and

$$\|(1-\beta)f(x)+f(xy^2)\| \leqslant 6M^2\delta. \tag{59}$$

Eliminating $f(xy^2)$ from (59) and the alternatives in $\mathscr{P}f_{y^2}^{(1,\beta,1)}(xy^2)$, we get

$$\begin{aligned}&\|(3-2\beta)f(x)+f(xy^4)\| \leqslant 13M^2\delta \text{ or}\\&\|(\beta^2-\beta+1)f(x)+f(xy^4)\| \leqslant 7M^3\delta.\end{aligned} \tag{60}$$

By (58) and the the alternatives in $\mathscr{P}f_y^{(1,\beta,1)}(xy^2)$, we have

$$\begin{aligned}&\|f(x)+2f(xy^2)-f(xy^3)\| \leqslant 6M\delta \text{ or}\\&\|f(x)-\beta f(xy^2)-f(xy^3)\| \leqslant 6M\delta.\end{aligned} \tag{61}$$

Consider the alternatives in $\mathscr{P}f_y^{(1,\beta,1)}(xy^3)$ as follows.

- If $\mathscr{J}_y(xy^3) \leqslant \delta_1$, then we eliminate $f(xy^3)$ from (61) and $\mathscr{J}_y(xy^3) \leqslant \delta_1$ to get

  $$\begin{aligned}&\|2f(x)+3f(xy^2)-f(xy^4)\| \leqslant 13M\delta \text{ or}\\&\|2f(x)-(1+2\beta)f(xy^2)-f(xy^4)\| \leqslant 13M\delta.\end{aligned} \tag{62}$$

  By (59) and (62), we obtain

  $$\begin{aligned}&\|(1-3\beta)f(x)+f(xy^4)\| \leqslant 31M^2\delta \text{ or}\\&\|(2\beta^2-\beta-3)f(x)+f(xy^4)\| \leqslant 31M^3\delta.\end{aligned} \tag{63}$$

  By (60) and (63), we conclude that

  $$\|f(x)\| \leqslant 44M^5\delta \quad \text{or} \quad \|(3–2\beta)f(x)\| \leqslant 44M^4\delta.$$

  In the case when $\beta \neq \dfrac{3}{2}$, we get

  $$\|f(x)\| \leqslant 44M^5\delta.$$

  Suppose $\beta = \dfrac{3}{2}$. Hence (49), (59), (60), (61) and (63) become

  $$\|f(xy^{-2})+4f(x)+2f(xy)\| \leqslant 3\delta. \tag{64}$$

  $$\left\|-\frac{1}{2}f(x)+f(xy^2)\right\| \leqslant 6M^2\delta, \tag{65}$$

  $$\begin{aligned}&\|f(xy^4)\| \leqslant 13M^2\delta \text{ or}\\&\|\tfrac{7}{4}f(x)+f(xy^4)\| \leqslant 7M^3\delta,\end{aligned} \tag{66}$$

  and

  $$\begin{aligned}&\left\|-\tfrac{7}{2}f(x)+f(xy^4)\right\| \leqslant 31M^2\delta \text{ or}\\&\|f(xy^4)\| \leqslant 31M^3\delta,\end{aligned} \tag{67}$$

  respectively. By (66) and (67), we get

  $$\|f(xy^4)\| \leqslant 31M^3\delta. \tag{68}$$

  Eliminating $f(xy^4)$ from $\mathscr{P}f_{y^2}^{(1,\frac{3}{2},1)}(xy^4)$ and (68), we obtain

  $$\|f(xy^2)+f(xy^6)\| \leqslant 63M^3\delta. \tag{69}$$

  By $\mathscr{P}f_{y^4}^{(1,\frac{3}{2},1)}(xy^2)$ and (69), we have

  $$\begin{aligned}&\|f(xy^{-2})-3f(xy^2)\| \leqslant 64M^3\delta \text{ or}\\&\left\|f(xy^{-2})+\tfrac{1}{2}f(xy^2)\right\| \leqslant 64M^3\delta.\end{aligned} \tag{70}$$

By (65) and (70), we get

$$\left\|f(xy^{-2})-\tfrac{3}{2}f(x)\right\|\leqslant 82M^3\delta \text{ or } \left\|f(xy^{-2})+\tfrac{1}{4}f(x)\right\|\leqslant 67M^3\delta. \quad (71)$$

Eliminating $f(xy^{-2})$ from (64) and (71), we have

$$\left\|\tfrac{11}{2}f(x)+2f(xy)\right\|\leqslant 85M^3\delta \text{ or } \left\|\tfrac{15}{4}f(x)+2f(xy)\right\|\leqslant 70M^3\delta. \quad (72)$$

By (58) and (72), we conclude that

$$\|f(x)\|\leqslant 46M^3\delta.$$

- If $\mathscr{F}_y^{(1,\beta,1)}(xy^3)\leqslant\delta_2$, then we eliminate $f(xy^3)$ from (61) and $\mathscr{F}_y^{(1,\beta,1)}(xy^3)\leqslant\delta_2$ to get

$$\|\beta f(x)+(1+2\beta)f(xy^2)+f(xy^4)\|\leqslant 7M^2\delta \text{ or } \|\beta f(x)+(1-\beta^2)f(xy^2)+f(xy^4)\|\leqslant 7M^2\delta. \quad (73)$$

By (59) and (73), we obtain

$$\|(2\beta^2-1)f(x)+f(xy^4)\|\leqslant 13M^3\delta \text{ or } \|(\beta^3-\beta^2-2\beta+1)f(x)-f(xy^4)\|\leqslant 13M^4\delta. \quad (74)$$

By (60) and (74), we conclude that

$$\|f(x)\|\leqslant 26M^7\delta.$$

The desired bound of $f(x)$ follows from the consideration of all cases above. □

Now we will give the bound of $\mathscr{J}_y(x)$ for a function $f\in\mathscr{A}_{(G,E)}^{(1,\beta,1)}$.

Lemma 7 *If $f\in\mathscr{A}_{(G,E)}^{(1,\beta,1)}$, then $\mathscr{J}_y(x)\leqslant 139M^8\delta$ for all $x,y\in G$.*

Proof: Let $f\in\mathscr{A}_{(G,E)}^{(1,\beta,1)}$ and $x,y\in G$. Suppose $\mathscr{J}_y(x)>\delta_1$. From the alternatives in $\mathscr{P}f_y^{(1,\beta,1)}(x)$, we get $\mathscr{F}_y^{(1,\beta,1)}(x)\leqslant\delta_2$. The alternatives in $\mathscr{P}f_y^{(1,\beta,1)}(xy^{-1})$ will be considered as follows.

Case (i). Assume that $\mathscr{J}_y(xy^{-1})\leqslant\delta_1$. By Lemma 5 and Lemma 6, we conclude that

$$\|f(x)\|\leqslant 46M^7\delta. \quad (75)$$

By $\mathscr{F}_y^{(1,\beta,1)}(x)\leqslant\delta_2$ and (75), we conclude that $\mathscr{J}_y(x)\leqslant 139M^8\delta$ as desired.

Case (ii). Assume that $\mathscr{F}_y^{(1,\beta,1)}(xy^{-1})\leqslant\delta_2$. Consider the alternatives in $\mathscr{P}f_y^{(1,\beta,1)}(xy)$. If $\mathscr{F}_y^{(1,\beta,1)}(xy)\leqslant\delta_2$, then Lemma 5 gives $\|f(x)\|\leqslant 4M^3\delta$. Thus the desired proof is similar to the above case. If $\mathscr{J}_y(xy)\leqslant\delta_1$, then the proof is as in Case (i) after replacing y by y^{-1} and x by xy^{-1}. □

HYERS-ULAM STABILITY

We will next provide the following lemma which eventually be used in the main theorem.

Lemma 8 *If $f\in\mathscr{A}_{(G,E)}^{(\alpha,\beta,\gamma)}$, then $\mathscr{J}_y(x)\leqslant 139M^9\delta$ for all $x,y\in G$.*

Proof: Let $f\in\mathscr{A}_{(G,E)}^{(\alpha,\beta,\gamma)}$. If $\alpha\neq\gamma$, then by Lemma 3 and Lemma 4, we conclude that $\mathscr{J}_y(x)\leqslant 56M^5\delta$ for all $x,y\in G$. If $\alpha=\gamma$, then we consider two cases as follows:

Case (i). Assume that $\alpha=0$. Lemma 1 gives $\mathscr{J}_y(x)\leqslant 12M\delta$ for all $x,y\in G$.

Case (ii) Assume that $\alpha\neq 0$. Hence $f\in\mathscr{A}_{(G,E)}^{(1,\alpha^{-1}\beta,1)}$ and Lemma 7 gives

$$\mathscr{J}_y(x)\leqslant 139M^8\max\{\delta_1,|\alpha|^{-1}\delta_2\}\leqslant 139M^9\delta$$

for all $x,y\in G$. □

Now we will prove the Hyers-Ulam stability of the alternative Jensen's functional equation (3). For the stability results of Jensen's functional equation, it can be found in, for instance, Kominek [14] or Jung [15].

Theorem 1 *Let G be an abelian group. If $f\in\mathscr{A}_{(G,E)}^{(\alpha,\beta,\gamma)}$, then there exists a unique Jensen's mapping $J:G\to E$ satisfying (2) with $J(0)=f(0)$ such that*

$$\|f(x)-J(x)\|\leqslant\varepsilon$$

for all $x\in G$ when $\varepsilon=278M^9\delta$. Moreover, the mapping J is given by

$$J(x)=f(0)+\lim_{n\to\infty}\frac{1}{2^n}(f(x^{2^n})-f(0))$$

for all $x\in G$.

Proof: Assume that $f\in\mathscr{A}_{(G,E)}^{(\alpha,\beta,\gamma)}$. By Lemma 8, we obtain $\mathscr{J}_y(x)\leqslant 139M^9\delta$ for all $x,y\in G$. The Hyers-Ulam stability of the Jensen's functional equation can be proved by the so-called direct method and it can be seen in Srisawat [16]. Hence the rest of the proof can be omitted. □

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