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A refinement of Hardy type inequality on the *n*-spheres

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ABSTRACT: We give a refinement of Hardy type inequality with the best constant on the sphere. This improves the result of Xiao [J Math Inequal **10** (2016):793–805].

KEYWORDS: Hardy inequality, the best constant, sphere

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INTRODUCTION

The classical Hardy inequality states that, for $n \geqslant 3$ and all $f \in C_c^\infty(\mathbb{R}^n \backslash \{0\})$,

$$\int_{\mathbb{R}^n} |\nabla f|^2 \,\mathrm{d}x \geqslant \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{f^2}{|x|^2} \,\mathrm{d}x.$$

The constant $(n-2)^2/4$ is optimal and not attained in the Sobolev space $W^{1,2}(\mathbb{R}^n \backslash \{0\})$. There has been a lot of research concerning Hardy inequality on the Euclidean space because of its applications to partial differential equations involving singular potentials. We can refer to [1–3] and the references therein.

The validity of Hardy inequality on a manifold and its best constants allows people to obtain qualitative properties on the manifold. In [4], Carron studied the weighted L^2-Hardy inequalities on a Riemannian manifold under some geometric assumptions on the weight function and obtained

$$\int_M \rho^\alpha |\nabla f|^2 \,\mathrm{d}V \geqslant \frac{(C+\alpha-1)^2}{4} \int_M \rho^\alpha \frac{f^2}{\rho^2} \,\mathrm{d}V,$$

for any $f \in C_0^\infty(M)$, where the weight function ρ satisfies $|\nabla \rho| = 1$ and $\Delta \rho \geqslant \frac{C}{\rho}$. Along this line, we refer to [5–7] and so on. Particularly, in [7] Kombe and Özaydin obtained the improved Hardy inequalities in the Poincaré conformal disc model

$$\int_{\mathbb{B}^n} |\nabla f|^2 \,\mathrm{d}V \geqslant \frac{(n-2)^2}{4} \int_{\mathbb{B}^n} \frac{f^2}{\rho^2} \,\mathrm{d}V,$$

where $f \in C_c^\infty(\mathbb{B}^n \backslash \{0\})$ and $\rho = \log[(1+|x|)/(1-|x|)]$ is the geodesic distance. Furthermore, the constant $(n-2)^2/4$ is sharp.

By comparison with the results above, the results of Hardy inequality on the sphere are relatively few. Recently, Xiao [8] studied the Hardy type inequality on the sphere and derived the following inequality

$$C \int_{\mathbb{S}^n} f^2 \,\mathrm{d}V + \int_{\mathbb{S}^n} |\nabla f|^2 \,\mathrm{d}V \geqslant \frac{(n-2)^2}{4} \left(\int_{\mathbb{S}^n} \frac{f^2}{d(p,x)^2} \mathrm{d}V + \int_{\mathbb{S}^n} \frac{f^2}{(\pi - d(p,x))^2} \mathrm{d}V \right) \quad (1)$$

for any function $f \in C^\infty(\mathbb{S}^n)$ and some constant C, where $d(p,x)$ is the geodesic distance from p on $\mathbb{S}^n$, and the constant $(n-2)^2/4$ is sharp.

In [9] the author used the tangent function and obtained another type Hardy inequality on the sphere as follows.

$$\frac{n-2}{2} \int_{\mathbb{S}^n} f^2 \,\mathrm{d}V + \int_{\mathbb{S}^n} |\nabla f|^2 \,\mathrm{d}V \geqslant \frac{(n-2)^2}{4} \int_{\mathbb{S}^n} \frac{f^2}{\tan^2 d(p,x)} \,\mathrm{d}V.$$

The results of the L^p Hardy inequalities were discussed in [10, 11], respectively.

In this short note, we still focus on Inequality (1). We observe that for any $0 \leqslant R \leqslant \pi$, Inequality (1) can be changed by

$$C \int_{\mathbb{S}^n} f^2 \,\mathrm{d}V + \int_{\mathbb{S}^n} |\nabla f|^2 \,\mathrm{d}V \geqslant \frac{(n-2)^2}{4} \times \left(\int_{B_p(R)} \frac{f^2}{d(p,x)^2} \mathrm{d}V + \int_{\mathbb{S}^n \backslash B_p(R)} \frac{f^2}{(\pi - d(p,x))^2} \mathrm{d}V \right). \quad (2)$$

However it is not easy to see whether the constant $(n-2)^2/4$ is sharp. Clearly, if it is not sharp, then (2) boils down to very little significance. The other observation is that the first term in the left-hand side in (1) has no effect on the sharpness of the constant $(n-2)^2/4$ regardless of the choice of C, but cannot be removed because it leads to a contradiction if f is a nonzero constant function. This may be the most remarkable difference from that in Euclidean spaces and some other Riemannian manifolds. As a consequence, it is very interesting and important to prove the sharpness of the constant $(n-2)^2/4$ in (2), and consider how to determine and reduce that constant C. Specifically, by choosing $R=\pi/2$, we adapt the inequality (1) to the following form.

Theorem 1 *Let $\mathbb{S}^n$ be the n dimensional sphere with $n \geqslant 3$. Then for any function $f \in C^\infty(\mathbb{S}^n)$, it holds*

$$\frac{2(n-1)(n-2)}{\pi^2}\int_{\mathbb{S}^n} f^2\,\mathrm{d}V + \int_{\mathbb{S}^n} |\nabla f|^2\,\mathrm{d}V \geqslant \frac{(n-2)^2}{4}\left(\int_{B_p(\frac{\pi}{2})}\frac{f^2}{d(p,x)^2}\,\mathrm{d}V + \int_{B_q(\frac{\pi}{2})}\frac{f^2}{d(q,x)^2}\,\mathrm{d}V\right),$$

where p and q are the antipodal points, and $B_p(\pi/2)$ (resp., $B_q(\pi/2)$) denotes the geodesic ball centered at p (resp., q) of radius $\pi/2$. Moreover, the constant $(n-2)^2/4$ is sharp. That is,

$$\frac{(n-2)^2}{4} = \inf_{f\in C^\infty(\mathbb{S}^n)\setminus\{0\}} \frac{\int_{\mathbb{S}^n}|\nabla f|^2\mathrm{d}V + \frac{2(n-1)(n-2)}{\pi^2}\int_{\mathbb{S}^n} f^2\mathrm{d}V}{\int_{B_p(\frac{\pi}{2})}\frac{f^2}{d(p,x)^2}\mathrm{d}V + \int_{B_q(\frac{\pi}{2})}\frac{f^2}{d(q,x)^2}\mathrm{d}V}.$$

Our proof is based on Lemma 1 below and a new construction of the auxiliary functions. Then, by using symmetry of spheres and standard discussions, Theorem 1 is proved.

THE PROOF OF THE MAIN RESULT

We first establish a useful lemma as follows.

Lemma 1

$$\frac{1}{3} \leqslant \frac{1-r\cot r}{r^2} \leqslant \frac{4}{\pi^2}, \quad r\in(0,\frac{\pi}{2}].$$

Proof: Let $F(r) = (1-r\cot r)/r^2$. Then $F(\pi/2) = 4/\pi^2$, and by L'Hospital's rule

$$\begin{aligned}\lim_{r\to 0^+} F(r) &= \lim_{r\to 0^+}\frac{1-r\cot r}{r^2} = \lim_{r\to 0^+}\frac{\sin r - r\cos r}{r^2\sin r}\\ &= \lim_{r\to 0^+}\frac{\sin r - r\cos r}{r^3}\\ &= \lim_{r\to 0^+}\frac{\cos r-\cos r + r\sin r}{3r^2} = \frac{1}{3}.\end{aligned}$$

To prove the result, it suffices to show $F'(r)\geqslant 0$. Since

$$F'(r) = \frac{r^2\csc^2 r + r\cot r - 2}{r^3},$$

we only need to show

$$r^2\csc^2 r + r\cot r - 2 \geqslant 0, \quad r\in(0,\frac{\pi}{2}].$$

That is to prove

$$r^2 + r\sin r\cos r \geqslant 2\sin^2 r, \quad r\in(0,\frac{\pi}{2}].$$

Namely,

$$r^2 + \frac{1}{2}r\sin 2r \geqslant 1-\cos 2r, \quad r\in(0,\frac{\pi}{2}].$$

Notice that the inequality above becomes the equality when $r=0$. Therefore, the inequality is valid if

$$2r + r\cos 2r \geqslant \frac{3}{2}\sin 2r, \quad r\in(0,\frac{\pi}{2}].$$

By a similar argument, it simply requires that

$$1 \geqslant r\sin 2r + \cos 2r, \quad r\in(0,\frac{\pi}{2}].$$

Then it is is sufficient to prove

$$\sin 2r \geqslant 2r\cos 2r, \quad r\in(0,\frac{\pi}{2}].$$

The inequality follows from $\tan 2r \geqslant 2r$ if $r\in(0,\pi/4)$, and is obviously true if $r\in(\pi/4,\pi/2]$. This ends the proof. □

We are now in a position to prove Theorem 1 in the following.

Proof: Let $r_p(x) = d(p,x)$ denote the distance function from the fixed point $p\in\mathbb{S}^n$. Next we follow the arguments in [7] (see also [8,9]). Let $f = r_p^\alpha\varphi$ with $\alpha<0$. Then $\nabla f = \varphi\nabla r_p^\alpha + r_p^\alpha\nabla\varphi$ and

$$\begin{aligned}|\nabla f|^2 &= \varphi^2|\nabla r_p^\alpha|^2 + r_p^{2\alpha}|\nabla\varphi|^2 + 2r_p^\alpha\varphi\langle\nabla r_p^\alpha,\nabla\varphi\rangle\\ &\geqslant \varphi^2\alpha^2 r_p^{2\alpha-2} + \tfrac{1}{2}\langle\nabla r_p^{2\alpha},\nabla\varphi^2\rangle\\ &= \varphi^2\alpha^2 r_p^{2\alpha-2} + \tfrac{1}{2}\mathrm{div}(\varphi^2\nabla r_p^{2\alpha}) - \tfrac{1}{2}\varphi^2\Delta r_p^{2\alpha}, \end{aligned} \quad (3)$$

where

$$\begin{aligned}\Delta r_p^{2\alpha} &= \mathrm{div}(\nabla r_p^{2\alpha}) = \mathrm{div}(2\alpha r_p^{2\alpha-1}\nabla r_p)\\ &= 2\alpha r_p^{2\alpha-1}\Delta r_p + 2\alpha(2\alpha-1)r_p^{2\alpha-2}\\ &= 2(n-1)\alpha r_p^{2\alpha-1}\cot r_p + 2\alpha(2\alpha-1)r_p^{2\alpha-2}.\end{aligned} \quad (4)$$

The last equality holds because $\Delta r_p = (n-1)\cot r_p$ in the sphere. Therefore, from (3) and (4), we have

$$|\nabla f|^2 \geqslant \tfrac{1}{2}\mathrm{div}(\varphi^2\nabla r_p^{2\alpha}) + \alpha(1-\alpha)\tfrac{f^2}{r_p^2} - (n-1)\alpha\tfrac{f^2}{r_p}\cot r_p.$$

Integrating both sides of the inequality above on $B_p(\pi/2)$ gives

$$\int_{B_p(\frac{\pi}{2})}|\nabla f|^2\,\mathrm{d}V \geqslant \frac{1}{2}\int_{B_p(\frac{\pi}{2})}\mathrm{div}(\varphi^2\nabla r_p^{2\alpha})\,\mathrm{d}V + \alpha(1-\alpha)\int_{B_p(\frac{\pi}{2})}\frac{f^2}{r_p^2}\mathrm{d}V - (n-1)\alpha\int_{B_p(\frac{\pi}{2})}\frac{f^2}{r_p}\cot r_p\mathrm{d}V. \quad (5)$$

Let q be the antipodal point of p. Then $r_q(x) = d(q,x) = \pi - r_p$ for any $x \in \mathbb{S}^n$. Set $f = r_q^{\alpha}\phi$. Then by similar arguments, we also have

$$\int_{B_q(\frac{\pi}{2})} |\nabla f|^2 \,\mathrm{d}V \geqslant \frac{1}{2}\int_{B_q(\frac{\pi}{2})} \mathrm{div}(\phi^2\nabla r_q^{2\alpha})\,\mathrm{d}V + \alpha(1-\alpha)\int_{B_q(\frac{\pi}{2})} \frac{f^2}{r_q^2}\mathrm{d}V - (n-1)\alpha\int_{B_q(\frac{\pi}{2})} \frac{f^2}{r_q}\cot r_q \mathrm{d}V. \quad (6)$$

Note that $\partial B_p(\pi/2) = \partial B_q(\pi/2)$ and $\nabla r_p = -\nabla r_q$, $r_p = r_q = \pi/2$ on $\partial B_p(\pi/2)$. By Stokes theorem, we obtain

$$\begin{aligned}
&\int_{B_p(\frac{\pi}{2})} \mathrm{div}(\varphi^2\nabla r_p^{2\alpha})\,\mathrm{d}V + \int_{B_q(\frac{\pi}{2})} \mathrm{div}(\phi^2\nabla r_q^{2\alpha})\,\mathrm{d}V \\
&= \int_{\partial B_p(\frac{\pi}{2})} \langle \varphi^2\nabla r_p^{2\alpha}, \mathbf{n}\rangle\,\mathrm{d}\nu + \int_{\partial B_q(\frac{\pi}{2})} \langle \phi^2\nabla r_q^{2\alpha}, \mathbf{n}\rangle\,\mathrm{d}\nu \\
&= 2\alpha\int_{\partial B_p(\frac{\pi}{2})} \frac{f^2}{r_p}\langle \nabla r_p, \mathbf{n}\rangle\,\mathrm{d}\nu + 2\alpha\int_{\partial B_q(\frac{\pi}{2})} \frac{f^2}{r_q}\langle \nabla r_q, \mathbf{n}\rangle\,\mathrm{d}\nu \\
&= \frac{4\alpha}{\pi}\int_{\partial B_p(\frac{\pi}{2})} f^2\langle \nabla r_p + \nabla r_q, \mathbf{n}\rangle\,\mathrm{d}\nu = 0, \quad (7)
\end{aligned}$$

where $\mathbf{n}$ is a fixed normal vector along $\partial B_p(\pi/2)$ and $\mathrm{d}\nu$ is the induced volume form with respect to $\mathbf{n}$. Therefore, it follows from (5)–(7) that

$$\begin{aligned}
\int_{\mathbb{S}^n} |\nabla f|^2\,\mathrm{d}V \geqslant\ & \alpha(1-\alpha)\int_{B_p(\frac{\pi}{2})} \frac{f^2}{r_p^2}\,\mathrm{d}V \\
&+\alpha(1-\alpha)\int_{B_q(\frac{\pi}{2})} \frac{f^2}{r_q^2}\mathrm{d}V - (n-1)\alpha\int_{B_p(\frac{\pi}{2})} \frac{f^2}{r_p}\cot r_p \mathrm{d}V \\
&-(n-1)\alpha\int_{B_q(\frac{\pi}{2})} \frac{f^2}{r_q}\cot r_q\,\mathrm{d}V,
\end{aligned}$$

which shows

$$\begin{aligned}
&\int_{\mathbb{S}^n} |\nabla f|^2\,\mathrm{d}V - (n-1)\alpha\int_{B_p(\frac{\pi}{2})} f^2\frac{1-r_p\cot r_p}{r_p^2}\,\mathrm{d}V \\
&\qquad -(n-1)\alpha\int_{B_q(\frac{\pi}{2})} f^2\frac{1-r_q\cot r_q}{r_q^2}\,\mathrm{d}V \\
&\qquad \geqslant \alpha(2-n-\alpha)\int_{B_p(\frac{\pi}{2})} \frac{f^2}{r_p^2}\mathrm{d}V + \alpha(2-n-\alpha)\int_{B_q(\frac{\pi}{2})} \frac{f^2}{r_q^2}\mathrm{d}V.
\end{aligned}$$

Using Lemma 1 and letting $\alpha = -(n-2)/2$, we deduce that

$$\int_{\mathbb{S}^n} |\nabla f|^2\,\mathrm{d}V + \frac{2(n-1)(n-2)}{\pi^2}\int_{\mathbb{S}^n} f^2\,\mathrm{d}V \geqslant \frac{(n-2)^2}{4}\left(\int_{B_p(\frac{\pi}{2})} \frac{f^2}{r_p^2}\,\mathrm{d}V + \int_{B_q(\frac{\pi}{2})} \frac{f^2}{r_q^2}\,\mathrm{d}V\right).$$

In what follows, we show the constant $(n-2)^2/4$ is sharp. The skill is borrowed from [9] (see also [8, 12]). Let $\eta : \mathbb{R} \to [0,1]$ be a smooth function such that $0 \leqslant \eta \leqslant 1$ and

$$\eta(t) = \begin{cases} 1, & t \in [-1,1]; \\ 0, & |t| \geqslant 2. \end{cases}$$

Let $H(t) = 1 - \eta(t)$, and for sufficient small $\varepsilon > 0$ we construct

$$f_\varepsilon(r) = \begin{cases} 0, & r = 0; \\ H(\frac{r}{\varepsilon})r^{\frac{2-n}{2}}, & 0 < r \leqslant \frac{\pi}{2}; \\ H(\frac{\pi-r}{\varepsilon})(\pi-r)^{\frac{2-n}{2}}, & \frac{\pi}{2} \leqslant r < \pi; \\ 0, & r = \pi. \end{cases}$$

Observe that $f_\varepsilon(r)$ can be approximated by smooth functions on the sphere $\mathbb{S}^n$. Compute

$$\begin{aligned}
\int_{\mathbb{S}^n} f_\varepsilon^2\,\mathrm{d}V &= \int_{B_p(\frac{\pi}{2})} f_\varepsilon^2\,\mathrm{d}V + \int_{B_q(\frac{\pi}{2})} f_\varepsilon^2\,\mathrm{d}V \\
&= \mathrm{Vol}(\mathbb{S}^{n-1})\int_\varepsilon^{\frac{\pi}{2}} H^2(\frac{r_p}{\varepsilon})r_p^{2-n}(\sin r_p)^{n-1}\,\mathrm{d}r \\
&\quad +\mathrm{Vol}(\mathbb{S}^{n-1})\int_{\frac{\pi}{2}}^{\pi-\varepsilon} H^2(\frac{\pi-r_p}{\varepsilon})(\pi-r_p)^{2-n}(\sin(\pi-r_p))^{n-1}\mathrm{d}r \\
&= \mathrm{Vol}(\mathbb{S}^{n-1})\int_\varepsilon^{\frac{\pi}{2}} H^2(\frac{r_p}{\varepsilon})r_p^{2-n}(\sin r_p)^{n-1}\,\mathrm{d}r \\
&\quad +\mathrm{Vol}(\mathbb{S}^{n-1})\int_\varepsilon^{\frac{\pi}{2}} H^2(\frac{r_q}{\varepsilon})r_q^{2-n}(\sin r_q)^{n-1}\,\mathrm{d}r \\
&= 2\mathrm{Vol}(\mathbb{S}^{n-1})\int_\varepsilon^{\frac{\pi}{2}} H^2(\frac{r_p}{\varepsilon})r_p^{2-n}(\sin r_p)^{n-1}\,\mathrm{d}r \\
&\leqslant 2\mathrm{Vol}(\mathbb{S}^{n-1})\int_\varepsilon^{\frac{\pi}{2}} r_p^{2-n}r_p^{n-1}\mathrm{d}r = \left(\frac{\pi^2}{4} - \varepsilon^2\right)\mathrm{Vol}(\mathbb{S}^{n-1}). \quad (8)
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
\int_{B_p(\frac{\pi}{2})} \frac{f_\varepsilon^2}{r_p^2}\,\mathrm{d}V &= \mathrm{Vol}(\mathbb{S}^{n-1})\int_\varepsilon^{\frac{\pi}{2}} H^2(\frac{r_p}{\varepsilon})r_p^{-n}(\sin r_p)^{n-1}\,\mathrm{d}r \\
&\geqslant \mathrm{Vol}(\mathbb{S}^{n-1})\int_{2\varepsilon}^{\frac{\pi}{2}} H^2(\frac{r_p}{\varepsilon})r_p^{-n}(\sin r_p)^{n-1}\,\mathrm{d}r \\
&= \mathrm{Vol}(\mathbb{S}^{n-1})\int_{2\varepsilon}^{\frac{\pi}{2}} r_p^{-n}(\sin r_p)^{n-1}\,\mathrm{d}r,
\end{aligned}$$

$$\int_{B_q(\frac{\pi}{2})}\frac{f_\varepsilon^2}{r_q^2}\,\mathrm{d}V$$
$$=\mathrm{Vol}(\mathbb{S}^{n-1})\int_{\frac{\pi}{2}}^{\pi-\varepsilon}H^2(\tfrac{\pi-r_p}{\varepsilon})(\pi-r_p)^{-n}(\sin(\pi-r_p))^{n-1}\mathrm{d}r$$
$$=\mathrm{Vol}(\mathbb{S}^{n-1})\int_{\varepsilon}^{\frac{\pi}{2}}H^2(\tfrac{r_q}{\varepsilon})r_q^{-n}(\sin r_q)^{n-1}\mathrm{d}r$$
$$\geqslant\mathrm{Vol}(\mathbb{S}^{n-1})\int_{2\varepsilon}^{\frac{\pi}{2}}r_q^{-n}(\sin r_q)^{n-1}\mathrm{d}r.$$

Therefore, combining the above two inequalities, we obtain

$$\int_{B_p(\frac{\pi}{2})}\frac{f_\varepsilon^2}{r_p^2}\,\mathrm{d}V+\int_{B_q(\frac{\pi}{2})}\frac{f_\varepsilon^2}{r_q^2}\,\mathrm{d}V \geqslant 2\mathrm{Vol}(\mathbb{S}^{n-1})\int_{2\varepsilon}^{\frac{\pi}{2}}r_p^{-n}(\sin r_p)^{n-1}\,\mathrm{d}r. \quad (9)$$

Next we are to estimate

$$\int_{\mathbb{S}^n}|\nabla f_\varepsilon|^2\,\mathrm{d}V=\int_{B_p(\frac{\pi}{2})}|\nabla f_\varepsilon|^2\,\mathrm{d}V+\int_{B_q(\frac{\pi}{2})}|\nabla f_\varepsilon|^2\,\mathrm{d}V.$$

A straightforward calculation yields

$$\left(\int_{B_p(\frac{\pi}{2})}|\nabla f_\varepsilon|^2\,\mathrm{d}V\right)^{\frac{1}{2}}=\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\left(\int_{\varepsilon}^{\frac{\pi}{2}}\left|H'(\frac{r_p}{\varepsilon})\frac{1}{\varepsilon}r_p^{\frac{2-n}{2}}+\frac{2-n}{2}H(\frac{r_p}{\varepsilon})r_p^{-\frac{n}{2}}\right|^2(\sin r_p)^{n-1}\mathrm{d}r\right)^{\frac{1}{2}}$$
$$\leqslant\frac{\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}}{\varepsilon}\left(\int_{\varepsilon}^{\frac{\pi}{2}}\left|H'(\frac{r_p}{\varepsilon})\right|^2r_p^{2-n}(\sin r_p)^{n-1}\mathrm{d}r\right)^{\frac{1}{2}}+\frac{n-2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\left(\int_{\varepsilon}^{\frac{\pi}{2}}H^2(\frac{r_p}{\varepsilon})r_p^{-n}(\sin r_p)^{n-1}\mathrm{d}r\right)^{\frac{1}{2}}$$
$$=\frac{\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}}{\varepsilon}\left(\int_{\varepsilon}^{2\varepsilon}\left|H'(\frac{r_p}{\varepsilon})\right|^2r_p^{2-n}(\sin r_p)^{n-1}\mathrm{d}r\right)^{\frac{1}{2}}+\frac{n-2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\left(\int_{\varepsilon}^{\frac{\pi}{2}}H^2(\frac{r_p}{\varepsilon})r_p^{-n}(\sin r_p)^{n-1}\mathrm{d}r\right)^{\frac{1}{2}}$$
$$\leqslant\frac{\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}}{\varepsilon}\max_{t\in[0,2]}H'(t)\left(\int_{\varepsilon}^{2\varepsilon}r_p\,\mathrm{d}r\right)^{\frac{1}{2}}+\frac{n-2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\left(\int_{\varepsilon}^{\frac{\pi}{2}}r_p^{-n}(\sin r_p)^{n-1}\,\mathrm{d}r\right)^{\frac{1}{2}}$$
$$=\sqrt{\tfrac{3}{2}}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\max_{t\in[0,2]}H'(t)+\frac{n-2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\left(\int_{\varepsilon}^{\frac{\pi}{2}}r_p^{-n}(\sin r_p)^{n-1}\,\mathrm{d}r\right)^{\frac{1}{2}},$$

$$\left(\int_{B_q(\frac{\pi}{2})}|\nabla f_\varepsilon|^2\,\mathrm{d}V\right)^{\frac{1}{2}}$$
$$=\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\left(\int_{\frac{\pi}{2}}^{\pi-\varepsilon}\left|H'(\tfrac{\pi-r_p}{\varepsilon})\tfrac{-1}{\varepsilon}(\pi-r_p)^{\frac{2-n}{2}}+\tfrac{2-n}{2}H(\tfrac{\pi-r_p}{\varepsilon})(\pi-r_p)^{-\frac{n}{2}}\right|^2(\sin(\pi-r_p))^{n-1}\mathrm{d}r\right)^{\frac{1}{2}}$$
$$=\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\left(\int_{\varepsilon}^{\frac{\pi}{2}}\left|H'(\frac{r_q}{\varepsilon})\frac{-1}{\varepsilon}r_q^{\frac{2-n}{2}}+\frac{2-n}{2}H(\frac{r_q}{\varepsilon})r_q^{-\frac{n}{2}}\right|^2(\sin r_q)^{n-1}\,\mathrm{d}r\right)^{\frac{1}{2}}$$
$$\leqslant\sqrt{\tfrac{3}{2}}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\max_{t\in[0,2]}H'(t)+\frac{n-2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\left(\int_{\varepsilon}^{\frac{\pi}{2}}r_q^{-n}(\sin r_q)^{n-1}\,\mathrm{d}r\right)^{\frac{1}{2}},$$

Thus, we have

$$\int_{\mathbb{S}^n}|\nabla f_\varepsilon|^2\,\mathrm{d}V\leqslant 3\mathrm{Vol}(\mathbb{S}^{n-1})(\max_{t\in[0,2]}H'(t))^2+\frac{(n-2)^2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})\int_{\varepsilon}^{\frac{\pi}{2}}r_q^{-n}(\sin r_q)^{n-1}\,\mathrm{d}r+\sqrt{\tfrac{3}{2}}(n-2)\,\mathrm{Vol}(\mathbb{S}^{n-1})\times\max_{t\in[0,2]}H'(t)\left(\int_{\varepsilon}^{\frac{\pi}{2}}r_q^{-n}(\sin r_q)^{n-1}\,\mathrm{d}r\right)^{\frac{1}{2}}. \quad (10)$$

Since $f_\varepsilon(r)$ can be approximated by smooth functions on the sphere $\mathbb{S}^n$, then, by (8)–(10), it holds that

$$C:=\inf_{f\in C^\infty(\mathbb{S}^n)\setminus\{0\}}\frac{\int_{\mathbb{S}^n}|\nabla f|^2\,\mathrm{d}V+\frac{2(n-1)(n-2)}{\pi^2}\int_{\mathbb{S}^n}f^2\,\mathrm{d}V}{\int_{B_p(\frac{\pi}{2})}\frac{f^2}{r_p^2}\,\mathrm{d}V+\int_{B_q(\frac{\pi}{2})}\frac{f^2}{r_q^2}\,\mathrm{d}V}$$
$$\leqslant\frac{\int_{\mathbb{S}^n}|\nabla f_\varepsilon|^2\,\mathrm{d}V+\frac{2(n-1)(n-2)}{\pi^2}\int_{\mathbb{S}^n}f_\varepsilon^2\,\mathrm{d}V}{\int_{B_p(\frac{\pi}{2})}\frac{f_\varepsilon^2}{r_p^2}\,\mathrm{d}V+\int_{B_q(\frac{\pi}{2})}\frac{f_\varepsilon^2}{r_q^2}\,\mathrm{d}V}$$
$$\leqslant\frac{\frac{2(n-1)(n-2)}{\pi^2}(\frac{\pi^2}{4}-\varepsilon^2)}{2\int_{2\varepsilon}^{\frac{\pi}{2}}r_p^{-n}(\sin r_p)^{n-1}\,\mathrm{d}r}+\frac{3(\max_{t\in[0,2]}H'(t))^2}{2\int_{2\varepsilon}^{\frac{\pi}{2}}r_p^{-n}(\sin r_p)^{n-1}\,\mathrm{d}r}+\frac{(n-2)^2}{4}\frac{\int_{\varepsilon}^{\frac{\pi}{2}}r_q^{-n}(\sin r_q)^{n-1}\,\mathrm{d}r}{\int_{2\varepsilon}^{\frac{\pi}{2}}r_p^{-n}(\sin r_p)^{n-1}\,\mathrm{d}r}$$
$$+\frac{\sqrt{\frac{3}{2}}(n-2)\mathrm{Vol}(\mathbb{S}^{n-1})\max_{t\in[0,2]}H'(t)\left(\int_{\varepsilon}^{\frac{\pi}{2}}r_q^{-n}(\sin r_q)^{n-1}\mathrm{d}r\right)^{\frac{1}{2}}}{2\int_{2\varepsilon}^{\frac{\pi}{2}}r_p^{-n}(\sin r_p)^{n-1}\,\mathrm{d}r}$$
$$:=I+II+III+IV.$$

Note that

$$\lim_{\varepsilon\to 0}\int_{2\varepsilon}^{\frac{\pi}{2}} r_p^{-n}(\sin r_p)^{n-1}\,\mathrm{d}r = \infty,$$

and by L'Hospital rule,

$$\lim_{\varepsilon\to 0}\frac{\int_{\varepsilon}^{\frac{\pi}{2}} r_q^{-n}(\sin r_q)^{n-1}\,\mathrm{d}r}{\int_{2\varepsilon}^{\frac{\pi}{2}} r_p^{-n}(\sin r_p)^{n-1}\,\mathrm{d}r} = 1.$$

This implies that $I = II = IV = 0$, and $C \leqslant (n-2)^2/4$. The reverse inequality follows from the Hardy inequalty in Theorem 1. This completes the proof. □

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