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On meromorphic solutions of certain type of nonlinear differential-difference equations

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ABSTRACT: In this article, we analyze the meromorphic solutions of the following two types of nonlinear differential-difference equation:

$$f^n f' + p(z)f(z+c) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}$$

and

$$f^n f' + q(z)f(z+c)e^{Q(z)} = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z},$$

where p_1, p_2 and α_1, α_2 are nonzero constants, $p(z)$, $Q(z)$ are non-vanishing polynomials and $q(z)$ is a rational function.

KEYWORDS: meromorphic solution, exponential polynomial, nonlinear differential-difference equation, Nevanlinna theory

MSC2020: 30D35 39B32

INTRODUCTION AND RESULTS

Let $\mathbb{C}$ denote the complex plane and $f(z)$ be a meromorphic function on $\mathbb{C}$. The fundamental results and standard notations associated with the Nevanlinna's value distribution theory are assumed to be known to the reader, see [1–4], for example, the characteristic function $T(r,f)$, the proximity function $m(r,f)$, the counting function $N(r,f)$, the reduced counting function $\overline{N}(r,f)$ and so on. We use $S(r,f)$ to denote any quantity which satisfies $S(r,f)=o(T(r,f))$ as $r\to\infty$, possibly outside of an exceptional set E of finite linear measure. The set E is not necessarily the same at each occurrence. We shall call a meromorphic function $\alpha(z)$ is a small function with respect to f if $T(r,\alpha(z)) = S(r,f)$. We use $\sigma(f)$ and $\lambda(f)$ to denote the order of growth and the exponent of convergence of zeros sequence of f, respectively. Moreover,

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

and
$$\lambda_2(f) = \limsup_{r\to\infty} \frac{\log\log N(r,1/f)}{\log r}$$

stand for the hyper-order and the hyper-exponent of convergence of zeros sequence of $f(z)$, respectively.

Nevanlinna's value distribution theory of meromorphic functions has been widely applied to investigate the solvability and existence of entire or meromorphic solutions of complex differential equations, difference equations and differential-difference equations, see [5–16]. It is a significant and tough problem to study the existence of meromorphic solutions of complex differential equations, especially for nonlinear ones.

In the past decades, the related results of nonlinear differential equations have been studied in [17–19]. Yang and Li [18] showed that the equation

$$4f^3 + 3f'' = -\sin 3z \tag{1}$$

admits exactly three nonconstant entire solutions, namely $f_1(z) = \sin z$, $f_2(z) = \frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$ and $f_3(z) = -\frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$. Later, Li and Yang [20] also obtained the following Theorem 1 that is the generalization of the study of entire solutions of equation (1).

Theorem 1 ([20]) *Let $n\geqslant 4$ be an integer, and $P(f)$ denote an algebraic differential polynomial in f of degree $d\leqslant n-3$. Let p_1 and p_2 be two nonzero polynomials, α_1 and α_2 be two nonzero constants with $(\alpha_1/\alpha_2)\neq$ rational. Then the differential equation*

$$f^n(z) + P(f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} \tag{2}$$

has no transcendental entire solutions.

In Theorem 1, if p_1 and p_2 are nonzero constants, and the degree d of $P(f)$ with $d\leqslant n-3$ is reduced to $d\leqslant n-2$, then every meromorphic solution of equation (2) can be expressed as a special form, see Li [9]. Liao, Yang and Zhang [10] generalized the above results to the following.

Theorem 2 ([10]) *Let $n\geqslant 3$ be an integer and $Q_d(z,f)$ be a differential polynomial in f of degree d with rational functions as its coefficients. Suppose that $p_1(z)$, $p_2(z)$ are rational functions and $\alpha_1(z)$, $\alpha_2(z)$ are polynomials. If $d\leqslant n-2$ and the differential equation*

$$f^n(z) + Q_d(z,f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)} \tag{3}$$

admits a meromorphic solution f with finitely many poles, then $\alpha_1'(z)/\alpha_2'(z)$ is a rational number. Furthermore, only one of the following four cases holds:

(i) $f(z) = q(z)\,\mathrm{e}^{P(z)}$ and $\alpha_1'(z)/\alpha_2'(z) = 1$, where $q(z)$ is a rational function and $P(z)$ is a polynomial with $nP'(z) = \alpha_1'(z) = \alpha_2'(z)$;

(ii) $f(z) = q(z)\,\mathrm{e}^{P(z)}$ and either $\alpha_1'(z)/\alpha_2'(z) = k/n$ or $\alpha_1'(z)/\alpha_2'(z) = n/k$, where $q(z)$ is a rational function, k is an integer with $1 \leqslant k \leqslant d$ and $P(z)$ is a polynomial with $nP'(z) = \alpha_1'(z)$ or $nP'(z) = \alpha_2'(z)$;

(iiii) f satisfies the first order linear differential equation

$$f' = \left(\frac{1}{n}\frac{p_2'(z)}{p_2(z)} + \frac{1}{n}\alpha_2'(z)\right)f + \psi,$$

and $\alpha_1'(z)/\alpha_2'(z) = (n-1)/n$ or f satisfies the first order linear differential equation

$$f' = \left(\frac{1}{n}\frac{p_1'(z)}{p_1(z)} + \frac{1}{n}\alpha_1'(z)\right)f + \psi,$$

and $\alpha_1'(z)/\alpha_2'(z) = n/(n-1)$, where ψ is a rational function;

(iv) $f(z) = \gamma_1(z)(z)\,\mathrm{e}^{\beta_1(z)} + \gamma_2(z)(z)\,\mathrm{e}^{-\beta_1(z)}$ and $\alpha_1'(z)/\alpha_2'(z) = -1$, where $\gamma_1(z)$, $\gamma_2(z)$ are rational functions and $\beta_1(z)$ is a polynomial with $n\beta_1'(z) = \alpha_1'(z)$ or $n\beta_1'(z) = \alpha_2'(z)$.

A natural problem arises whether similar conclusions are valid in a more general case, with $\alpha_1(z)$, $\alpha_2(z)$ being any nonconstant entire functions, $p_1(z)$, $p_2(z)$ being nonzero small functions of $f(z)$ and the coefficients of $Q_d(z,f)$ in equation (3) also being small functions. Lu, Liao and Wang [21] proved the corresponding results.

In another way, if the left side of equation (3) adds a term $R(z)f^{n-1}(z)$, where $R(z)$ is a rational function, then Liao, Yang and Zhang [10] proved that similar conclusions still hold. If the term f^n is replaced by $f^n f'$ in the differential equation (3), then it is quite difficult to find out accurate meromorphic solutions. Recently, Liao [22, 23] obtained some related results. In 2017, Zhang, Xu and Liao [15] gave the result as follows.

Theorem 3 ([15]) *Let $n \geqslant 3$ be an integer and $Q_d(z,f)$ be a differential polynomial in f of degree d with rational functions as its coefficients. Suppose that $p_1(z)$, $p_2(z)$ are nonzero rational functions, and $\alpha_1(z)$, $\alpha_2(z)$ are nonconstant polynomials. If $d \leqslant n-2$ and the following differential equation:*

$$f^n f' + Q_d(z,f) = p_1(z)\,\mathrm{e}^{\alpha_1(z)} + p_2(z)\,\mathrm{e}^{\alpha_2(z)} \tag{4}$$

admits a meromorphic solution f with finitely many poles, then $\alpha_1'(z)/\alpha_2'(z)$ is a rational function and f is of the following form:

$$f(z) = q(z)\,\mathrm{e}^{P(z)},$$

where $q(z)$ is a rational function, and $P(z)$ is a nonconstant polynomial. Furthermore, only one of the following two cases hold:

(i) $\alpha_1'(z)/\alpha_2'(z) = 1$, $Q_d(z,f) \equiv 0$ and $(n+1)P'(z) = \alpha_1'(z) = \alpha_2'(z)$;

(ii) $\alpha_1'(z)/\alpha_2'(z) = (n+1)/k$, k is an integer with $1 \leqslant k \leqslant d$, $Q_d(z,f) \equiv p_2(z)e^{\alpha_2(z)}$ and $(n+1)P'(z) = \alpha_1'(z)$, or $\alpha_1'(z)/\alpha_2'(z) = k/(n+1)$, k is an integer with $1 \leqslant k \leqslant d$, $Q_d(z,f) \equiv p_1(z)\,\mathrm{e}^{\alpha_1(z)}$ and $(n+1)P'(z) = \alpha_2'(z)$.

Yang and Laine [14] initiated to investigate finite order entire solutions f of the nonlinear differential-difference equations of the form

$$f^n(z) + L(z,f) = h(z). \tag{5}$$

Here n is an integer with $n \geqslant 2$, $L(z,f)$ is a linear differential-difference polynomial in f with small meromorphic functions as its coefficients, and $h(z)$ is a given non-vanishing meromorphic function of finite order. Then they obtained a particular case that the equation $f^2(z) + q(z)f(z+1) = p(z)$ admits no transcendental entire solutions of finite order, where $p(z)$, $q(z)$ are polynomials.

Inspired by Theorem 3 and the result of Yang and Laine [14], we consider the meromorphic solution of nonlinear differential-difference equations and obtain the following result.

Theorem 4 *Let $n \geqslant 3$ be an integer, $p(z)$ be a non-vanishing polynomial and p_1, p_2, α_1, α_2 be nonzero constants such that $\alpha_1 \neq \alpha_2$. Suppose that $\alpha_1/\alpha_2 \neq n+1$ and $\alpha_2/\alpha_1 \neq n+1$. If the equation*

$$f^n f' + p(z)f(z+c) = p_1\,\mathrm{e}^{\alpha_1 z} + p_2\,\mathrm{e}^{\alpha_2 z} \tag{6}$$

admits a meromorphic solution $f(z)$ with hyper order $\sigma_2(f) < 1$, then $n = 3$ and f satisfies $\overline{\lambda}(f) = \sigma(f)$.

Wen, Heittokangas and Laine [12] studied and classified the finite order entire solutions f of the following equation

$$f^n(z) + q(z)\,\mathrm{e}^{Q(z)}f(z+c) = P(z) \tag{7}$$

in terms of growth and zero distribution, where $q(z)$, $Q(z)$, $P(z)$ are polynomials, $n \geqslant 2$ is an integer and $c \in \mathbb{C}\backslash\{0\}$. Later, they obtained the following Theorem 5.

Recall that a function f of the form

$$f(z) = P_1(z)\,\mathrm{e}^{Q_1(z)} + \cdots + P_k(z)\,\mathrm{e}^{Q_k(z)}, \tag{8}$$

where $P_j(z)$ and $Q_j(z)$ are polynomials in z is called an exponential polynomial. Furthermore, two classes of transcendental entire functions are defined as follows:

$$\Gamma_1 = \left\{ e^{\alpha(z)} + d \,:\, d \in \mathbb{C} \text{ and } \alpha(z) \text{ is a nonconstant polynomial} \right\},$$

$$\Gamma_0 = \left\{ e^{\alpha(z)} :\ \alpha(z) \text{ is a nonconstant polynomial} \right\}.$$

Theorem 5 *Let $n \geqslant 2$ be an integer, let $c \in \mathbb{C}\backslash\{0\}$, and let $q(z)$, $Q(z)$, $P(z)$ be polynomials such that $Q(z)$ is not a constant and $q(z) \not\equiv 0$. Then we identify the finite order entire solutions f of equation (7) as follows:*

(i) Every solution f satisfies $\sigma(f) = \deg Q$ and is of mean type.

(ii) Every solution f satisfies $\lambda(f) = \sigma(f)$ if and only if $P(z) \not\equiv 0$.

(iii) A solution f belongs to Γ_0 if and only if $P(z) \equiv 0$. In particular, this is the case if $n \geqslant 3$.

(iv) If a solution f belongs to Γ_0 and if g is any other finite order entire solution to (7), then $f = \eta g$, where $\eta^{n-1} = 1$.

(v) If f is an exponential polynomial solution of the form (8), then $f \in \Gamma_1$. Moreover, if $f \in \Gamma_1 \backslash \Gamma_0$, then $\sigma(f) = 1$.

A natural question is that whether we can get the existence and form of the solution of the equation if we replace $Q_d(z, f)$ with $q(z)e^{Q(z)}f(z+c)$ in equation (4), where $q(z)$ is a rational function and $Q(z)$ is a nonconstant polynomial. The next, we consider the problem and give the form of meromorphic solutions of the following nonlinear differential-difference equation.

Theorem 6 *Let $n \geqslant 4$ be an integer, $q(z)$ be a rational function and $Q(z)$ is a nonconstant polynomial. Suppose that c, p_1, p_2, α_1, $\alpha_2 \in \mathbb{C}\backslash\{0\}$ with $\alpha_1 \neq \alpha_2$. If the equation*

$$f^n f' + q(z) f(z+c) e^{Q(z)} = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \qquad (9)$$

admits a finite order meromorphic solution $f(z)$ with $\lambda(f) < \sigma(f)$ and $\lambda(1/f) < \sigma(f)$, then $f(z)$ satisfies $\sigma(f) = \deg Q = 1$. Furthermore, $f(z) = A_1 e^{\frac{\alpha_2}{n+1}z}$, $Q(z) = \left(\alpha_1 - \frac{\alpha_2}{n+1}\right)z + a_0$ or $f(z) = A_2 e^{\frac{\alpha_1}{n+1}z}$, $Q(z) = \left(\alpha_2 - \frac{\alpha_1}{n+1}\right)z + a_0$, where A_1, A_2, $a_0 \in \mathbb{C}$.

Example 1 Obviously, $f(z) = e^{\frac{1}{2}z}$ is a meromorphic solution of the differential-difference equation

$$f^5 f' + 2f(z - \ln 9) e^{-\frac{11}{2}z + \ln 3} = \tfrac{1}{2} e^{3z} + 2e^{-5z},$$

where $n = 5$, $\alpha_1 = 3$, $\alpha_2 = -5$ and $a_0 = \ln 3$, then $f(z) = A_2 e^{\frac{\alpha_1}{n+1}z} = e^{\frac{1}{2}z}$, $Q(z) = (\alpha_2 - \frac{\alpha_1}{n+1})z + a_0 = -\frac{11}{2}z + \ln 3$ and $\sigma(f) = \deg Q = 1$.

AUXILIARY LEMMAS

In order to prove our results, we first give some lemmas as follows.

Lemma 1 ([2]) *Let $f(z)$ be a meromorphic function and let $k \in \mathbb{N}$. Then*

$$m\left(r, \tfrac{f^{(k)}}{f}\right) = S(r, f),$$

where $S(r, f) = O(\log T(r, f) + \log r)$, possibly outside a set $E_1 \subset [0, \infty)$ of a finite linear measure. If $f(z)$ is a finite order of growth, then

$$m\left(r, \tfrac{f^{(k)}}{f}\right) = O(\log r).$$

Lemma 2 ([7, 24, 25]) *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a finite order meromorphic function. Let σ be the order of $f(z)$, then for each $\varepsilon > 0$, we have*

$$m\left(r, \tfrac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O(r^{\sigma-1+\varepsilon}).$$

Lemma 3 ([14]) *Let $f(z)$ be a transcendental meromorphic function with finite order, and $P(z, f)$, $Q(z, f)$ be two differential-difference polynomials of $f(z)$. If*

$$f^n(z) P(z, f) = Q(z, f)$$

holds and if the total degree of $Q(z, f)$ in $f(z)$ and its derivatives and their shifts is at most n, then

$$m(r, P(z, f)) = S(r, f).$$

Lemma 4 ([4]) *Suppose that $f_1(z)$, $f_1(z)$, ..., $f_n(z)$ ($n \geqslant 2$) are meromorphic functions and $g_1(z)$, $g_2(z)$, ..., $g_n(z)$ are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv 1$;

(ii) $g_j(z) - g_k(z)$ are not constants for $1 \leqslant j < k \leqslant n$;

(iii) For $1 \leqslant j \leqslant n$, $1 \leqslant h < k \leqslant n$, $T(r, f_j) = o(T(r, e^{g_h - g_k}))$, $r \to \infty$ possibly outside a set E.

Then $f_j(z) \equiv 0$ $(j = 1, 2, \ldots, n)$.

PROOF OF Theorem 4

Suppose that (6) has a meromorphic solution $f(z)$ satisfying the conditions of Theorem 4. Firstly, we will prove that (6) does not exists meromorphic solutions with $\sigma_2(f) < 1$ when $n \geqslant 4$. We will arrive at contradictions by considering the following two cases.

Case 1: Suppose that $f(z)$ has at least one pole. Let z_0 be a pole of $f(z)$ with the multiplicity $q(\geqslant 1)$. If $c = 0$, then we can get a contradiction at once by comparing multiplicities of the pole z_0 at both sides of (6). If $c \neq 0$, then we deduce from (6) that $z_0 + c$ is a pole of $f(z)$ with multiplicities at least $(n+1)q+1$. Substituting $z_0 + c$ for z in (6), we have

$$f^n(z_0+c)f'(z_0+c)+p(z_0+c)f(z_0+2c) = p_1 e^{\alpha_1(z_0+c)} + p_2 e^{\alpha_2(z_0+c)}. \quad (10)$$

Similarly, it follows from (10) that z_0+2c is also a pole of $f(z)$ with multiplicities at least $(n+1)^2q+(n+1)+1$. Repeating the above steps, we find that for arbitrary integer $j\,(j \geqslant 1)$, the point $z_0 + jc$ is a pole of $f(z)$ with multiplicities at least $(n+1)^j q + (n+1)^{j-1} + \cdots + 1$. Hence, for each integer m, we have

$$n(m|c|+|z_0|+1, f) \geqslant q + \sum_{j=1}^{m} [(n+1)^j q + (n+1)^{j-1} + \cdots + 1].$$

Thus, we have

$$\begin{aligned}\sigma_2(f) \geqslant \lambda_2(\frac{1}{f}) &= \limsup_{r\to\infty} \frac{\log\log n(r,f)}{\log r} \\ &\geqslant \limsup_{m\to\infty} \frac{\log\log n(m|c|+|z_0|+1, f)}{\log(m|c|+|z_0|+1)} \\ &\geqslant \limsup_{m\to\infty} \frac{\log\log(n+1)^m}{\log m} = 1,\end{aligned}$$

which contradicts with $\sigma_2(f) < 1$.

Case 2: Suppose that $f(z)$ has no poles, that is to say, $f(z)$ is an entire function. If $f(z)$ is a polynomial, by comparing the growth of both sides of equation (6), we find that the order of growth of the left side of (6) is 0. However, the order of growth of the right side of (6) is 1, this is not valid. Hence, we can obtain that $f(z)$ is transcendental.

Set $L = f^n f'$, $P = p(z)f(z+c)$. Then we rewrite equation (6) as follows:

$$L + P = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}. \quad (11)$$

Differentiating both sides of equation (11), we get

$$L' + P' = p_1\alpha_1 e^{\alpha_1 z} + p_2\alpha_2 e^{\alpha_2 z}. \quad (12)$$

Eliminating $e^{\alpha_1 z}$ from (11) and (12), we obtain

$$(\alpha_1 L - L') + (\alpha_1 P - P') = p_2(\alpha_1 - \alpha_2) e^{\alpha_2 z}. \quad (13)$$

Differentiating (13) yields

$$(\alpha_1 L' - L'') + (\alpha_1 P' - P'') = p_2(\alpha_1 - \alpha_2)\alpha_2 e^{\alpha_2 z}. \quad (14)$$

Eliminating $e^{\alpha_2 z}$ from (14) and (13), we have

$$\alpha_1\alpha_2 L - (\alpha_1+\alpha_2)L' + L'' = -[\alpha_1\alpha_2 P - (\alpha_1+\alpha_2)P' + P'']. \quad (15)$$

Note that

$$\begin{aligned}L' &= (f^n f')' = nf^{n-1}(f')^2 + f^n f'' \\ L'' &= (f^n f')'' \\ &= n(n-1)f^{n-2}(f')^3 + 3nf^{n-1}f'f'' + f^n f'''. \quad (16)\end{aligned}$$

Combining (15) with (16), we have

$$f^{n-2}\varphi = -[\alpha_1\alpha_2 P - (\alpha_1+\alpha_2)P' + P''], \quad (17)$$

where

$$\begin{aligned}\varphi = {}&\alpha_1\alpha_2 f^2 f' - (\alpha_1+\alpha_2)(f(f')^2 + f^2 f'') \\ &+ n(n-1)(f')^3 + 3nff'f'' + f^2 f'''. \quad (18)\end{aligned}$$

Since f is an entire function, then φ is also an entire function. Thus $T(r,\varphi) = m(r,\varphi)$. If $\varphi \neq 0$, then by (17), $n \geqslant 4$, and Lemma 3, we have

$$\begin{aligned}T(r,\varphi) = m(r,\varphi) = S(r,f); \\ T(r,f\varphi) = m(r,f\varphi) = S(r,f).\end{aligned} \quad (19)$$

We can deduce from the above two equalities and the first fundamental theorem that

$$T(r,f) \leqslant T(r,f\varphi) + T(r,\tfrac{1}{\varphi}) = T(r,\varphi) + S(r,f) = S(r,f).$$

This is impossible. Thus $\varphi = 0$. Using (17) yields $\alpha_1\alpha_2 P - (\alpha_1+\alpha_2)P' + P'' = 0$.

From $\alpha_1\alpha_2 P - (\alpha_1+\alpha_2)P' + P'' = 0$, we get that P has the form

$$P = p(z)f(z+c) = \widetilde{d}_1 e^{\alpha_1 z} + \widetilde{d}_2 e^{\alpha_2 z},$$

where $\widetilde{d}_1$ and $\widetilde{d}_2$ are constants. So

$$f(z) = \widetilde{t}_1(z) e^{\alpha_1 z} + \widetilde{t}_2(z) e^{\alpha_2 z}, \quad (20)$$

where $\widetilde{t}_1(z) = \frac{\widetilde{d}_1 e^{-\alpha_1 c}}{p(z-c)}$, $\widetilde{t}_2(z) = \frac{\widetilde{d}_2 e^{-\alpha_2 c}}{p(z-c)}$ are rational functions.

In a similar way, we can also deduce from (15) and $f^{n-2}\varphi \equiv 0$ that $\alpha_1\alpha_2 L - (\alpha_1+\alpha_2)L' + L'' = 0$, and then by solving a second-order differential equation, we can get that

$$f^n f' = \widetilde{b}_1 e^{\alpha_1 z} + \widetilde{b}_2 e^{\alpha_2 z}, \quad (21)$$

where $\widetilde{b}_1$ and $\widetilde{b}_2$ are constants. From (20) and (21), we obtain

$$\begin{aligned}&\widetilde{b}_1 e^{\alpha_1 z} + \widetilde{b}_2 e^{\alpha_2 z} \\ &= \left(\widetilde{t}_1(z) e^{\alpha_1 z} + \widetilde{t}_2(z) e^{\alpha_2 z}\right)^n \left(\widetilde{t}_1(z) e^{\alpha_1 z} + \widetilde{t}_2(z) e^{\alpha_2 z}\right)' \\ &= \sum_{j=0}^{n} C_n^j \left(\widetilde{t}_1(z) e^{\alpha_1 z}\right)^j \left(\widetilde{t}_2(z) e^{\alpha_2 z}\right)^{n-j} \\ &\qquad \times \left(\widetilde{t}_1(z) e^{\alpha_1 z} + \widetilde{t}_2(z) e^{\alpha_2 z}\right)'. \quad (22)\end{aligned}$$

Dividing both sides of equation (22) by $e^{\alpha_2 z}$, we have

$$\widetilde{b}_1 e^{(\alpha_1-\alpha_2)z}+\widetilde{b}_2=(\widetilde{t}_1'(z)+\alpha_1\widetilde{t}_1(z))\times\Big(\sum_{j=0}^{n-1}C_n^j\widetilde{t}_1^j(z)\widetilde{t}_2^{n-j}(z)e^{(j+1)\alpha_1 z+(n-j-1)\alpha_2 z}+\widetilde{t}_1^n(z)e^{(n+1)\alpha_1 z-\alpha_2 z}\Big)+(\widetilde{t}_2'(z)+\alpha_2\widetilde{t}_2(z))\times\sum_{j=0}^{n}C_n^j\widetilde{t}_1^j(z)\widetilde{t}_2^{n-j}(z)e^{j\alpha_1 z+(n-j)\alpha_2 z}. \quad (23)$$

Since $\frac{\alpha_1}{\alpha_2}\neq n+1$, $\frac{\alpha_2}{\alpha_1}\neq 1, n+1$, from (23) and Lemma 4, we obtain that $\widetilde{t}_1(z)\equiv 0$. Similarly, dividing both side of (22) by $e^{\alpha_1 z}$, we can get $\widetilde{t}_2=0$. Then $f=0$, which is a contradiction.

Combining Case 1 and Case 2, we obtain that equation (6) does not possess any meromorphic solutions of $\sigma_2(f)<1$ when $n\geqslant 4$.

Next, we need to prove $\overline{\lambda}(f)=\sigma(f)$ when $n=3$. We can rewrite (17) as

$$f\phi=-[\alpha_1\alpha_2 P-(\alpha_1+\alpha_2)P'+P''], \quad (24)$$

where $\phi=\varphi$.

If $\phi\equiv 0$, then we get a contradiction by the similar proof of Case 2. If $\phi\not\equiv 0$, then from (24) and Lemma 3, we have

$$T(r,\phi)=m(r,\phi)+N(r,\phi)=S(r,f). \quad (25)$$

Furthermore, we conclude from the lemma of logarithmic derivatives that

$$3m\left(r,\tfrac{1}{f}\right)\leqslant m\left(r,\tfrac{1}{\phi}\right)+m\left(r,\tfrac{\phi}{f^3}\right)\leqslant m\left(r,\tfrac{1}{\phi}\right)+S(r,f). \quad (26)$$

From (18), if z_0 be a multiple zero of f, then z_0 is also a zero of ϕ and it follows that

$$N\left(r,\tfrac{1}{f}\right)\leqslant \overline{N}\left(r,\tfrac{1}{f}\right)+N\left(r,\tfrac{1}{\phi}\right)+S(r,f). \quad (27)$$

Equation (25) yields $m(r,1/\phi)+N(r,1/\phi)=S(r,f)$, and then by (26) and (27), we obtain

$$\begin{aligned}T(r,f)&=T\left(r,\tfrac{1}{f}\right)+S(r,f)\\&=m\left(r,\tfrac{1}{f}\right)+N\left(r,\tfrac{1}{f}\right)+S(r,f)\\&\leqslant \tfrac{1}{3}m\left(r,\tfrac{1}{\phi}\right)+\overline{N}\left(r,\tfrac{1}{f}\right)+N\left(r,\tfrac{1}{\phi}\right)+S(r,f)\\&\leqslant \overline{N}\left(r,\tfrac{1}{f}\right)+S(r,f).\end{aligned}$$

This yields $\sigma(f)\leqslant\overline{\lambda}(f)$, and hence $\overline{\lambda}(f)=\sigma(f)$.

PROOF OF Theorem 6

Let $f(z)$ be a transcendental meromorphic solution of finite order of equation (9) with $\lambda(f)<\sigma(f)$ and $\lambda(\frac{1}{f})<\sigma(f)$. Firstly, we prove that $\sigma(f)=1$.

Case 1: If $\sigma(f)<1$, then by (9), Lemma 1, Lemma 2 and $q(z)$ is a rational function, we have

$$\begin{aligned}T(r,e^{Q(z)})&=m(r,e^{Q(z)})=m\left(r,\frac{p_1e^{\alpha_1 z}+p_2e^{\alpha_2 z}-f^nf'}{q(z)f(z+c)}\right)\\&\leqslant m\left(r,\tfrac{1}{q(z)f(z+c)}\right)+m(r,p_1e^{\alpha_1 z}+p_2e^{\alpha_2 z})+m(r,f^nf')+S(r,f)\\&\leqslant m\left(r,\tfrac{f}{q(z)f(z+c)}\right)+m\left(r,\tfrac{1}{f}\right)+m(r,p_1e^{\alpha_1 z}+p_2e^{\alpha_2 z})+m(r,f^n)+m\left(r,\tfrac{f'}{f}\right)+m(r,f)+S(r,f)\\&\leqslant(n+1)T(r,f)+T\left(r,\tfrac{1}{f}\right)+m(r,p_1e^{\alpha_1 z}+p_2e^{\alpha_2 z})+S(r,f)\\&\leqslant T(r,p_1e^{\alpha_1 z}+p_2e^{\alpha_2 z})+S(r,p_1e^{\alpha_1 z}+p_2e^{\alpha_2 z}). \quad (28)\end{aligned}$$

Therefore, $\deg Q\leqslant 1$.

Note that $\deg Q\geqslant 1$, therefore $\deg Q=1$. Denote $Q(z)=a_1z+a_0$, where a_0, a_1 are constants and $a_1\neq 0$. Rewriting (9) as follows:

$$f^nf'+q(z)f(z+c)e^{a_1z+a_0}=p_1e^{\alpha_1 z}+p_2e^{\alpha_2 z}. \quad (29)$$

Differentiating (29), we get

$$nf^{n-1}(f')^2+f^nf''+\left[\left(q(z)f(z+c)\right)'+a_1q(z)f(z+c)\right]e^{a_1z+a_0}=\alpha_1p_1e^{\alpha_1 z}+\alpha_2p_2e^{\alpha_2 z}. \quad (30)$$

Eliminating $e^{\alpha_1 z}$ from (29) and (30) yields

$$nf^{n-1}(f')^2+f^nf''-\alpha_1f^nf'+\left[\left(q(z)f(z+c)\right)'+a_1q(z)f(z+c)-\alpha_1q(z)f(z+c)\right]e^{a_1z+a_0}=(\alpha_2-\alpha_1)p_2e^{\alpha_2 z}. \quad (31)$$

Subcase 1.1: If $a_1\neq\alpha_2$, then from Lemma 4 and (31), we have

$$\alpha_2-\alpha_1\equiv 0,$$

which contradicts with $\alpha_1\neq\alpha_2$.

Subcase 1.2: If $a_1=\alpha_2$, then by (31), we obtain

$$nf^{n-1}(f')^2+f^nf''-\alpha_1f^nf'+\left[\left(q(z)f(z+c)\right)'e^{a_0}+a_1q(z)f(z+c)e^{a_0}-\alpha_1q(z)f(z+c)e^{a_0}-(\alpha_2-\alpha_1)p_2\right]e^{\alpha_2 z}=0. \quad (32)$$

By Lemma 4 and (32), we have

$$nf^{n-1}(f')^2+f^nf''-\alpha_1f^nf'\equiv 0. \quad (33)$$

From (33), we obtain

$$n(f')^2+ff''-\alpha_1 ff'\equiv 0. \quad (34)$$

Dividing with f^2 on both sides in equation (34) and recalling $f''/f=(f'/f)'+(f'/f)^2$, we get a Bernoulli equation

$$t'-\alpha_1 t=-(n+1)t^2, \quad (35)$$

where $t=f'/f$.

A routine computation yields

$$t=\frac{\alpha_1 e^{\alpha_1 z}}{(n+1)e^{\alpha_1 z}+\alpha_1 C_1}=\frac{1}{n+1}\frac{((n+1)e^{\alpha_1 z}+\alpha_1 C_1)'}{(n+1)e^{\alpha_1 z}+\alpha_1 C_1},$$

so

$$f^{n+1}=C_2[(n+1)e^{\alpha_1 z}+\alpha_1 C_1],$$

where C_1, C_2 are constants.

If $C_2=0$, then $f^{n+1}=0$. This is a contradiction. If $C_2\neq 0$, then $\sigma(f)=1$, which contradicts with $\sigma(f)<1$.

Case 2: Suppose that $\sigma(f)>1$. Denoting $P(z)=p_1 e^{\alpha_1 z}+p_2 e^{\alpha_2 z}$ and $H(z)=q(z)f(z+c)$. Equation (9) can be written as

$$f^n f'+H(z)e^{Q(z)}=P(z). \quad (36)$$

Differentiating (36), we get

$$nf^{n-1}(f')^2+f^n f''+(H'(z)+Q'(z)H(z))e^{Q(z)}=P'(z). \quad (37)$$

Eliminating $e^{Q(z)}$ from (36) and (37) yields

$$nH(z)f^{n-1}(f')^2+H(z)f^n f''+\big(H'(z)+Q'(z)H(z)\big)(P(z)-f^n f')=H(z)P'(z). \quad (38)$$

Thus, we have

$$f^{n-1}(nH(z)(f')^2+H(z)ff''-(H'(z)+Q'(z)H(z))ff')=H(z)P'(z)-(H'(z)+Q'(z)H(z))P(z). \quad (39)$$

Since $\deg[H(z)P'(z)-(H'(z)+Q'(z)H(z))P(z)]=1$ in $f(z)$ and $n\geqslant 4$, by Lemma 3 and (39) we obtain

$$m(r,nH(z)(f')^2+H(z)ff''-(H'(z)+Q'(z)H(z))ff')=S(r,f)$$

and

$$m(r,f(nH(z)(f')^2+H(z)ff''-(H'(z)+Q'(z)H(z))ff'))=S(r,f).$$

If $nH(z)(f')^2+H(z)ff''-(H'(z)+Q'(z)H(z))ff'\not\equiv 0$ and $N(r,f)=S(r,f)$, then

$$\begin{aligned}T(r,f)&=m(r,f)+N(r,f)\\&=m\Big(r,\tfrac{f(nH(z)(f')^2+H(z)ff''-(H'(z)+Q'(z)H(z))ff')}{nH(z)(f')^2+H(z)ff''-(H'(z)+Q'(z)H(z))ff'}\Big)+S(r,f)\\&=S(r,f),\end{aligned}$$

which yields a contradiction.

If $nH(z)(f')^2+H(z)ff''-(H'(z)+Q'(z)H(z))ff'\equiv 0$, then

$$n\frac{f'}{f}+\frac{f''}{f'}=\frac{H'(z)}{H(z)}+Q'(z).$$

By integration, we have

$$Cf^n f'=H(z)e^{Q(z)}=q(z)f(z+c)e^{Q(z)},\ C\in\mathbb{C}\backslash\{0\}. \quad (40)$$

From (40) and (9), we can get

$$(1+C)f^n f'=p_1 e^{\alpha_1 z}+p_2 e^{\alpha_2 z}. \quad (41)$$

If $C=-1$, then $p_1=p_2=0$. This is a contradiction. If $C\neq -1$, (41) yields f is an entire function and $\sigma(f)\leqslant 1$ when $C\neq -1$. A contradiction follows because we assume that $\sigma(f)>1$.

Combining Case 1 and Case 2, we have $\sigma(f)=1$. By Lemma 1, Lemma 2 and from (9), we have

$$\begin{aligned}T(r,e^{Q(z)})&=m(r,e^{Q(z)})=m\Big(r,\tfrac{p_1 e^{\alpha_1 z}+p_2 e^{\alpha_2 z}-f^n f'}{q(z)f(z+c)}\Big)\\&\leqslant m\Big(r,\tfrac{1}{q(z)f(z+c)}\Big)+m(r,p_1e^{\alpha_1 z}+p_2e^{\alpha_2 z})\\&\quad+m(r,f^n f')+S(r,f)\\&\leqslant m\Big(r,\tfrac{f}{q(z)f(z+c)}\Big)+m(r,\tfrac{1}{f})+m(r,p_1e^{\alpha_1 z}+p_2e^{\alpha_2 z})\\&\quad+m\Big(r,\tfrac{f^n f'}{f}\Big)+m(r,f)+S(r,f)\\&\leqslant(n+2)T(r,f)+T(r,p_1e^{\alpha_1 z}+p_2e^{\alpha_2 z})\\&\quad+S(r,f)+S(r,p_1e^{\alpha_1 z}+p_2e^{\alpha_2 z}).\end{aligned}$$

Thus, we have $\deg Q=1$. Denote $Q(z)=a_1 z+a_0$, where a_1, a_0 are constants and $a_1\neq 0$. From (9), we have

$$f^n f'+q_1(z)f(z+c)e^{a_1 z}=p_1 e^{\alpha_1 z}+p_2 e^{\alpha_2 z}, \quad (42)$$

where $q_1(z)=q(z)e^{a_0}$.

According to Hadamard decomposition theorem, $f(z)$ is of the following form

$$f(z)=h(z)e^{az}.$$

Substituting $f(z)=h(z)e^{az}$ into the (42) yields

$$T_1 e^{((n+1)a-\alpha_2)z}+T_2 e^{(a_1+a-\alpha_2)z}=p_1 e^{(\alpha_1-\alpha_2)z}+p_2, \quad (43)$$

where

$$T_1=(h'+ah)h^n,$$
$$T_2=q_1(z)h(z+c)e^{ac}.$$

Now we consider four cases as below.

Case A: Suppose that $(n+1)a-\alpha_2=0$ and $a_1+a-\alpha_2=0$. By Lemma 4 and (43), we have $p_1=0$, this is a contradiction.

Case B: Suppose that $(n+1)a-\alpha_2=0$ and $a_1+a-\alpha_2\neq 0$.

If $a_1+a-\alpha_2=\alpha_1-\alpha_2$, then $a=\alpha_2/(n+1)$ and $a_1=\alpha_1-\alpha_2/(n+1)$. Substituting these into (43), we have

$$T_1-p_2+(T_2-p_1)\,\mathrm{e}^{(\alpha_1-\alpha_2)z}=0, \tag{44}$$

by Lemma 4 and (44), we have $T_1-p_2\equiv T_2-p_1\equiv 0$, then $h(z)$, $q(z)$ reduce to non-zero constant, $f(z)=A_1\,\mathrm{e}^{\frac{\alpha_2}{n+1}z}$, and $Q(z)=(\alpha_1-\frac{\alpha_2}{n+1})z+a_0$.

If $a_1+a-\alpha_2\neq\alpha_1-\alpha_2$, then by Lemma 4 and (43), we have $p_1\equiv 0$. This is a contradiction.

Case C: Suppose that $(n+1)a-\alpha_2\neq 0$ and $a_1+a-\alpha_2=0$.

If $(n+1)a-\alpha_2=\alpha_1-\alpha_2$, then $a=\alpha_1/(n+1)$ and $a_1=\alpha_2-\alpha_1/(n+1)$. Substituting these into (43), we have

$$T_2-p_2+(T_1-p_1)\,\mathrm{e}^{(\alpha_1-\alpha_2)z}=0. \tag{45}$$

By Lemma 4 and (45), we have $T_2-p_2\equiv T_1-p_1\equiv 0$. Clearly, $h(z)$ and $q(z)$ reduce to non-zero constants, and hence we have $f(z)=A_2\,\mathrm{e}^{\frac{\alpha_1}{n+1}z}$ and $Q(z)=(\alpha_2-\frac{\alpha_1}{n+1})z+a_0$.

If $(n+1)a-\alpha_2\neq\alpha_1-\alpha_2$, then by Lemma 4 and (43), we have $p_1\equiv 0$. This is a contradiction.

Case D: Suppose that $(n+1)a-\alpha_2\neq 0$ and $a_1+a-\alpha_2\neq 0$.

If $(n+1)a-\alpha_2$, $a_1+a-\alpha_2$ and $\alpha_1-\alpha_2$ are pairwise distinct from each other, then by Lemma 4 and (43), we have $p_1\equiv p_2\equiv q(z)\equiv 0$. This is a contradiction.

If only two of $(n+1)a-\alpha_2$, $a_1+a-\alpha_2$ and $\alpha_1-\alpha_2$ coincide, without loss of generality, suppose that $(n+1)a-\alpha_2=a_1+a-\alpha_2\neq\alpha_1-\alpha_2$, then we write (43) as

$$(T_1+T_2)\,\mathrm{e}^{(a_1+a-\alpha_2)z}=p_1\,\mathrm{e}^{(\alpha_1-\alpha_2)z}+p_2.$$

It follows from Lemma 4 that $p_1\equiv p_2\equiv 0$, this is a contradiction.

If $(n+1)a-\alpha_2=a_1+a-\alpha_2=\alpha_1-\alpha_2$, then (43) can be written as

$$(T_1+T_2-p_1)\,\mathrm{e}^{(\alpha_1-\alpha_2)z}=p_2.$$

From the above equality and using Lemma 4, we have $p_2\equiv 0$, which implies a contradiction.

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REFERENCES

1. Gross F (1973) *Factorization of Meromorphic Functions*, Mathematics Research Center, Naval Research Laboratory, Washington, DC.
2. Hayman WK (1964) *Meromorphic Functions*, Clarendon Press, Oxford.
3. Laine I (2011) *Nevanlinna Theory and Complex Differential Equations*, Walter De Gruyter, Berlin.
4. Yang CC, Yi HX (2003) *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers Group, Dordrecht.
5. Chen MF, Gao ZS, Zhang JL (2019) Entire solutions of certain type of non-linear difference equations. *Comput Methods Funct Theory* **19**, 17–36.
6. Chen W, Hu PC, Zhang YY (2016) On solutions to some nonlinear difference and differential equations. *J Korean Math Soc* **53**, 835–846.
7. Chiang YM, Feng SJ (2008) On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane. *Ramanujan J* **16**, 105–129.
8. Latreuch Z (2017) On the existence of entire solutions of certain class of nonlinear difference equations. *Mediterranean J Math* **3**, ID 115.
9. Li P (2011) Entire solutions of certain type of differential equations II. *J Math Anal Appl* **375**, 310–319.
10. Liao LW, Yang CC, Zhang JJ (2013) On meromorphic solutions of certain type of non-linear differential equations. *Ann Acad Sci Fenn Math* **38**, 581–593.
11. Rong JX, Xu JF (2019) Three results on the nonlinear differential equations and differential-difference equations. *Mathematics* **7**, ID 539.
12. Wen ZT, Heittokangas J, Laine I (2012) Exponential polynomials as solutions of certain nonlinear difference equations. *Acta Math Sin* **28**, 1295–1306.
13. Wu N, Xuan, ZX (2019) On the growth of solutions to linear complex differential equations on annuli. *Bull Korean Math Soc* **56**, 461–470.
14. Yang CC, Laine I (2010) On analogies between nonlinear difference and differential equations. *Proc Japan Acad* **86**, 10–14.
15. Zhang JJ, Xu XP, Liao LW (2017) Meromorphic solutions of nonlinear complex differential equations. *Sci Sin Math* **47**, 919–932. [in Chinese]
16. Zhang RR, Huang ZB (2018) On meromorphic solutions of non-linear difference equations. *Comput Methods Funct Theory* **18**, 389–408.
17. Heittokangas J, Korhonen R, Laine I (2002) On meromorphic solutions of certain nonlinear differential equations. *Bull Austral Math Soc* **66**, 331–343.
18. Yang CC, Li P (2004) On the transcendental solutions of a certain type of nonlinear differential equations. *Arch Math* **82**, 442–448.
19. Yang CC (2001) On entire solutions of a certain type of nonlinear differential equations. *Bull Austral Math Soc* **64**, 377–380.
20. Li P, Yang CC (2006) On the non-existence of entire solutions of certain type of nonlinear differential equations. *J Math Anal Appl* **320**, 827–835.
21. Lu XQ, Liao LW, Wang J (2017) On meromorphic solutions of a certain type of nonlinear differential equations. *Acta Math Sin* **33**, 1597–1608.
22. Liao LW (2015) The new developments in the research of nonlinear complex differential equations. *J Jiangxi Norm Univ Nat Sci* **39**, 331–339. [in Chinese]
23. Liao LW (2014) On solutions to nonhomogeneous algebraic differential equations and their application. *J Aust Math Soc* **97**, 391–403.
24. Halburd RG, Korhonen RJ (2006) Difference analogue of the lemma on the logarithmic derivative with applications to difference equations. *J Math Anal Appl* **314**, 477–487.
25. Halburd RG, Korhonen RJ (2006) Nevanlinna theory for the difference operator. *Ann Acad Sci Fenn Math* **31**, 463–478.