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Nonlinear mixed Lie triple derivations on prime $*$-rings

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ABSTRACT: Let $\mathscr{R}$ be a 2-torsion free unital prime $*$-ring containing a nontrivial symmetric idempotent. We prove that if a map $\delta : \mathscr{R} \to \mathscr{R}$ satisfies $\delta([[A,B]_*,C]) = [[\delta(A),B]_*,C] + [[A,\delta(B)]_*,C] + [[A,B]_*,\delta(C)]$ for all $A,B,C \in \mathscr{R}$, then δ is an additive $*$-derivation.

KEYWORDS: mixed Lie triple derivation, $*$-derivation, prime $*$-ring

MSC2020: 16W25 46L10

INTRODUCTION

Let $\mathscr{A}$ be an algebra. For $A,B \in \mathscr{A}$, denote by $A\circ B = AB + BA$ and $[A,B] = AB - BA$ the Jordan product and Lie product of A and B, respectively. In some sense, the Jordan product and Lie product are used to characterize the algebraic structure. There are many papers in the literature related to the Jordan product and Lie product, see for example [1–7]. A map $\delta : \mathscr{A} \to \mathscr{A}$ (without the linearity assumption) is called a nonlinear Lie triple derivation if

$$\delta([[A,B],C]) = [[\delta(A),B],C] + [[A,\delta(B)],C] + [[A,B],\delta(C)]$$

for all $A,B,C \in \mathscr{A}$. Ji, Liu and Zhao [8] obtained the structure of nonlinear Lie triple derivations on triangular algebras. Chen and Xiao [9] characterized nonlinear Lie triple derivations on parabolic subalgebras of finite-dimensional simple Lie algebras.

Let $\mathscr{A}$ be a $*$-algebra. For $A,B \in \mathscr{A}$, denote by $[A,B]_* = AB - BA^*$ the skew Lie product of A and B. In the last decade, the study related to the skew Lie product has attracted several authors' attention, see for example [10–13]. A map $\delta : \mathscr{A} \to \mathscr{A}$ (without the linearity assumption) is called a nonlinear skew Lie triple derivation if

$$\delta([[A,B]_*,C]_*) = [[\delta(A),B]_*,C]_* + [[A,\delta(B)]_*,C]_* + [[A,B]_*,\delta(C)]_*$$

for all $A,B,C \in \mathscr{A}$. A map $\delta : \mathscr{A} \to \mathscr{A}$ is called an additive $*$-derivation if it is an additive derivation and satisfies $\delta(A^*) = \delta(A)^*$ for all $A \in \mathscr{A}$. Li, Zhao and Chen [14] proved that every nonlinear skew Lie triple derivation on factor von Neumann algebras is an additive $*$-derivation. Taghavi, Nouri and Darvish [15] proved that every nonlinear skew Lie triple derivation on prime $*$-algebras is additive. A map $\delta : \mathscr{A} \to \mathscr{A}$ (without the linearity assumption) is called a nonlinear mixed Lie triple derivation if

$$\delta([[A,B]_*,C]) = [[\delta(A),B]_*,C] + [[A,\delta(B)]_*,C] + [[A,B]_*,\delta(C)]$$

for all $A,B,C \in \mathscr{A}$. Liang and Zhang [16] gave concrete examples showing that nonlinear mixed Lie triple derivations are different from both nonlinear Lie triple derivations and nonlinear skew Lie triple derivations in general. They proved that every nonlinear mixed Lie triple derivation on factor von Neumann algebras is an additive $*$-derivation. Zhou, Yang and Zhang [17] generalized the above result to the case of prime $*$-algebras.

Let $\mathscr{R}$ be a ring. $\mathscr{R}$ is called a $*$-ring if there is a map $* : \mathscr{R} \to \mathscr{R}$ satisfying $(AB)^* = B^*A^*$, $(A+B)^* = A^*+B^*$ and $(A^*)^* = A$ for all $A,B \in \mathscr{R}$. $\mathscr{R}$ is called prime when, for $A,B \in \mathscr{R}$, $A\mathscr{R}B = \{0\}$ implies $A = 0$ or $B = 0$. $\mathscr{R}$ is called 2-torsion free when, for $A \in \mathscr{R}$, $2A = 0$ implies $A = 0$. An element $A \in \mathscr{R}$ is called symmetric if $A^* = A$. Note that the imaginary number unit i played an important role in [16, 17]. However, $*$-rings do not contain the imaginary number unit i. Motivated by the above mentioned works, we will study nonlinear mixed Lie triple derivations on prime $*$-rings.

MAIN RESULT

The main result is the following theorem.

Theorem 1 *Let $\mathscr{R}$ be a 2-torsion free unital prime $*$-ring containing a nontrivial symmetric idempotent. If a map $\delta : \mathscr{R} \to \mathscr{R}$ satisfies*

$$\delta([[A,B]_*,C]) = [[\delta(A),B]_*,C] + [[A,\delta(B)]_*,C] + [[A,B]_*,\delta(C)]$$

for all $A,B,C \in \mathscr{R}$, then δ is an additive $*$-derivation.

Let $P \in \mathscr{R}$ be a nontrivial symmetric idempotent. Write $P_1 = P$, $P_2 = I - P_1$, $\mathscr{R}_{ij} = P_i\mathscr{R}P_j$ $(i,j = 1,2)$. Then $\mathscr{R}=\mathscr{R}_{11}+\mathscr{R}_{12}+\mathscr{R}_{21}+\mathscr{R}_{22}$. For every $A \in \mathscr{R}$, $A = A_{11} + A_{12} + A_{21} + A_{22}$, $A_{ij} \in \mathscr{R}_{ij}$ $(i,j = 1,2)$.

Lemma 1
(a) $\delta(P_i)^* = \delta(P_i)$, $i = 1, 2$;
(b) $P_i\delta(P_i)P_j = -P_i\delta(P_j)P_j$, $1 \leqslant i \neq j \leqslant 2$.

Proof: **(a)**: Let $1 \leqslant i \neq j \leqslant 2$. It is clear that $\delta(0) = 0$. Since $[[P_j, P_i]_*, P_j] = 0$, we have

$$\begin{aligned}0 &= \delta([[P_j, P_i]_*, P_j]) \\ &= [[\delta(P_j), P_i]_*, P_j] + [[P_j, \delta(P_i)]_*, P_j] + [[P_j, P_i]_*, \delta(P_j)] \\ &= -P_i\delta(P_j)^*P_j - P_j\delta(P_j)P_i + 2P_j\delta(P_i)P_j \\ &\quad - \delta(P_i)P_j - P_j\delta(P_i). \end{aligned} \tag{1}$$

Multiplying (1) by P_i from the left and by P_j from the right, we have

$$P_i\delta(P_j)^*P_j = -P_i\delta(P_i)P_j, \tag{2}$$

which yields that

$$P_j\delta(P_j)P_i = -P_j\delta(P_i)^*P_i. \tag{3}$$

Multiplying (1) by P_j from the left and by P_i from the right, we have

$$P_j\delta(P_j)P_i = -P_j\delta(P_i)P_i. \tag{4}$$

Comparing (3) and (4), we get

$$P_j\delta(P_i)^*P_i = P_j\delta(P_i)P_i. \tag{5}$$

Since $[[P_i, P_i]_*, P_i] = 0$, we have

$$\begin{aligned}0 &= \delta([[P_i, P_i]_*, P_i]) \\ &= [[\delta(P_i), P_i]_*, P_i] + [[P_i, \delta(P_i)]_*, P_i] + [[P_i, P_i]_*, \delta(P_i)] \\ &= -P_i\delta(P_i)^*P_i + P_i\delta(P_i)^* + P_i\delta(P_i)P_i - P_i\delta(P_i). \end{aligned} \tag{6}$$

Multiplying (6) by P_i from the left and by P_j from the right, we have

$$P_i\delta(P_i)^*P_j = P_i\delta(P_i)P_j. \tag{7}$$

For every $X_{ij} \in \mathscr{R}_{ij}$, it follows from $[[X_{ij}, P_i]_*, P_i] = 0$ that

$$\begin{aligned}0 &= \delta([[X_{ij}, P_i]_*, P_i]) \\ &= [[\delta(X_{ij}), P_i]_*, P_i] + [[X_{ij}, \delta(P_i)]_*, P_i] + [[X_{ij}, P_i]_*, \delta(P_i)] \\ &= \delta(X_{ij})P_i - P_i\delta(X_{ij})^*P_i - P_i\delta(X_{ij})P_i + P_i\delta(X_{ij})^* \\ &\quad + X_{ij}\delta(P_i)P_i - \delta(P_i)X_{ij}^* - X_{ij}\delta(P_i) + P_i\delta(P_i)X_{ij}^*. \end{aligned} \tag{8}$$

Multiplying (8) by P_i from the left and by P_j from the right, we have

$$P_i\delta(X_{ij})^*P_j = X_{ij}\delta(P_i)P_j. \tag{9}$$

Multiplying (8) by P_j from the left and by P_i from the right, we have

$$P_j\delta(X_{ij})P_i = P_j\delta(P_i)X_{ij}^*. \tag{10}$$

It follows from (10) that

$$P_i\delta(X_{ij})^*P_j = X_{ij}\delta(P_i)^*P_j. \tag{11}$$

Comparing (9) and (11), we get

$$X_{ij}(P_j\delta(P_i)P_j - P_j\delta(P_i)^*P_j) = 0$$

for all $X_{ij} \in \mathscr{R}_{ij}$. By $\mathscr{R}$ is prime, we have

$$P_j\delta(P_i)^*P_j = P_j\delta(P_i)P_j. \tag{12}$$

Since $[[P_i, P_i]_*, X_{ij}] = 0$, we have

$$\begin{aligned}0 &= \delta([[P_i, P_i]_*, X_{ij}]) \\ &= [[\delta(P_i), P_i]_*, X_{ij}] + [[P_i, \delta(P_i)]_*, X_{ij}] + [[P_i, P_i]_*, \delta(X_{ij})] \\ &= \delta(P_i)X_{ij} - P_i\delta(P_i)^*X_{ij} - X_{ij}\delta(P_i)P_i + P_i\delta(P_i)X_{ij} \\ &\quad - \delta(P_i)X_{ij} + X_{ij}\delta(P_i)P_i. \end{aligned} \tag{13}$$

Multiplying (13) by P_i from the left and by P_j from the right, we have

$$(P_i\delta(P_i)P_i - P_i\delta(P_i)^*P_i)X_{ij} = 0.$$

This gives that

$$P_i\delta(P_i)^*P_i = P_i\delta(P_i)P_i. \tag{14}$$

From (5), (7), (12) and (14), we get that (a) holds.

(b): It follows from (2) and (a) that (b) holds. □

Lemma 2 *For every $X_{ij} \in \mathscr{R}_{ij}$ $(1 \leqslant i \neq j \leqslant 2)$, we have*

$$P_j\delta(X_{ij})P_i = 0.$$

Proof: It follows from $[[P_i, X_{ij}]_*, X_{ij}] = 0$ and Lemma 1(a) that

$$\begin{aligned}0 &= \delta([[P_i, X_{ij}]_*, X_{ij}]) \\ &= [[\delta(P_i), X_{ij}]_*, X_{ij}] + [[P_i, \delta(X_{ij})]_*, X_{ij}] + [[P_i, X_{ij}]_*, \delta(X_{ij})] \\ &= -2X_{ij}\delta(P_i)X_{ij} + P_i\delta(X_{ij})X_{ij} - 2\delta(X_{ij})X_{ij} \\ &\quad + X_{ij}\delta(X_{ij})P_i + X_{ij}\delta(X_{ij}). \end{aligned} \tag{15}$$

Multiplying (15) by P_i from both sides and by $\mathscr{R}$ is 2-torsion free, we have $X_{ij}\delta(X_{ij})P_i = 0$ for all $X_{ij} \in \mathscr{R}_{ij}$. Hence $P_j\delta(X_{ij})P_i = 0$. □

Lemma 3 *For every $A_{ij} \in \mathscr{R}_{ij}, B_{ji} \in \mathscr{R}_{ji}$ $(1 \leqslant i \neq j \leqslant 2)$, we have*

$$\delta(A_{ij} + B_{ji}) = \delta(A_{ij}) + \delta(B_{ji}).$$

Proof: Let $T = \delta(A_{ij} + B_{ji}) - \delta(A_{ij}) - \delta(B_{ji})$. We show that $T = 0$. Since $[[A_{ij}, P_i]_*, P_j] = 0$, we have

$$\begin{aligned}&[[\delta(A_{ij} + B_{ji}), P_i]_*, P_j] + [[A_{ij} + B_{ji}, \delta(P_i)]_*, P_j] \\ &\qquad + [[A_{ij} + B_{ji}, P_i]_*, \delta(P_j)] \\ &= \delta([[A_{ij} + B_{ji}, P_i]_*, P_j]) \\ &= \delta([[A_{ij}, P_i]_*, P_j]) + \delta([[B_{ji}, P_i]_*, P_j]) \\ &= [[\delta(A_{ij}) + \delta(B_{ji}), P_i]_*, P_j] + [[A_{ij} + B_{ji}, \delta(P_i)]_*, P_j] \\ &\qquad + [[A_{ij} + B_{ji}, P_i]_*, \delta(P_j)],\end{aligned}$$

which implies that $[[T, P_i]_*, P_j] = 0$, and so $T_{ji} = 0$. Similarly, $T_{ij} = 0$.

For every $X_{ji} \in \mathscr{R}_{ji}$, we have $[[X_{ji}, A_{ij}]_*, P_i] = 0$. It follows that

$$\begin{aligned}
&[[\delta(X_{ji}), A_{ij} + B_{ji}]_*, P_i] + [[X_{ji}, \delta(A_{ij} + B_{ji})]_*, P_i] \\
&\qquad\qquad + [[X_{ji}, A_{ij} + B_{ji}]_*, \delta(P_i)] \\
&= \delta([[X_{ji}, A_{ij} + B_{ji}]_*, P_i]) \\
&= \delta([[X_{ji}, A_{ij}]_*, P_i]) + \delta([[X_{ji}, B_{ji}]_*, P_i]) \\
&= [[\delta(X_{ji}), A_{ij} + B_{ji}]_*, P_i] + [[X_{ji}, \delta(A_{ij}) + \delta(B_{ji})]_*, P_i] \\
&\quad + [[X_{ji}, A_{ij} + B_{ji}]_*, \delta(P_i)].
\end{aligned}$$

This gives that $[[X_{ji}, T]_*, P_i] = 0$, and so $T_{ii} = 0$. Similarly, $T_{jj} = 0$. □

Lemma 4 $P_j \delta(P_i) P_j = P_i \delta(P_i) P_i = 0$ $(1 \leqslant i \neq j \leqslant 2)$.

Proof: From (9) and Lemma 2, we have

$$X_{ij} \delta(P_i) P_j = P_i \delta(X_{ij})^* P_j = (P_j \delta(X_{ij}) P_i)^* = 0$$

for all $X_{ij} \in \mathscr{R}_{ij}$. Hence $P_j \delta(P_i) P_j = 0$.

For every $X_{ji} \in \mathscr{R}_{ji}$, it follows from $[[X_{ji}, P_i]_*, P_i] = X_{ji} + X_{ji}^*$, Lemma 1(a) and Lemma 3 that

$$\begin{aligned}
&\delta(X_{ji}) + \delta(X_{ji}^*) = \delta([[X_{ji}, P_i]_*, P_i]) \\
&= [[\delta(X_{ji}), P_i]_*, P_i] + [[X_{ji}, \delta(P_i)]_*, P_i] + [[X_{ji}, P_i]_*, \delta(P_i)] \\
&= \delta(X_{ji}) P_i - P_i \delta(X_{ji})^* P_i - P_i \delta(X_{ji}) P_i + P_i \delta(X_{ji})^* \\
&\quad + X_{ji} \delta(P_i) P_i + P_i \delta(P_i) X_{ji}^* + (X_{ji} - X_{ji}^*) \delta(P_i) \\
&\quad - \delta(P_i)(X_{ji} - X_{ji}^*). \qquad (16)
\end{aligned}$$

Multiplying (16) by P_j from the left and by P_i from the right, we have

$$P_j \delta(X_{ji}^*) P_i = 2 X_{ji} \delta(P_i) P_i - P_j \delta(P_i) X_{ji}.$$

By Lemma 2, $P_j \delta(P_i) P_j = 0$ and $\mathscr{R}$ is 2-torsion free, we get that $X_{ji} \delta(P_i) P_i = 0$. Hence $P_i \delta(P_i) P_i = 0$. □

Let $T = P_1 \delta(P_1) P_2 - P_2 \delta(P_1) P_1$. It follows from Lemma 1(a) that $T^* = -T$. Let $\Delta : \mathscr{R} \to \mathscr{R}$ be a map defined by

$$\Delta(X) = \delta(X) - (XT - TX).$$

Remark 1 By Lemmas 1–4, it is easy to verify that Δ is also a nonlinear mixed Lie triple derivation and satisfies:

(a) for every $A_{ij} \in \mathscr{R}_{ij}, B_{ji} \in \mathscr{R}_{ji}$ $(1 \leqslant i \neq j \leqslant 2)$, we have
$$\Delta(A_{ij} + B_{ji}) = \Delta(A_{ij}) + \Delta(B_{ji});$$

(b) for every $X_{ij} \in \mathscr{R}_{ij}$ $(1 \leqslant i \neq j \leqslant 2)$, we have
$$P_j \Delta(X_{ij}) P_i = 0;$$

(c) $\Delta(P_i) = 0$, $i = 1, 2$.

Lemma 5 $\Delta(\mathscr{R}_{ij}) \subseteq \mathscr{R}_{ij}$, $i, j = 1, 2$.

Proof: For every $A_{ii} \in \mathscr{R}_{ii}$ $(i = 1, 2)$, it follows from $[[P_i, A_{ii}]_*, P_i] = 0$ and $\Delta(P_i) = 0$ that

$$\begin{aligned}
0 &= \Delta([[P_i, A_{ii}]_*, P_i]) \\
&= [[P_i, \Delta(A_{ii})]_*, P_i] \\
&= 2 P_i \Delta(A_{ii}) P_i - \Delta(A_{ii}) P_i - P_i \Delta(A_{ii}). \qquad (17)
\end{aligned}$$

Multiplying (17) by P_i from the left and by P_j from the right, we have

$$P_i \Delta(A_{ii}) P_j = 0. \qquad (18)$$

Multiplying (17) by P_j from the left and by P_i from the right, we have

$$P_j \Delta(A_{ii}) P_i = 0. \qquad (19)$$

For every $X_{ij} \in \mathscr{R}_{ij}$ $(1 \leqslant i \neq j \leqslant 2)$, it follows from $[[X_{ij}, A_{ii}]_*, P_j] = 0$ and $\Delta(P_j) = 0$ that

$$\begin{aligned}
0 &= \Delta([[X_{ij}, A_{ii}]_*, P_j]) \\
&= [[\Delta(X_{ij}), A_{ii}]_*, P_j] + [[X_{ij}, \Delta(A_{ii})]_*, P_j] \\
&= -A_{ii} \Delta(X_{ij})^* P_j - P_j \Delta(X_{ij}) A_{ii} + X_{ij} \Delta(A_{ii}) P_j \\
&\quad + P_j \Delta(A_{ii}) X_{ij}^*. \qquad (20)
\end{aligned}$$

Multiplying (20) by P_i from the left and by P_j from the right, and by Remark 1(b), we have

$$X_{ij} \Delta(A_{ii}) P_j = A_{ii} \Delta(X_{ij})^* P_j = A_{ii} (P_j \Delta(X_{ij}) P_i)^* = 0.$$

Hence

$$P_j \Delta(A_{ii}) P_j = 0. \qquad (21)$$

From (18), (19) and (21), we have $\Delta(\mathscr{R}_{ii}) \subseteq \mathscr{R}_{ii}$ for $i = 1, 2$.

For every $A_{ij} \in \mathscr{R}_{ij}$ $(1 \leqslant i \neq j \leqslant 2)$, since $A_{ij} = [[P_i, A_{ij}]_*, P_j]$ and $\Delta(P_i) = \Delta(P_j) = 0$, we have

$$\begin{aligned}
\Delta(A_{ij}) = \Delta([[P_i, A_{ij}]_*, P_j]) &= [[P_i, \Delta(A_{ij})]_*, P_j] \\
&= P_i \Delta(A_{ij}) P_j + P_j \Delta(A_{ij}) P_i. \qquad (22)
\end{aligned}$$

Multiplying (22) by P_i from both sides, we have

$$P_i \Delta(A_{ij}) P_i = 0. \qquad (23)$$

Multiplying (22) by P_j from both sides, we have

$$P_j \Delta(A_{ij}) P_j = 0. \qquad (24)$$

By Remark 1(b), we have

$$P_j \Delta(A_{ij}) P_i = 0. \qquad (25)$$

It follows from (23), (24) and (25), we have $\Delta(\mathscr{R}_{ij}) \subseteq \mathscr{R}_{ij}$ for $1 \leqslant i \neq j \leqslant 2$. □

Lemma 6 *For every* $A_{ii}\in\mathscr{R}_{ii}$, $B_{ij}\in\mathscr{R}_{ij}$, $C_{ji}\in\mathscr{R}_{ji}$, $D_{jj}\in\mathscr{R}_{jj}$ $(1\leqslant i\neq j\leqslant 2)$, *we have*

$$\Delta(A_{ii}+B_{ij}+C_{ji}+D_{jj})=\Delta(A_{ii})+\Delta(B_{ij})+\Delta(C_{ji})+\Delta(D_{jj}).$$

Proof: Let

$$T=\Delta(A_{ii}+B_{ij}+C_{ji}+D_{jj})-\Delta(A_{ii})-\Delta(B_{ij})-\Delta(C_{ji})-\Delta(D_{jj}).$$

From $[[P_i,A_{ii}+B_{ij}+C_{ji}+D_{jj}]_*,P_i]=-C_{ji}-B_{ij}$, Remark 1(a), $\Delta(P_i)=0$ and Lemma 5, we have

$$\begin{aligned}
&[[P_i,\Delta(A_{ii}+B_{ij}+C_{ji}+D_{jj})]_*,P_i]\\
&\quad=\Delta([[P_i,A_{ii}+B_{ij}+C_{ji}+D_{jj}]_*,P_i])\\
&\quad=\Delta(-C_{ji}-B_{ij})\\
&\quad=\Delta(-C_{ji})+\Delta(-B_{ij})\\
&\quad=\Delta([[P_i,B_{ij}]_*,P_i])+\Delta([[P_i,C_{ji}]_*,P_i])\\
&\quad=[[P_i,\Delta(B_{ij})]_*,P_i]+[[P_i,\Delta(C_{ji})]_*,P_i]\\
&\quad=[[P_i,\Delta(A_{ii})+\Delta(B_{ij})+\Delta(C_{ji})+\Delta(D_{jj})]_*,P_i].
\end{aligned}$$

This implies that $[[P_i,T]_*,P_i]=0$. Then $T_{ij}=T_{ji}=0$.

For every $X_{ij}\in\mathscr{R}_{ij}$, from $[[X_{ij},A_{ii}+B_{ij}+C_{ji}+D_{jj}]_*,P_j]=[[X_{ij},D_{jj}]_*,P_j]$, $\Delta(P_j)=0$ and Lemma 5, we have

$$\begin{aligned}
&[[\Delta(X_{ij}),D_{jj}]_*,P_j]+[[X_{ij},\Delta(A_{ii}+B_{ij}+C_{ji}+D_{jj})]_*,P_j]\\
&\quad=[[\Delta(X_{ij}),A_{ii}+B_{ij}+C_{ji}+D_{jj}]_*,P_j]\\
&\qquad+[[X_{ij},\Delta(A_{ii}+B_{ij}+C_{ji}+D_{jj})]_*,P_j]\\
&\quad=\Delta([[X_{ij},A_{ii}+B_{ij}+C_{ji}+D_{jj}]_*,P_j])\\
&\quad=\Delta([[X_{ij},D_{jj}]_*,P_j])\\
&\quad=[[\Delta(X_{ij}),D_{jj}]_*,P_j]+[[X_{ij},\Delta(D_{jj})]_*,P_j]\\
&\quad=[[\Delta(X_{ij}),D_{jj}]_*,P_j]\\
&\qquad+[[X_{ij},\Delta(A_{ii})+\Delta(B_{ij})+\Delta(C_{ji})+\Delta(D_{jj})]_*,P_j]
\end{aligned}$$

This implies that $[[X_{ij},T]_*,P_j]=0$. That is,

$$X_{ij}TP_j+P_jTX_{ij}^*=0. \qquad (26)$$

Multiplying (26) by P_j from the right, we have $X_{ij}TP_j=0$, and so $T_{jj}=0$. Similarly, $T_{ii}=0$. □

Lemma 7 *For every* $A_{ij},B_{ij}\in\mathscr{R}_{ij}$ $(1\leqslant i\neq j\leqslant 2)$, *we have*

$$\Delta(A_{ij}+B_{ij})=\Delta(A_{ij})+\Delta(B_{ij}).$$

Proof: From Lemmas 5 and 6, $\Delta(P_i)=\Delta(P_j)=0$ and $[[P_i+A_{ij},P_j+B_{ij}]_*,P_j]=A_{ij}+B_{ij}+A_{ij}^*$, we have

$$\begin{aligned}
&\Delta(A_{ij}+B_{ij})+\Delta(A_{ij}^*)=\Delta([[P_i+A_{ij},P_j+B_{ij}]_*,P_j])\\
&\quad=[[\Delta(A_{ij}),P_j+B_{ij}]_*,P_j]+[[P_i+A_{ij},\Delta(B_{ij})]_*,P_j]\\
&\quad=\Delta(A_{ij})+\Delta(B_{ij})+\Delta(A_{ij})^*. \qquad (27)
\end{aligned}$$

Multiplying (27) by P_i from the left and by P_j from the right, and by Lemma 5, we have $\Delta(A_{ij}+B_{ij})=\Delta(A_{ij})+\Delta(B_{ij})$. □

Lemma 8 *For every* $A_{ii},B_{ii}\in\mathscr{R}_{ii}$ $(i=1,2)$, *we have*

$$\Delta(A_{ii}+B_{ii})=\Delta(A_{ii})+\Delta(B_{ii}).$$

Proof: For every $X_{ij}\in\mathscr{R}_{ij}$ $(1\leqslant i\neq j\leqslant 2)$, by Lemma 5, $\Delta(P_j)=0$, and $[[A_{ii},X_{ij}]_*,P_j]=A_{ii}X_{ij}$, we have

$$\begin{aligned}
\Delta(A_{ii}X_{ij})&=\Delta([[A_{ii},X_{ij}]_*,P_j])\\
&=[[\Delta(A_{ii}),X_{ij}]_*,P_j]+[[A_{ii},\Delta(X_{ij})]_*,P_j]\\
&=\Delta(A_{ii})X_{ij}+A_{ii}\Delta(X_{ij}). \qquad (28)
\end{aligned}$$

It follows from Lemma 7 and (28) that

$$\begin{aligned}
&\Delta(A_{ii}+B_{ii})X_{ij}+(A_{ii}+B_{ii})\Delta(X_{ij})\\
&\quad=\Delta((A_{ii}+B_{ii})X_{ij})\\
&\quad=\Delta(A_{ii}X_{ij})+\Delta(B_{ii}X_{ij})\\
&\quad=\Delta(A_{ii})X_{ij}+A_{ii}\Delta(X_{ij})+\Delta(B_{ii})X_{ij}+B_{ii}\Delta(X_{ij}),
\end{aligned}$$

which implies that $(\Delta(A_{ii}+B_{ii})-\Delta(A_{ii})-\Delta(B_{ii}))X_{ij}=0$. Hence $\Delta(A_{ii}+B_{ii})=\Delta(A_{ii})+\Delta(B_{ii})$ by Lemma 5. □

Lemma 9 *For every* $A_{ii},B_{ii}\in\mathscr{R}_{ii}$, $A_{ij},B_{ij}\in\mathscr{R}_{ij}$, $B_{ji}\in\mathscr{R}_{ji}$, $B_{jj}\in\mathscr{R}_{jj}$ $(1\leqslant i\neq j\leqslant 2)$, *we have*

(a) $\Delta(A_{ii}B_{ij})=\Delta(A_{ii})B_{ij}+A_{ii}\Delta(B_{ij})$;
(b) $\Delta(A_{ii}B_{ii})=\Delta(A_{ii})B_{ii}+A_{ii}\Delta(B_{ii})$;
(c) $\Delta(A_{ij}B_{ji})=\Delta(A_{ij})B_{ji}+A_{ij}\Delta(B_{ji})$;
(d) $\Delta(A_{ij}B_{jj})=\Delta(A_{ij})B_{jj}+A_{ij}\Delta(B_{jj})$.

Proof: **(a)**: It follows from (28) that (a) holds.

(b): For every $X_{ij}\in\mathscr{R}_{ij}$ $(1\leqslant i\neq j\leqslant 2)$, we have from (a) that

$$\begin{aligned}
&\Delta(A_{ii}B_{ii})X_{ij}+A_{ii}B_{ii}\Delta(X_{ij})\\
&\quad=\Delta(A_{ii}B_{ii}X_{ij})\\
&\quad=\Delta(A_{ii})B_{ii}X_{ij}+A_{ii}\Delta(B_{ii}X_{ij})\\
&\quad=\Delta(A_{ii})B_{ii}X_{ij}+A_{ii}\Delta(B_{ii})X_{ij}+A_{ii}B_{ii}\Delta(X_{ij}).
\end{aligned}$$

It follows that $(\Delta(A_{ii}B_{ii})-\Delta(A_{ii})B_{ii}-A_{ii}\Delta(B_{ii}))X_{ij}=0$, and so (b) holds.

(c): For every $X_{ij}\in\mathscr{R}_{ij}$ $(1\leqslant i\neq j\leqslant 2)$, from (a), Lemma 5, and $[[A_{ij},B_{ji}]_*,X_{ij}]=A_{ij}B_{ji}X_{ij}$, we have

$$\begin{aligned}
&\Delta(A_{ij}B_{ji})X_{ij}+A_{ij}B_{ji}\Delta(X_{ij})\\
&\quad=\Delta(A_{ij}B_{ji}X_{ij})\\
&\quad=\Delta([[A_{ij},B_{ji}]_*,X_{ij}])\\
&\quad=[[\Delta(A_{ij}),B_{ji}]_*,X_{ij}]+[[A_{ij},\Delta(B_{ji})]_*,X_{ij}]\\
&\qquad+[[A_{ij},B_{ji}]_*,\Delta(X_{ij})]\\
&\quad=\Delta(A_{ij})B_{ji}X_{ij}+A_{ij}\Delta(B_{ji})X_{ij}+A_{ij}B_{ji}\Delta(X_{ij}).
\end{aligned}$$

Hence $(\Delta(A_{ij}B_{ji})-\Delta(A_{ij})B_{ji}-A_{ij}\Delta(B_{ji}))X_{ij}=0$, and so (c) holds.

(d): For every $X_{ji} \in \mathscr{R}_{ji}$ $(1 \leqslant i \neq j \leqslant 2)$, it follows from (a), (c), and Lemma 5 that

$$\begin{aligned}\Delta(A_{ij}B_{jj})X_{ji} + A_{ij}B_{jj}\Delta(X_{ji}) &= \Delta(A_{ij}B_{jj}X_{ji}) \\ &= \Delta(A_{ij})B_{jj}X_{ji} + A_{ij}\Delta(B_{jj}X_{ji}) \\ &= \Delta(A_{ij})B_{jj}X_{ji} + A_{ij}\Delta(B_{jj})X_{ji} + A_{ij}B_{jj}\Delta(X_{ji}).\end{aligned}$$

This implies that $(\Delta(A_{ij}B_{jj}) - \Delta(A_{ij})B_{jj} - A_{ij}\Delta(B_{jj}))X_{ji} = 0$. Hence (d) holds. □

Proof of Theorem 1

It is easy to verify that Δ is additive on $\mathscr{R}$ by Lemmas 6–8 and that Δ is an additive derivation on $\mathscr{R}$ by Lemmas 5 and 9. For every $A_{ij} \in \mathscr{R}_{ij}$ $(1 \leqslant i \neq j \leqslant 2)$, from Lemma 5, $\Delta(P_j) = 0$, and $[[A_{ij}, P_j]_*, P_j] = A_{ij} + A_{ij}^*$, we have

$$\begin{aligned}\Delta(A_{ij}) + \Delta(A_{ij}^*) &= \Delta([[A_{ij}, P_j]_*, P_j]) \\ &= [[\Delta(A_{ij}), P_j]_*, P_j] = \Delta(A_{ij}) + \Delta(A_{ij})^*,\end{aligned}$$

which yields that

$$\Delta(A_{ij}^*) = \Delta(A_{ij})^*. \tag{29}$$

For every $A_{ii} \in \mathscr{R}_{ii}$ $(i = 1, 2)$ and $X_{ij} \in \mathscr{R}_{ij}$ $(1 \leqslant i \neq j \leqslant 2)$, from Lemmas 5 and 9(a), $\Delta(P_i) = 0$, and $[[A_{ii}, P_i]_*, X_{ij}] = A_{ii}X_{ij} - A_{ii}^*X_{ij}$, we have

$$\begin{aligned}\Delta(A_{ii})X_{ij} + A_{ii}\Delta(X_{ij}) - \Delta(A_{ii}^*)X_{ij} - A_{ii}^*\Delta(X_{ij}) &= \Delta(A_{ii}X_{ij}) - \Delta(A_{ii}^*X_{ij}) \\ &= \Delta([[A_{ii}, P_i]_*, X_{ij}]) \\ &= [[\Delta(A_{ii}), P_i]_*, X_{ij}] + [[A_{ii}, P_i]_*, \Delta(X_{ij})] \\ &= \Delta(A_{ii})X_{ij} - \Delta(A_{ii})^*X_{ij} + A_{ii}\Delta(X_{ij}) - A_{ii}^*\Delta(X_{ij}).\end{aligned}$$

This implies that $(\Delta(A_{ii}^*) - \Delta(A_{ii})^*)X_{ij} = 0$, and so

$$\Delta(A_{ii}^*) = \Delta(A_{ii})^*. \tag{30}$$

For every $A \in \mathscr{R}$, we have $A = \sum_{i,j=1}^{2} A_{ij}$. It follows from (29), (30), and the additivity of Δ that

$$\Delta(A^*) = \sum_{i,j=1}^{2} \Delta(A_{ij}^*) = \sum_{i,j=1}^{2} \Delta(A_{ij})^* = \Delta(A)^*.$$

Hence Δ is an additive $*$-derivation. By the definition of Δ, δ is an additive $*$-derivation.

From Theorem 1, we have the following corollaries.

Corollary 1 ([16], Theorem 2.1) *Let $\mathscr{A}$ be a factor von Neumann algebra on a complex Hilbert space H with* $\dim(\mathscr{A}) > 1$. *If a map $\delta : \mathscr{A} \to \mathscr{A}$ satisfies $\delta([[A,B]_*, C]) = [[\delta(A), B]_*, C] + [[A, \delta(B)]_*, C] + [[A,B]_*, \delta(C)]$ for all $A, B, C \in \mathscr{A}$, then δ is an additive $*$-derivation.*

Corollary 2 ([17], Theorem 2.1) *Let $\mathscr{M}$ be a unite prime $*$-algebra containing a nontrivial projection. If a map $\delta : \mathscr{M} \to \mathscr{M}$ satisfies $\delta([[A,B]_*, C]) = [[\delta(A), B]_*, C] + [[A, \delta(B)]_*, C] + [[A,B]_*, \delta(C)]$ for all $A, B, C \in \mathscr{M}$, then δ is an additive $*$-derivation.*

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