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Hyperstability of an alternative equation of Jensen type

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ABSTRACT: We studied the stability and hyperstability of the alternative functional equation

$$\|f(x-y)-2f(x)+f(x+y)\|\,\|f(x-y)-\lambda f(x)+f(x+y))\|=0$$

for all $x,y\in G$ using an algebraic approach. The functions are defined as $f:G\to B$ where G is a commutative group and B is a Banach space with $\lambda\notin\{-2,-1,0,2\}$. We have obtained some necessary conditions on which sufficient conditions for hyperstability were established.

KEYWORDS: alternative functional equation, Jensen equation, generalized stability, hyperstability

MSC2020: 39B52 39B82

INTRODUCTION

The study of alternative functional equations can be viewed as a generalization of an original equation. For instance, Cauchy (additive) functional equation $f(x+y)=f(x)+f(y)$ can be generalized to $(f(x+y))^2=(f(x)+f(y))^2$, which may be rewritten as

$$(f(x+y)+f(x)+f(y))(f(x+y)-f(x)-f(y))=0.$$

Several researchers have studied this equation along with the functional equations (see [1])

$$(f(x+y))(f(x+y)-f(x)-f(y))=0$$
$$(f(x+y)-f(x)-f(y))(f(x+y)-af(x)-bf(y))=0.$$

This kind of generalizations is later recognized as alternative functional equations, in the sense that for each point in the designated domain, either the original equation or a modified counterpart is satisfied. Some recent studies on alternative functional equations can be found in [2–5].

Regarding the study of alternative equations of Jensen type, Nakmahachalasint [6] studied the solutions of

$$f(x)+2f(x+y)+f(x+2y)=0 \quad \text{or} \quad f(x)-2f(x+y)+f(x+2y)=0$$

when f is a function from a semigroup to a uniquely divisible commutative group. Srisawat et al [7] have proved that

$$f(x-y)-2f(x)+f(x+y)=0 \quad \text{or} \quad f(x-y)-\lambda f(x)+f(x+y)=0 \tag{1}$$

is equivalent to

$$f(x-y)-2f(x)+f(x+y)=0 \tag{2}$$

when $\lambda\notin\{-2,-1,0,2\}$ is a fixed integer and f is a function from a group to a uniquely divisible commutative group. In 2020, Kitisin and Srisawat[8] analyzed the solutions of

$$f(x-y)-2f(x)+f(x+y)=0 \quad \text{or} \quad \alpha f(x-y)+\beta f(x)+\gamma f(x+y)=0$$

where f is a function on groups.

Following the stability in the sense of Hyers and Ulam [9] on Cauchy functional equation, many researchers have studied and have extended the concept of stability of related functional equations [10–12]. Hyers-Ulam type stability of (1) has been investigated in [13].

Hyperstability of a functional equation can be regarded as a stronger version of stability. Stability problem of a functional equation is the study of conditions under which "if a function nearly satisfies the equation, that function can be approximated by a solution of the said equation", and also of the error bounds of said approximation. On the other hand, hyperstability problem deals with the condition under which "if a function at least nearly satisfies the equation, that function must actually satisfies it" (see [14] for a detailed definition). Studies regarding hyperstability of Cauchy and Jensen functional equations can be found in [15–17] and more.

In this article, we will extend the stability result of (1) to the Găvruţa type of stability (whered the control function is not necessarily of some specific forms) and consequently to hyperstability. Our study is focused on the alternative inequality

$$\|f(x-y)-2f(x)+f(x+y)\| \leqslant \varphi(x,y) \quad \text{or} \quad \|f(x-y)-\lambda f(x)+f(x+y)\| \leqslant \varphi(x,y) \quad (3)$$

where $\varphi : G^2 \to [0,\infty)$ is a function with certain properties.

FRAMEWORK

In this and the following sections, let G be a commutative group with e as the additive identity, let B be a Banach space, and let $\lambda \notin \{-2,-1,0,2\}$ be a fixed real number. Also let $\mathbb{N}$ denote the set of all positive integers.

Furthermore, for any $\alpha \in \mathbb{R}$, denote

$$\mathscr{F}_y^{(\alpha)}(x) := f(x-y)-\alpha f(x)+f(x+y).$$

The inequality (3) can be written as

$$\|\mathscr{F}_y^{(2)}(x)\| \leqslant \varphi(x,y) \quad \text{or} \quad \|\mathscr{F}_y^{(\lambda)}(x)\| \leqslant \varphi(x,y).$$

For each $x, y \in G$, denote the statement

$$\mathscr{P}_y^{(\alpha)}(x) := \Big(\|\mathscr{F}_y^{(\alpha)}(x)\| \leqslant \varphi(x,y)\Big).$$

Next, we explicitly list some properties for the control functions for future references. Let $K = (\{0,1,2\}\times\{1,2\})\cup\{(-1,1),(3,1)\}$ and

$$C = \left\{\varphi : G^2 \to B \;\middle|\; \sum_{n=0}^{\infty} \frac{\varphi(2^n x, 2^n y)}{2^n} < \infty,\ \forall x,y \in G\right\}.$$

For $\varphi \in C$, we let

$$\Lambda\varphi(x,y) := \frac{1}{2}\sum_{k=0}^{\infty}\frac{\varphi(2^k x, 2^k y)}{2^k}.$$

For any functions $\varphi \in C$, $\gamma : G \to G^{\mathbb{N}}$, and $h : G \to G$ we denote the following properties:

H(i) $\lim_{n\to\infty} \varphi(x, \gamma x(n)) = 0$ for every $x \in G$;

H(ii) $\lim_{n\to\infty} \Lambda\varphi(k_1(x+l\gamma x(n)), k_2(x+l\gamma x(n))) = 0$ for every $(k_1,k_2) \in K$, $l \in \{-1,1\}$ and every $x \in G$;

H(iii) for each x, there exists an increasing sequence (k_n) of positive integers such that $\lim_{n\to\infty} \varphi(2^{k_n}x, (2^{k_n}-1)x) = 0$;

and

J(i) $\lim_{n\to\infty} \varphi(x+nh(x), nh(x)) = 0$ for every $x \in G$;

J(ii) $\lim_{n\to\infty} \Lambda\varphi(k_1(x+lnh(x)), k_2(x+lnh(x))) = 0$ for every $(k_1,k_2) \in K$, $l \in \{1,2\}$ and every $x \in G$;

J(iii) $\lim_{n\to\infty} \varphi(x, nh(x)) = 0$ for every $x \in G$.

The set C of functions will be used in the stability problem. All other conditions will be used in the hyperstability problem. The properties H(i)–H(iii) will be one criterion for hyperstability, and the properties J(i)–J(iii) will be another criterion.

The following proposition is straightforward.

Proposition 1 *Let $\varphi_1 : G^2 \to B$, $\gamma : G \to G^{\mathbb{N}}$, $h : G \to G$, and let $a_1, a_2, a_3, \ldots, a_8 \in \mathbb{R}$. Also let $\varphi_2 : G^2 \to B$ be defined by*

$$\begin{aligned}\varphi_2(x,y) = {}& a_1\varphi_1(x-2y,y)+a_2\varphi_1(x-y,y)+a_3\varphi_1(x,y)\\ &+a_4\varphi_1(x+y,y)+a_5\varphi_1(x+2y,y)\\ &+a_6\varphi_1(x-y,2y)+a_7\varphi_1(x,2y)+a_8\varphi_1(x+y,2y)\end{aligned}$$

for all $x, y \in G$. Then

(i) If $\varphi_1 \in C$, then $\varphi_2 \in C$.

(ii) If φ_1 and γ satisfy H(ii), then $\lim_{n\to\infty} \Lambda\varphi_2(x+l\gamma x(n), x+l\gamma x(n)) = 0$ for all $x \in G$ and $l \in \{-1,1\}$.

(iii) If φ and h satisfy J(ii), then $\lim_{n\to\infty} \Lambda\varphi_2(x+lnh(x), x+lnh(x)) = 0$ for all $x \in G$ and $l \in \{1,2\}$.

From this point onward, we will assume that $\varphi(x,e) = 0$ for every $x \in G$. This always holds without loss of generality since $\mathscr{F}_e^{(2)}(x) = 0$.

MAIN RESULTS

Găvruţa-type Stability

We will use a similar approach as Srisawat [13] and will provide a short proof to clarify its validity for our setting. The idea of the following lemma is the use of the identities

$$\mathscr{F}_y^{(2)}(x-y)+2\mathscr{F}_y^{(2)}(x)+\mathscr{F}_y^{(2)}(x+y)-\mathscr{F}_{2y}^{(2)}(x) = 0$$

and $\mathscr{F}_y^{(\lambda)}(x) = \mathscr{F}_y^{(2)}(x)+(2-\lambda)f(x)$. For example, if $\mathscr{F}_y^{(\lambda)}(x-y)$, $\mathscr{F}_y^{(\lambda)}(x)$, $\mathscr{F}_y^{(\lambda)}(x+y)$, and $\mathscr{F}_{2y}^{(2)}(x)$ are bounded, then

$$\begin{aligned}&(2-\lambda)(f(x-y)+2f(x)+f(x+y))\\ &\quad= \mathscr{F}_y^{(\lambda)}(x-y)+2\mathscr{F}_y^{(\lambda)}(x)+\mathscr{F}_y^{(\lambda)}(x+y)-\mathscr{F}_{2y}^{(2)}(x)\end{aligned}$$

is bounded. A linear combination between $\mathscr{F}_y^{(\lambda)}(x)$ and the left hand side of the equation can yield $\mathscr{F}_y^{(2)}(x)$, so it is also bounded.

With arguments similar to the above for each of all possible cases of (3) with (x,y) substituted by $(x-2y,y)$, $(x-y,y)$, (x,y), $(x+y,y)$, $(x+2y,y)$, $(x-y,2y)$, $(x,2y)$, $(x+y,2y)$, we obtained the linear combinations in the following lemma.

Lemma 1 *Let $\varphi : G\times G \to [0,\infty)$ and let $f : G \to B$ satisfy (3) for all $x,y \in G$. Then*

$$\begin{aligned}\|\mathscr{F}_y^{(2)}(x)\| \leqslant {}& (1+M)\varphi(x-2y,y)+(2+3M)\varphi(x-y,y)\\ &+(1+3M)\varphi(x,y)+(2+3M)\varphi(x+y,y)\\ &+(1+M)\varphi(x+2y,y)+(1+M)\varphi(x-y,2y)\\ &+(1+M)\varphi(x,2y)+(1+M)\varphi(x+y,2y)\end{aligned}$$

for all $x, y \in G$, where $M = \max\{\frac{1}{|2+\lambda|}, \frac{1}{|1+\lambda|}, \frac{1}{|\lambda|}, \frac{1}{|2-\lambda|}\}$.

Proof: Let $x, y \in G$. Suppose that $\mathscr{P}_y^{(\lambda)}(x)$.

Case I: $\mathscr{P}_y^{(\lambda)}(x-y)$ and $\mathscr{P}_y^{(2)}(x+y)$. Then one of the following linear combinations is applicable.

$$\begin{aligned}
\mathscr{F}_y^{(2)}(x) &= -2\mathscr{F}_y^{(2)}(x+y)-\mathscr{F}_y^{(2)}(x+2y)+\mathscr{F}_{2y}^{(2)}(x+y)\\
&= \tfrac{1}{1+\lambda}\Big(-\mathscr{F}_y^{(\lambda)}(x-y)+\mathscr{F}_y^{(\lambda)}(x)-\mathscr{F}_y^{(2)}(x+y)+\mathscr{F}_{2y}^{(2)}(x)\\
&\quad+\mathscr{F}_y^{(\lambda)}(x)+2\mathscr{F}_y^{(2)}(x+y)+\mathscr{F}_y^{(2)}(x+2y)-\mathscr{F}_{2y}^{(\lambda)}(x+y)\Big)\\
&= \tfrac{1}{2-\lambda}\Big(\mathscr{F}_y^{(\lambda)}(x-y)+2\mathscr{F}_y^{(\lambda)}(x)+\mathscr{F}_y^{(2)}(x+y)-\mathscr{F}_{2y}^{(2)}(x)\\
&\quad+\tfrac{1}{2}\mathscr{F}_y^{(\lambda)}(x)+\tfrac{\lambda}{2}\mathscr{F}_y^{(2)}(x+y)+\tfrac{1}{2}\mathscr{F}_y^{(\lambda)}(x+2y)-\tfrac{1}{2}\mathscr{F}_{2y}^{(2)}(x+y)\Big)\\
&= \tfrac{1}{2-\lambda}\Big(\mathscr{F}_y^{(\lambda)}(x-y)+2\mathscr{F}_y^{(\lambda)}(x)+\mathscr{F}_y^{(2)}(x+y)-\mathscr{F}_{2y}^{(2)}(x)\\
&\quad+\mathscr{F}_y^{(\lambda)}(x)+\lambda\mathscr{F}_y^{(2)}(x+y)+\mathscr{F}_y^{(\lambda)}(x+2y)-\mathscr{F}_{2y}^{(\lambda)}(x+y)\Big)\\
&= \tfrac{1}{\lambda}\Big(-\mathscr{F}_y^{(\lambda)}(x-y)-\mathscr{F}_y^{(2)}(x+y)+\mathscr{F}_{2y}^{(\lambda)}(x)-\mathscr{F}_y^{(\lambda)}(x)\\
&\quad-2\mathscr{F}_y^{(2)}(x+y)-\mathscr{F}_y^{(2)}(x+2y)+\mathscr{F}_{2y}^{(\lambda)}(x+y)\Big)\\
&= \tfrac{1}{1+\lambda}\Big(-\mathscr{F}_y^{(\lambda)}(x-y)+\mathscr{F}_y^{(\lambda)}(x)-\mathscr{F}_y^{(2)}(x+y)+\mathscr{F}_{2y}^{(\lambda)}(x)\\
&\quad-\tfrac{1}{2}\mathscr{F}_y^{(\lambda)}(x)-\tfrac{\lambda}{2}\mathscr{F}_y^{(2)}(x+y)-\tfrac{1}{2}\mathscr{F}_y^{(\lambda)}(x+2y)+\tfrac{1}{2}\mathscr{F}_{2y}^{(2)}(x+y)\Big)\\
&= \tfrac{1}{1+\lambda}\Big(-\mathscr{F}_y^{(\lambda)}(x-y)+\mathscr{F}_y^{(\lambda)}(x)-\mathscr{F}_y^{(2)}(x+y)+\mathscr{F}_{2y}^{(\lambda)}(x)\\
&\quad-\mathscr{F}_y^{(\lambda)}(x)-\lambda\mathscr{F}_y^{(2)}(x+y)-\mathscr{F}_y^{(\lambda)}(x+2y)+\mathscr{F}_{2y}^{(\lambda)}(x+y)\Big).
\end{aligned}$$

Using the triangle inequality, we got

$$\begin{aligned}
\|\mathscr{F}_y^{(2)}(x)\| &\leqslant M\varphi(x-y,y)+3M\varphi(x,y)\\
&\quad+(2+3M)\varphi(x+y,y)+(1+M)\varphi(x+2y,y)\\
&\quad+M\varphi(x,2y)+(1+M)\varphi(x+y,2y).
\end{aligned}$$

Case II: $\mathscr{P}_y^{(2)}(x-y)$ and $\mathscr{P}_y^{(\lambda)}(x+y)$. This case is analogous to Case I.

$$\begin{aligned}
\|\mathscr{F}_y^{(2)}(x)\| &\leqslant M\varphi(x+y,y)+3M\varphi(x,y)\\
&\quad+(2+3M)\varphi(x-y,y)+(1+M)\varphi(x-2y,y)\\
&\quad+M\varphi(x,2y)+(1+M)\varphi(x-y,2y).
\end{aligned}$$

Case III: Other cases. This is actually the most straightforward.

$$\begin{aligned}
\mathscr{F}_y^{(2)}(x) &= -\tfrac{1}{2}\mathscr{F}_y^{(2)}(x-y)-\tfrac{1}{2}\mathscr{F}_y^{(2)}(x+y)+\tfrac{1}{2}\mathscr{F}_{2y}^{(2)}(x)\\
&= -\mathscr{F}_y^{(2)}(x-y)-\mathscr{F}_y^{(\lambda)}(x)-\mathscr{F}_y^{(2)}(x+y)+\mathscr{F}_{2y}^{(\lambda)}(x)\\
&= \tfrac{1}{2+\lambda}\Big(-\mathscr{F}_y^{(\lambda)}(x-y)+2\mathscr{F}_y^{(\lambda)}(x)-\mathscr{F}_y^{(\lambda)}(x+y)+\mathscr{F}_{2y}^{(2)}(x)\Big)\\
&= \tfrac{1}{1+\lambda}\Big(-\mathscr{F}_y^{(\lambda)}(x-y)+\mathscr{F}_y^{(\lambda)}(x)-\mathscr{F}_y^{(\lambda)}(x+y)+\mathscr{F}_{2y}^{(\lambda)}(x)\Big).
\end{aligned}$$

So

$$\begin{aligned}
\|\mathscr{F}_y^{(2)}(x)\| &\leqslant (1+M)\varphi(x-y,y)+(1+2M)\varphi(x,y)\\
&\quad+(1+M)\varphi(x+y,y)+(1+M)\varphi(x,2y).
\end{aligned}$$

In all 3 cases, we have

$$\begin{aligned}
\|\mathscr{F}_y^{(2)}(x)\| &\leqslant (1+M)\varphi(x-2y,y)+(2+3M)\varphi(x-y,y)\\
&\quad+(1+3M)\varphi(x,y)+(2+3M)\varphi(x+y,y)\\
&\quad+(1+M)\varphi(x+2y,y)+(1+M)\varphi(x-y,2y)\\
&\quad+(1+M)\varphi(x,2y)+(1+M)\varphi(x+y,2y).
\end{aligned}$$

□

Let $a_1 = 1+M$, $a_2 = 2+3M$, and $a_3 = 1+3M$ and

$$\begin{aligned}
\varphi'(x,y) &= a_1\varphi(-x,x)+a_2\varphi(0,x)+a_3\varphi(x,x)\\
&\quad+a_2\varphi(2x,x)+a_1\varphi(3x,x)+a_1\varphi(0,2x)\\
&\quad+a_1\varphi(x,2x)+a_1\varphi(2x,x).
\end{aligned}$$

We have

$$\|f(x-y)-2f(x)+f(x+y)\| \leqslant \varphi'(x,y)$$

for all $x, y \in G$. Proposition 1 implies that $\varphi' \in C$ whenever $\varphi \in C$. Using Theorem 3.1 of [18], we got the following theorem.

Theorem 1 *Let $\varphi \in C$ and let $f : G \to B$ satisfy (3). Then there exists $A : G \to B$ satisfying (1) and*

$$\|f(x)-A(x)-f(e)\| \leqslant \Lambda\varphi'(x,x)$$

for all $x \in G$. Moreover, A is defined by

$$A(x) := \lim_{k\to\infty}\frac{f(2^k x)}{2^k}.$$

By setting $\varphi(x,y) := 0$, Lemma 1 also implies that if $f : G \to B$ satisfies (1), then it also satisfies (2), and Theorem 1 implies that $f(x) = A(x)+f(e)$ where $A(x+y) = A(x)+A(y)$ for all $x, y \in G$.

Hyperstability

Firstly, let us see examples with implication of necessary conditions for hyperstability of (1).

Example 1 Let $\varphi_1, \varphi_2 : G^2 \to B$. Suppose that $x_0 \in G$ such that

$$\inf\{\varphi_1(x_0,y) \mid y \in G\backslash\{e\}\} = L_1 \neq 0$$

and

$$\inf\{\varphi_2(x_0+y,y) \mid y \in G\backslash\{e\}\} = L_2 \neq 0.$$

Let $M_1, M_2 \in B$ such that $\|M_1\| = (L_1/2)\max\left\{\left|\frac{1}{4-2\lambda}\right|, \left|\frac{1}{(2-\lambda)(1+\lambda)}\right|\right\}$ and $\|M_2\| = (L_2/2)\max\left\{\left|\frac{\lambda}{2-\lambda}\right|, \left|\frac{\lambda}{(2-\lambda)(1+\lambda)}\right|\right\}$. Define $f_1, f_2 : G \to B$ by

$$f_1(x) = \begin{cases} M_1, & x \neq x_0,\\ (\lambda-1)M_1, & x = x_0,\end{cases}$$

and

$$f_2(x) = \begin{cases} M_2, & x \neq x_0\\ \frac{2}{\lambda}M_2, & x = x_0.\end{cases}$$

Then the pairs (f_1, φ_1) and (f_2, φ_2) satisfy (3), but both f_1 and f_2 do not satisfy (1).

From Example 1, if we need (1) to possess hyperstability, our assumptions must at least imply

$$\inf\{\varphi_1(x,y) \mid y \in G\backslash\{e\}\} = 0$$
$$\text{and} \quad \inf\{\varphi_2(x+y,y) \mid y \in G\backslash\{e\}\} = 0$$

for every $x \in G$. Our main theorems will be based on stronger versions of these properties.

Also note that the condition H(ii) implies

$$\lim_{n\to\infty} \Lambda\varphi'(x+\gamma x(n), x+\gamma x(n)) = \lim_{n\to\infty} \Lambda\varphi'(x-\gamma x(n), x-\gamma x(n)) = 0$$

for all $x \in G$.

Lemma 2 *Let $\varphi \in C$ and let $f : G \to B$ satisfy (3). If there exists $\gamma : G \to G^{\mathbb{N}}$ which (φ,γ) satisfies H(i) and H(ii), then, for each $x \in G$, at least one of the following results holds*
(i) $f(x) = A(x) + f(e)$.
(ii) $f(x) = \frac{2}{\lambda}(A(x) + f(e))$.
The function $A : G \to B$ here is the same as what defined in Theorem 1.

Proof: Let $x \in G$. Consider (3) with $y = \gamma x(n)$, where $n \in \mathbb{N}$ is arbitrary. Since φ and γ satisfy H(ii), we consider 2 cases.

Case 1: $\|\mathscr{F}^{(2)}_{\gamma x(n)}(x)\| \leqslant \varphi(x,\gamma x(n))$ for all large n. Then

$$\begin{aligned}2\|f(x)-A(x)-f(e)\| &= \Big\|\mathscr{F}^{(2)}_{\gamma x(n)}(x) - \big(f(x-\gamma x(n)) - A(x-\gamma x(n)) - f(e)\big) \\ &\quad - \big(f(x+\gamma x(n)) - A(x+\gamma x(n)) - f(e)\big)\Big\| \\ &\leqslant \varphi(x,\gamma x(n)) + \Lambda\varphi'(x-\gamma x(n), x-\gamma x(n)) \\ &\quad + \Lambda\varphi'(x+\gamma x(n), x+\gamma x(n)).\end{aligned}$$

Taking limit $n \to \infty$ yields $f(x) = A(x) + f(e)$.

Case 2: There exists an increasing integer sequence (a_n) such that $\|\mathscr{F}^{(\lambda)}_{\gamma x(a_n)}(x)\| \leqslant \varphi(x,\gamma x(a_n))$ for all positive integers n. So

$$\begin{aligned}\|\lambda f(x) - 2A(x) - 2f(e)\| &= \Big\|\mathscr{F}^{(\lambda)}_{\gamma x(a_n)}(x) - (f(x-\gamma x(a_n)) - A(x-\gamma x(a_n)) - f(e)) \\ &\quad - (f(x+\gamma x(a_n)) - A(x+\gamma x(a_n)) - f(e))\Big\| \\ &\leqslant \varphi(x,\gamma x(a_n)) + \Lambda\varphi'(x-\gamma x(a_n), x-\gamma x(a_n)) \\ &\quad + \Lambda\varphi'(x+\gamma x(a_n), x+\gamma x(a_n)).\end{aligned}$$

Taking limit $n \to \infty$, the right-hand side approaches 0. So $f(x) = \frac{2}{\lambda}(A(x) + f(e))$. □

Theorem 2 *Let $\varphi \in C$ and let $f : G \to B$ satisfy (3). If there exists $\gamma : G \to G^{\mathbb{N}}$ such that (φ,γ) satisfies H(i), H(ii), and H(iii), then $f(x) = A(x) + f(e)$ for all $x \in G$, where A is defined as in Theorem 1.*

Proof: According to Lemma 2, we have

$$f(x) \in \left\{A(x) + f(e), \tfrac{2}{\lambda}(A(x) + f(e))\right\}$$

for all $x \in G$. Suppose there exists $x \in G$ such that $f(x) \neq A(x) + f(e)$. Since φ satisfies H(iii), there exists an increasing sequence (k_n) in $\mathbb{N}$ such that $\lim_{n\to\infty} \varphi(2^{k_n}x, (2^{k_n}-1)x) = 0$.

Since $\lim_{n\to\infty} f(2^n x)/2^n = A(x)$, we have $f(2^{k_n}x) = A(2^{k_n}x) + f(e)$ and $f((2^{k_n+1}-1)x) \in \{A((2^{k_n+1}-1)x) + f(e), \frac{2}{\lambda}\big(A((2^{k_n+1}-1)x) + f(e)\big)\}$ for all large enough n. Substituting $(2^{k_n}x, (2^{k_n}-1)x)$ into (3) and considering all possibilities, we have

$$\mathscr{F}^{(2)}_{(2^{k_n}-1)x}(2^{k_n}x) \in \left\{\tfrac{2-\lambda}{\lambda}(A(x)+f(e)), \tfrac{2(2-\lambda)}{\lambda}(2^{k_n}A(x)+f(e))\right\}$$
$$\mathscr{F}^{(\lambda)}_{(2^{k_n}-1)x}(2^{k_n}x) \in \left\{\tfrac{(2-\lambda)(2^{k_n}\lambda+1)}{\lambda}A(x) + \tfrac{(2-\lambda)(1+\lambda)}{\lambda}f(e), \tfrac{(2-\lambda)(2+\lambda)}{\lambda}(2^{k_n}A(x)+f(e))\right\}.$$

Since $\lim_{n\to\infty} \varphi(2^{k_n}x, (2^{k_n}-1)x) = 0$, we can conclude that $A(x) + f(e) = 0$. But then

$$A(x) + f(e) = \tfrac{2}{\lambda}(A(x) + f(e)) = f(x) \neq A(x) + f(e),$$

a contradiction. □

From the relatively loose conditions of Lemma 2 and Lemma 3, we will later give simpler conditions for hyperstability of (1) on Banach spaces.

Next is another criterion for hyperstability of (1).

Lemma 3 *Let $\varphi \in C$ and let $f : G \to B$ satisfy (3). Suppose that there exists $h : G \to G$ such that (φ,h) satisfies J(i) and J(ii). Then, for each $x \in G$, at least one of the following holds.*
(i) $f(x) = A(x) + f(e)$
(ii) $A(h(x)) = 0$ *and* $f(x) = (\lambda-1)(A(x) + f(e))$
where $A : G \to B$ is defined as in Theorem 1.

Proof: Let $x \in G$. We consider 2 cases.

Case 1: $\|\mathscr{F}^{(2)}_{nh(x)}(x+nh(x))\| \leqslant \varphi(x+nh(x), nh(x))$ for all large n. Then

$$\begin{aligned}\|f(x)-A(x)-f(e)\| &= \big\|\mathscr{F}^{(2)}_{nh(x)}(x+nh(x)) - f(x+2nh(x)) - A(x+2nh(x)) - f(e)) \\ &\quad + 2(f(x+nh(x)) - A(x+nh(x)) - f(e))\big\| \\ &\leqslant \varphi(x+nh(x), nh(x)) + \Lambda\varphi'(x+2nh(x), x+2nh(x)) \\ &\quad + 2\Lambda\varphi'(x+nh(x), x+nh(x))\end{aligned}$$

for all large n. Taking $n \to \infty$, we have $f(x) = A(x) + f(e)$.

Case 2: There exists an increasing sequence (k_n) of positive integers such that $\|\mathscr{F}^{(\lambda)}_{k_n h(x)}(x + k_n h(x))\| \leqslant$

$\varphi(x+k_nh(x),k_nh(x))$ for all n. Then

$$\begin{aligned}k_n(2-\lambda)A(h(x)) &= \mathscr{F}^{(\lambda)}_{k_nh(x)}(x+k_nh(x))\\ &\quad-(f(x+2k_nh(x))-A(x+2k_nh(x))-f(e))\\ &\quad+\lambda(f(x+k_nh(x))-A(x+k_nh(x))-f(e))\\ &\quad-(f(x)-(\lambda-1)(A(x)+f(e)))\\ k_n\|(2-\lambda)A(h(x))\| &\leqslant \varphi(x+k_nh(x),k_nh(x))\\ &\quad+\Lambda\varphi'(x+2k_nh(x),x+2k_nh(x))\\ &\quad+\lambda\Lambda\varphi'(x+k_nh(x),x+k_nh(x))\\ &\quad+\|f(x)-(\lambda-1)(A(x)+f(e))\|.\end{aligned}$$

The right-hand side of the above inequality is bounded, so $A(h(x))=0$ (otherwise the left-hand side would be unbounded). Hence

$$\begin{aligned}\|f(x)-(\lambda-1)(A(x)+f(e))\| &= \|\mathscr{F}^{(\lambda)}_{k_nh(x)}(x+k_nh(x))\\ &\quad-(f(x+2k_nh(x))-A(x+2k_nh(x))-f(e))\\ &\quad+\lambda(f(x+k_nh(x))-A(x+k_nh(x))-f(e))\|\\ &\leqslant \varphi(x+k_nh(x),k_nh(x))+\Lambda\varphi'(x+2k_nh(x),x+2k_nh(x))\\ &\quad+\lambda\Lambda\varphi'(x+k_nh(x),x+k_nh(x)).\end{aligned}$$

Taking $n\to\infty$, we got $f(x)=(\lambda-1)(A(x)+f(e))$. □

Theorem 3 *Let $\varphi\in C$ and let $f:G\to B$ satisfy (3). If there exists $h:G\to G$ such that φ and h satisfy J(i), J(ii), and J(iii). Then f satisfies (1) for all $x,y\in G$.*

Proof: Let $x\in G$ and suppose that $f(x)\neq A(x)+f(e)$. Then $f(x)=(\lambda-1)(A(x)+f(e))$ and $A(h(x))=0$. Also

$$\begin{aligned}\lim_{n\to\infty}&\|f(x+nh(x))-A(x+nh(x))-f(e)\|\\ &\leqslant \lim_{n\to\infty}\Lambda\varphi'(x+nh(x),x+nh(x))=0,\end{aligned}$$

so we have $f(x+nh(x))=A(x)+f(e)$ for all large n. We also have

$$f(x-nh(x))\in\{A(x)+f(e),(\lambda-1)(A(x)+f(e))\}$$

for all $n\in\mathbb{N}$. Consider the possible values

$$\begin{aligned}\left\|\mathscr{F}^{(2)}_{nh(x)}(x)\right\| &\in \big\{\|2(2-\lambda)(A(x)+f(e))\|,\\ &\qquad \|(2-\lambda)(A(x)+f(e))\|\big\}\\ \left\|\mathscr{F}^{(\lambda)}_{nh(x)}(x)\right\| &\in \big\{\|(2-\lambda)(1+\lambda)(A(x)+f(e))\|,\\ &\qquad \|\lambda(2-\lambda)(A(x)+f(e))\|\big\}\end{aligned}$$

for all large n. Since $\lim_{n\to\infty}\varphi(x,nh(x))=0$, at least one of these values must approach zero. So $A(x)+f(e)=0$, yielding

$$f(x)\neq A(x)+f(e)=0=(\lambda-1)(A(x)+f(e))=f(x),$$

a contradiction. □

Our next theorem here will be the case where G is cyclic.

Theorem 4 *Let G be cyclic, $\varphi\in C$ and let $f:G\to B$ satisfy (3). If there exists $h:G\to G$ such that (φ,h) satisfies J(i) and J(ii), $\inf\{\varphi(x,y)\mid y\in G\backslash\{e\}\}=0$ and*

$$\lim_{n\to\infty}\varphi(nx,nx)=0$$

for every $x\in G$. Then f satisfies (1) for all $x,y\in G$.

Proof: Suppose there exists $x_0\in G$ such that $f(x_0)\neq A(x_0)+f(e)$. According to Lemma 3, $f(x_0)=(\lambda-1)(A(x_0)+f(e))$ and $A(h(x_0))=0$. If $h(x_0)=e$, the condition J(ii) implies that

$$\Lambda\varphi'(x_0,x_0)=\lim_{n\to\infty}\Lambda\varphi'(x_0+nh(x),x_0+nh(x))=0,$$

which yields $f(x_0)=A(x_0)+f(e)$. So $h(x_0)\neq e$. Since G is cyclic, we can write $h(x_0)=k_0a$ where a is a generator of G and k_0 is a nonzero integer. With the additivity of A, we have $k_0A(a)=A(k_0a)=0$, so $A(x)=0$ for all $x\in G$.

Let $N\in\mathbb{N}$ such that

$$\begin{aligned}\varphi(na,na)<\min\big\{&\|(2-\lambda)f(e)\|,\\ &\|(1+\lambda)(2-\lambda)f(e)\|,\|\lambda(2-\lambda)f(e)\|\big\}\end{aligned}$$

for all $n>N$. Since all possible values of $\|\mathscr{F}^{(2)}_{na}(na)\|$ and $\|\mathscr{F}^{(\lambda)}_{na}(na)\|$ are greater than or equal to this value unless $f(na)=f(e)$, we conclude that $f(na)=f(e)$ for all $n>N$.

Let $k\in\mathbb{N}$ be such that $f(ka)\neq f(e)$ and $f(na)=f(e)$ for all $n\in\{k+1,k+2,\dots,N\}$. When $y\neq e$, we have

$$\begin{aligned}\|\mathscr{F}^{(2)}_{y}(ka)\| &\in\{\|2(2-\lambda)f(e)\|,\|(2-\lambda)f(e)\|\}\\ \|\mathscr{F}^{(2)}_{y}(ka)\| &\in\{\|(1+\lambda)(2-\lambda)f(e)\|,\|\lambda(2-\lambda)f(e)\|\}\end{aligned}$$

with the assumption $\inf\{\varphi(x,y)\mid y\in G\backslash\{e\}\}=0$, we have $f(e)=0$, which contradicts the assumption $f(ka)\neq f(e)$. □

Now we give a condition for hyperstability of (1) on Banach spaces.

Corollary 1 *Let B_1,B_2 be Banach spaces, $\lambda\notin\{-2,-1,0,2\}$, and let $\varphi\in C$. Suppose $f:B_1\to B_2$ satisfies (3) for all $x,y\in B_1$ and φ has all of the following properties.*

(i) *For each $x\in B_1$ there exists a sequence $\gamma x(n)$ in B_1 such that $\lim_{n\to\infty}\|\gamma x(n)\|=\infty$ and $\lim_{n\to\infty}\varphi(x,\gamma x(n))=0$.*
(ii) *For each $x\in B_1$ there exists $h(x)\in B_1$ such that $\lim_{n\to\infty}\varphi(x+nh(x),nh(x))=0$.*
(iii) *For each $(k_1,k_2)\in K$, $\lim_{s\to\infty}\varphi(sk_1x,sk_2x)=0$ converges uniformly on the set $\{x\in B_1\mid \|x\|=1\}$.*

Then f satisfies (1) for all $x,y\in B_1$.

Proof: Observe that

$$\lim_{n\to\infty} \Lambda\varphi'(x+\gamma x(n), x+\gamma x(n)) = 0$$
$$\lim_{n\to\infty} \Lambda\varphi'(x-\gamma x(n), x-\gamma x(n)) = 0$$
$$\lim_{n\to\infty} \Lambda\varphi'(x+nh(x), x+nh(x)) = 0$$
$$\lim_{n\to\infty} \Lambda\varphi'(x+2nh(x), x+2nh(x)) = 0$$

for every $x \in B_1$, so the conditions in Lemma 2 and Lemma 3 are met. Let $x \in B_1$ and suppose that $f(x) \neq A(x)+f(e)$. According to Lemma 2, $f(x) = \frac{2}{\lambda}(A(x)+f(e))$.

According to Lemma 3, $f(x) = (\lambda-1)(A(x)+1)$. Since $\frac{2}{\lambda} \neq \lambda - 1$, we have $A(x)+f(e) = 0$, which is a contradiction. □

Theorem 2 leads to a family of control functions which imply hyperstability of (1).

Corollary 2 *Let B_1, B_2 be Banach spaces over $\mathbb{R}$, $\lambda \notin \{-2,-1,0,2\}$ and let a, b, c, d be real numbers such that $b > \max\{0, a\}$. Let $\varphi : B_1 \times B_1 \to [0,\infty)$ be defined by*

$$\varphi(x,y) = \begin{cases} 0, & y = 0, \\ c\|y\|^{-b}, & y \neq 0 \text{ and } x = 0, \\ d\|x\|^{a}\|y\|^{-b}, & x, y \neq 0. \end{cases}$$

Suppose that $f : B_1 \to B$ satisfies (3). Then f satisfies (1).

The next example shows that a control function φ might have weaker properties than those in Corollary 1, but still imply hyperstability of (1).

Example 2 Let $G = B = \mathbb{R}$, and let $\lambda \notin \{-2,-1,0,2\}$. Let $\varphi : G \to B$ be defined by

$$\varphi(x,y) = \frac{|\sin(xy)|}{|x|+1}.$$

For each $x \in G$, there exists an increasing sequence $(\gamma x(n) = y_n)$ such that $\lim_{n\to\infty} |y_n| = \infty$ and

$$\frac{x y_n}{\pi} \in \mathbb{Z}$$

for every n. Then (φ, γ) satisfies H(i), H(ii), and H(iii). Hence this φ also is a suitable control function for hyperstability of (1).

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REFERENCES

1. Kannappan PL, Kuczma M (1974) On a functional equation related to the Cauchy equation. *Ann Polon Math* **30**, 49–55.
2. Faĭziev VA, Powers RC, Sahoo PK (2013) An alternative Cauchy functional equation on a semigroup. *Aequat Math* **85**, 131–163.
3. Ger R (2018) Solving alternative functional equations: What for? *Annales Univ Sci Budapest Sect Comp* **47**, 273–283.
4. Nakmahachalasint P (2015) An alternative quadratic functional equation on 2-divisible groups. *ScienceAsia* **41**, 136–140.
5. Batko B (2008) Stability of an alternative functional equation. *J Math Anal Appl* **339**, 303–311.
6. Nakmahachalasint P (2012) An alternative Jensen's functional equation on semigroups. *ScienceAsia* **38**, 408–413.
7. Srisawat C, Kitisin N, Nakmahachalasint P (2015) An alternative functional equation of Jensen type on groups. *ScienceAsia* **41**, 280–288.
8. Kitisin N, Srisawat C (2020) A general form of an alternative functional equation related to the Jensen's functional equation. *ScienceAsia* **46**, 368–375.
9. Hyers DH (1941) On the stability of linear functional equation. *Proc Natl Acad Sci USA* **27**, 222–224.
10. Bahyrycz A, Brzdęk J, Piszczek M, Sikorska J (2013) Hyperstability of the Fréchet equation and a characterization of inner product spaces. *J Funct Spaces* **2013**, ID 496361.
11. Dăianu DM (2014) Recursive procedure in the stability of Fréchet polynomials. *Adv Differ Equ* **2014**, ID 16.
12. Găvruţa P (1994) A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. *J Math Anal Appl* **184**, 431–436.
13. Srisawat C (2019) Hyers-Ulam stability of an alternative functional equation of Jensen type. *ScienceAsia* **45**, 275–278.
14. Brzdęk J, Ciepliński K (2013) Hyperstability and superstability. *Abstr Appl Anal* **2013**, ID 401756.
15. Khodaei H (2016) Hyperstability of the generalized polynomial functional equation of degree 5. *Mediterr J Math* **13**, 1829–1840.
16. Almahalebi M, Charifi S, Kabbaj S (2015) Hyperstability of a Cauchy functional equation. *J Nonlinear Anal Optim* **6**, 127–137.
17. El-Fassi I (2016) Hyperstability of an n-dimensional Jensen type functional equation. *Afr Mat* **27**, 1377–1389.
18. Kim GH, Dragomir SS (2006) On the stability of generalized d'Alembert and Jensen functional equations. *Int J Math Math Sci* **2006**, ID 043185