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# A lower bound of the rank of a signed graph in terms of order and maximum degree

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**ABSTRACT**: Let $\Gamma = (G, \sigma)$ be a signed graph of order $n$ with maximum degree $\Delta$. Denote by $r(\Gamma)$ the rank of $\Gamma$. We firstly prove that $r(K^{\sigma}_{a,b}) = 2$ $(a, b \geqslant 2)$ if and only if all the cycles of order 4 in $K^{\sigma}_{a,b}$ are balanced. Using this result, we also prove that $r(\Gamma) \geqslant \frac{n}{\Delta}$, and the equality holds if and only if $\Gamma = \frac{n}{2\Delta} K^{\sigma}_{\Delta,\Delta}$, and each cycle of order 4 in $K^{\sigma}_{\Delta,\Delta}$ is balanced. If $2\Delta \nmid n$, then $r(\Gamma) \geqslant \frac{n+1}{\Delta}$, and $r(\Gamma) = \frac{n+1}{\Delta}$ if and only if $\Gamma = \frac{n-2\Delta+1}{2\Delta} K^{\sigma}_{\Delta,\Delta} \cup K^{\sigma}_{(\Delta-1),\Delta}$, where each cycle of order 4 in $K^{\sigma}_{\Delta,\Delta}$ and $K^{\sigma}_{(\Delta-1),\Delta}$ is balanced.

**KEYWORDS**: signed graphs, rank of graphs, maximum degree

**MSC2010**: 05C35 05C50

## INTRODUCTION

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We use $N_G(v)$ to denote the *neighbor set* of a vertex $v \in V(G)$, and $d_G(v) = |N_G(v)|$ to denote the *degree* of $v$. Denote by $\Delta(G) = max\{d_G(v)\}$ (or $\Delta$) the *maximum degree* of $G$. If $d_G(v) = 1$, then $v$ is called a *pendant* vertex of $G$. We use $T_n$ to denote a *tree* of order $n$. Let $a$, $b$ be two positive integers. We use $K_{a,b}$ to denote the *complete bipartite graph* with $a$ and $b$ vertices on each part respectively.

Let $V(G) = \{v_1, v_2, \dots, v_n\}$. Then the *adjacency matrix* $A(G)$ of $G$ is a symmetric $n \times n$ matrix with entries $A(i, j) = 1$ (or written as $a_{ij} = 1$) if and only if $v_i v_j \in E(G)$ and zeros elsewhere. The rank of $A(G)$, denoted by $r(G)$, is called the *rank* of $G$. The multiplicity of 0 as an eigenvalue of $A(G)$, denoted by $\eta(G)$, is called the *nullity* of $G$. Obviously, $r(G) + \eta(G) = n$.

A *signed graph* $\Gamma = (G, \sigma)$ consists of a simple graph $G$ with edge set $E$ and a mapping $\sigma : E \rightarrow \{+, -\}$. $G$ is called the *underlying graph* of $\Gamma$. For convenience, sometimes we also use $G^{\sigma}$ to denote $\Gamma$. The *adjacency matrix* of $\Gamma$, denoted by $A(\Gamma) = a^{\sigma}_{ij} = \sigma(v_i v_j) a_{ij}$, where $a_{ij} \in A(G)$. We use $r(\Gamma)$ to denote the *rank* of a signed graph $\Gamma$.

Denote by $C^{\sigma}_n$ a signed cycle of order $n$. The *sign* $\mathrm{sgn}(C^{\sigma}_n)$ of $C^{\sigma}_n$ is defined as $\prod_{e \in E(C^{\sigma}_n)} \sigma(e)$. If $\mathrm{sgn}(C^{\sigma}_n) = +$ (or $\mathrm{sgn}(C^{\sigma}_n) = -$, respectively), then $C^{\sigma}_n$ is said to be *positive* (or *negative*, respectively). If all the cycles of $\Gamma$ are positive, then $\Gamma$ is *balanced*, and *unbalanced* otherwise.

Let $H^{\sigma}$ be a subgraph of $\Gamma$. Then $\Gamma - H^{\sigma}$ is the subgraph of $\Gamma$ with vertex set $V(G) \backslash V(H)$ and edge set $E(G) \backslash E(H)$ preserving the signs in $\Gamma$. Similarly, for $F \subset V(\Gamma)$, we use $\Gamma - F$ to denote the subgraph obtained from $\Gamma$ by removing all vertices in $F$ and all their incident edges. If there is a vertex $x$ which belongs to $V(\Gamma)$ but not $V(H^{\sigma})$, then we use $H^{\sigma} + x$ to denote the union of $H^{\sigma}$ and $x$, i.e., the graph with vertex set $V(H) \cup \{x\}$ and edge set $E(H^{\sigma})$.

Collatz et al [1] attempted to obtain all graphs of order $n$ with $r(G) < n$. Until today, this problem is still unsolved. In mathematics, the rank (or nullity) of a graph attracted a lot of researchers' attention, they focus on the relationship between the rank (or nullity) and some graph parameters, such as pendant vertices [2, 3], matching number [4–7], path cover number [8], and so on.

Song et al [9] proved that

$$r(G) \geqslant 2 + 2\ln_2 \Delta.$$

In 2018, Zhou et al [10] proved that

$$r(G) \geqslant \frac{n}{\Delta}.$$

The relationship between the rank $r_H(D_G)$ and maximum degree of a mixed graph $D_G$ was obtained by

Wei et al [11] as follows

$$r_H(D_G) \geqslant \frac{n}{\Delta}.$$

If we add some special conditions to the edge of a simple graph, then some special graphs will be obtained, such as signed graphs, oriented graphs, $\mathbb{T}$-gain graphs and so on. The rank of these special graphs are also worth studying.

For a signed graph $\Gamma = (G,\sigma)$, Fan et al [12] studied the nullity of unicyclic signed graphs. Fan et al [13] studied the nullity of bicyclic signed graphs. Let $\omega(G)$ be the number of connected components of $G$ and $d(G) = |E(G)| - |V(G)| + \omega(G)$. Lu et al [14] obtained the relationship between $r(\Gamma)$, $d(G)$ and $r(G)$, that is

$$r(G) - 2d(G) \leqslant r(\Gamma) \leqslant r(G) + 2d(G).$$

He et al [15] obtained the relationship between $r(\Gamma)$, $d(G)$ and $m(G)$ (matching number of $G$), that is

$$2m(G) - 2d(G) \leqslant r(\Gamma) \leqslant 2m(G) + d(G).$$

Li et al [16] obtained the bounds of the rank of a signed graph in terms of independence number. There are also some other papers about signed graphs. The readers can refer to [17–19].

For an oriented graph, in 2015, Li et al [20] first investigated the rank of oriented graphs. After that, there are a lot of related results. The most studied is the rank of oriented graph by using different parameters, such as $r(G)$ [21], $m(G)$ [22], bicyclic oriented graphs [23–25], independence number [26], and so on.

For a $\mathbb{T}$-gain graph $\Phi$, Yu et al [27]) first study the inertias of $\Phi$. They also gave some useful results. Lu et al [28] characterized all the $\mathbb{T}$-gain bicyclic graphs $\Phi$ satisfied $r(\Phi) = 2,3,4$. Lu et al [29] obtained the relationship between $r(\Phi)$, $d(G)$ and $r(G)$, that is

$$r(G) - 2d(G) \leqslant r(\Phi) \leqslant r(G) + 2d(G)$$

for a $\mathbb{T}$-gain graph $\Phi$. He et al [30] obtained the relationship between $r(\Phi)$, $d(G)$ and $m(G)$, that is

$$2m(G) - 2d(G) \leqslant r(\Phi) \leqslant 2m(G) + d(G).$$

## PRELIMINARIES

First, we will list some lemmas about the rank of signed graphs.

**Lemma 1 ([19])** *Let $\Gamma$ be a signed graph.*

(i) *If $\Gamma = \bigcup_{i=1}^{t} \Gamma_i$, where $\Gamma_i$ is the connected component of $\Gamma$, then $r(\Gamma) = \sum_{i=1}^{t} r(\Gamma_i)$.*
(ii) *$r(\Gamma) \geqslant 2$ if and only if $\Gamma$ contains at least on edge.*
(iii) *If $V(\Gamma_1) \subseteq V(\Gamma)$, then $r(\Gamma_1) \leqslant r(\Gamma)$.*

For a signed cycle $C_n^\sigma$, we have the following lemma.

**Lemma 2 ([12])** *For a signed cycle $C_n^\sigma$, if $C_n^\sigma$ is balanced, then*

$$r(C_n^\sigma) = \begin{cases} n-2, & \text{if } n \equiv 0 \,(\text{mod } 4),\\ n, & \text{otherwise.} \end{cases}$$

*If $C_n^\sigma$ is unbalanced, then*

$$r(C_n^\sigma) = \begin{cases} n-2, & \text{if } n \equiv 2 \,(\text{mod } 4),\\ n, & \text{otherwise.} \end{cases}$$

**Lemma 3 ([12])** *Let $\Gamma = (G,\sigma)$ be a signed graph. If $\Gamma$ has an edge $uv$ such that $d_\Gamma(u) = 1$, then $r(\Gamma) = r(\Gamma_1) + 2$ where $\Gamma_1 = \Gamma - u - v$.*

By Lemmas 2.4 and 3.1 of [10], we can get the following lemma.

**Lemma 4** *Let $\Gamma$ be a signed graph with $n$ vertices. If $r(\Gamma) = r$, then there is an induced subgraph $\Gamma_1$ of $\Gamma$ such that $r(\Gamma_1) = |V(\Gamma_1)| = r$.*

## MAIN RESULTS ABOUT $r(\Gamma)$

In this section, we will give our main results about the lower bound of $r(\Gamma)$.

**Lemma 5** *Let $\Gamma = K_{a,b}^\sigma$ $(a, b \geqslant 2)$ and $V(\Gamma) = V_1 \cup V_2$, $|V_1| = a, |V_2| = b$. Then $r(K_{a,b}^\sigma) = 2$ if and only if $\Gamma$ is balanced.*

*Proof*: *(Necessity)* Let

$$A(\Gamma) = \begin{pmatrix} 0 & A_1 \\ A_1^T & 0 \end{pmatrix}$$

be the adjacency matrix of $\Gamma$. Since $r(\Gamma) = 2$, we have $r(A_1) = 1$. Let $\alpha_1$, $\alpha_2$, $\dots$, $\alpha_a$ be the row vectors of $A_1$. Since $r(A_1) = 1$, we have that every maximal independent group of $\alpha_1$, $\alpha_2$, $\dots$, $\alpha_a$ has one vector. Without loss of generality, let $\alpha_i$ $(i \in \{1,2,\dots,a\})$ be the unique vector of the maximal linearly independent group and $\alpha_j = k_j\alpha_i$, $j = 1,2,\dots,i-1,i+1,\dots,a$, $k_j \neq 0$.

Let $x_1, x_2$ be any two vertices of $V_1$ and $y_1, y_2$ be any two vertices of $V_2$. For convenience, we assume $\alpha_1$, $\alpha_2$ be the vector corresponding to $x_1, x_2$

in $A_1$, respectively. Denote by $a_{ij}$ the element in $A_1$ corresponding to the edge $x_i y_j$, $i, j \in \{1,2\}$. Then

$$a_{11} = \frac{k_1}{k_2} a_{21}, \quad a_{12} = \frac{k_1}{k_2} a_{22}.$$

Let $C_4^\sigma$ be the signed cycle induced by $\{x_1, x_2, y_1, y_2\}$, then $\mathrm{sgn}(C_4^\sigma) = +$, that is, $C_4^\sigma$ is balanced.

*(Sufficiency)* Let $A_1$, $x_1, x_2$, $y_1, y_2$ and $a_{ij}$ be the same as described in the proof of "Necessity". Since all the cycles of order 4 in $\Gamma$ are balanced, so

$$a_{11} a_{12} a_{21} a_{22} = 1,$$

i.e.,

$$a_{11} a_{22} = \frac{1}{a_{12} a_{21}} = a_{12} a_{21},$$

since $a_{ij} = \pm 1$. So,

$$\frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}}.$$

Using the same method, we can get that the maximal linearly independent group of $\alpha_1$, $\alpha_2$, ..., $\alpha_a$ describe above has one vector, i.e., $r(A_1) = 1$, and then $r(\Gamma) = 2$. □

**Theorem 1** *Let $\Gamma = (G, \sigma)$ be a signed graph with $n$ vertices and minimum degree at least 1. Then*

$$r(\Gamma) \geqslant \frac{n}{\Delta}.$$

*Proof*: For convenience, let $r(\Gamma) = r$. Since $\Gamma$ has no isolated vertex, by Lemma 1(b), $r \geqslant 2$. By Lemma 4, there exists an nonsingular induced subgraph $\Gamma_1$ of $\Gamma$ and $r(\Gamma_1) = |\Gamma_1| = r$. Let $\Gamma_2$ be the signed graph obtained from $\Gamma - \Gamma_1$. Then, we can get the following claim.

**Claim 1** For any vertex $y \in V(\Gamma_2)$, there exists at least one vertex $x \in V(\Gamma_1)$ such that $xy \in E(\Gamma)$.

Suppose the contrary, let $u \in V(\Gamma_2)$ such that $d_{\Gamma_1}(u) = 0$. Since $\Gamma$ has no isolated vertex, there exists a vertex $v \in V(\Gamma_2)$ and $uv \in E(\Gamma_2)$. Let $\Gamma_3 = \Gamma_1 + u + v$. Then $u$ is a pendant vertex of $\Gamma_3$ with the unique neighbor $v$. By Lemma 3,

$$r(\Gamma_3) = r(\Gamma_1) + 2 = r + 2 > r,$$

a contradiction.

Let $E_1 = \{xy \mid x \in V(\Gamma_1), y \in V(\Gamma_2)$. Using the results of Claim 1,

$$n - r = |V(\Gamma_2)| \leqslant |E_1|. \tag{1}$$

Since $r(\Gamma_1) = |V(\Gamma_1)| = r$, we have

$$d_{\Gamma_1}(x) \geqslant 1, \tag{2}$$

$$d_{\Gamma}(x) \leqslant \Delta, \tag{3}$$

$$|E_1| = \sum_{x \in \Gamma_1} d_\Gamma(x) - \sum_{x \in \Gamma_1} d_{\Gamma_1}(x), \tag{4}$$

for each vertex $x \in V(\Gamma_1)$.

Combining with (1), (2), (3) and (4),

$$n - r \leqslant |E_1| \leqslant r\Delta - r, \tag{5}$$

so, we have

$$r(\Gamma) = r \geqslant \frac{n}{\Delta}.$$

□

In the following, the signed graphs $\Gamma$ satisfied $r(\Gamma) = \frac{n}{\Delta}$ will be characterized.

**Theorem 2** *Let $\Gamma = (G, \sigma)$ be a signed graph with $n$ vertices and minimum degree at least 1. Then $r(\Gamma) = \frac{n}{\Delta}$ if and only if $\Gamma = \frac{n}{2\Delta} K^\sigma_{\Delta,\Delta}$, and $K^\sigma_{\Delta,\Delta}$ is balanced.*

*Proof*: *(Sufficiency)* Let $\Gamma = \frac{n}{2\Delta} K^\sigma_{\Delta,\Delta}$, and each cycle (if any) of order 4 in $K^\sigma_{\Delta,\Delta}$ is balanced.

If $\Delta = 1$, then $r(K^\sigma_{\Delta,\Delta}) = r(K^\sigma_{1,1}) = 2$, and so $r(\Gamma) = n$, as desired.

If $\Delta \geqslant 2$, by Lemmas 1 and 5,

$$r(K^\sigma_{\Delta,\Delta}) = 2 \text{ and } r(\Gamma) = \frac{n}{\Delta}.$$

*(Necessity)* Since $r(\Gamma) = \frac{n}{\Delta}$, $\Delta | n$ and the inequalities (2), (3) and (5) all become equalities. Let $\Gamma_1$ be the same as described in Theorem 1 and so $|V(\Gamma_1)| = \frac{n}{\Delta}$, $r(\Gamma_1) = r(\Gamma)$. For each vertex $x \in V(\Gamma_1)$:
(i) $d_{\Gamma_1}(x) = 1$, i.e., $\Gamma_1 = \frac{n}{2\Delta} K^\sigma_{1,1}$;
(ii) $d_\Gamma(x) = \Delta$;
(iii) $|E_1| = n - r$.

If $\Delta = 1$, then $\Gamma = \frac{n}{2} K^\sigma_{1,1}$, as desired.

If $\Delta \geqslant 2$, let $x_1 y_1 \in E(\Gamma_1)$. By (ii), we have $d_\Gamma(x) = \Delta$ for each vertex $x \in V(\Gamma_1)$. Let

$$N_{\Gamma_2}(x_1) = \{y_2, y_3, \ldots, y_\Delta\},$$
$$N_{\Gamma_2}(y_1) = \{x_2, x_3, \ldots, x_\Delta\}.$$

For any $2 \leqslant i, j \leqslant \Delta$, by (iii), we have $x_i \neq y_j$. Now we will prove that $x_i$ is adjacent to $y_j$. Suppose to the contrary that $x_i y_j \notin E(\Gamma_2)$ (by (i) we have $x_i, y_j \in V(\Gamma_2)$). Let $\Gamma_4 = \Gamma_1 \cup \{x_i, y_j\}$, by Lemma 3,

$$\begin{aligned} r(\Gamma_4) &= r(\Gamma_4 - x_i - y_j - x_1 - y_1) + 4 \\ &= r(\Gamma_1 - x_1 - y_1) + 4 = r(\Gamma_1) + 2 > r(\Gamma), \end{aligned}$$

a contradiction. Hence, the signed graph obtained from $\{x_1, x_2, \ldots, x_\Delta, y_1, y_2, \ldots, y_\Delta\}$ is $K^\sigma_{\Delta,\Delta}$. By (i), we can get that $\Gamma = \frac{n}{2\Delta} K^\sigma_{\Delta,\Delta}$.

Since $r(\Gamma) = \frac{n}{\Delta}$, we have $r(K^\sigma_{\Delta,\Delta}) = 2$ for each $K^\sigma_{\Delta,\Delta}$. By Lemma 5, we can get that each cycle (if any) of order 4 in $K^\sigma_{\Delta,\Delta}$ is balanced. □

Let $\Gamma = (G, \sigma)$ be a signed graph with $n$ vertices. Next, we will determine the minimum rank of $\Gamma$ with the maximum degree $\Delta$ satisfying $2\Delta \nmid n$. All the corresponding extremal graphs are characterized.

**Theorem 3** *Let $\Gamma = (G, \sigma)$ be a signed graph with $n$ vertices and minimum degree at least 1, $2\Delta \nmid n$. Then $r(\Gamma) \geqslant \frac{n+1}{\Delta}$, and $r(\Gamma) = \frac{n+1}{\Delta}$ if and only if $\Gamma = \frac{n-2\Delta+1}{2\Delta} K^\sigma_{\Delta,\Delta} \cup K^\sigma_{(\Delta-1),\Delta}$, and $K^\sigma_{\Delta,\Delta}$, $K^\sigma_{(\Delta-1),\Delta}$ are balanced.*

*Proof*: Since $2\Delta \nmid n$, $\Delta \geqslant 2$. Assume $r(\Gamma) = r$. Using the results in Lemma 4, there exists a nonsingular induced subgraph $\Gamma_1$ of $\Gamma$ and $r(\Gamma_1) = |\Gamma_1| = r$. Let $\Gamma_2 = \Gamma - \Gamma_1$. Using the same methods in Theorem 1, we can get that for any vertex of $y \in V(\Gamma_2)$, there exists at least one vertex of $x \in V(\Gamma_1)$ satisfying $xy \in E(\Gamma)$.

Let $E_2 = \{xy \mid x \in V(\Gamma_1),\ y \in V(\Gamma_2)$. Then

$$|E_2| \geqslant |V(\Gamma_2)| = n - r. \tag{6}$$

Since $r(\Gamma_1) = |V(\Gamma_1)| = r$, we have

$$d_{\Gamma_1}(x) \geqslant 1, \quad \text{i.e.,} \sum_{x\in\Gamma_1} d_{\Gamma_1}(x) \geqslant r, \tag{7}$$

$$d_{\Gamma}(x) \leqslant \Delta, \quad \text{i.e.,} \sum_{x\in\Gamma_1} d_{\Gamma}(x) \leqslant r\Delta, \tag{8}$$

$$|E_2| = \sum_{x\in\Gamma_1} d_{\Gamma}(x) - \sum_{x\in\Gamma_1} d_{\Gamma_1}(x), \tag{9}$$

for each vertex $x \in V(\Gamma_1)$.

Combining with (6), (7), (8) and (9), we have

$$n - r \leqslant |E_2| \leqslant r\Delta - r, \tag{10}$$

so, we have

$$r(\Gamma) = r \geqslant \frac{n}{\Delta}.$$

If $r(\Gamma) = \frac{n}{\Delta}$, then by Trefth:3.3, $\Gamma = \frac{n}{2\Delta} K^\sigma_{\Delta,\Delta}$, a contradiction to $2\Delta \nmid n$.

Now, the following cases will be considered.

**Case 1:** Two inequalities in (6), (7) and (8) turn into equalities. In this case, we have

$$r(\Gamma) \geqslant \frac{n+1}{\Delta}.$$

**Case 2:** At most one inequality in (6), (7) and (8) turn into equality. In this case, we have

$$r(\Gamma) \geqslant \frac{n+2}{\Delta} > \frac{n+1}{\Delta}.$$

Combining with Cases 1 and 2, we have

$$r(\Gamma) \geqslant \frac{n+1}{\Delta}.$$

In the following, we will characterize the extremal signed graph $\Gamma$ with $r(\Gamma) = \frac{n+1}{\Delta}$.

*(Sufficiency)* Let $\Gamma = \frac{n-2\Delta+1}{2\Delta} K^\sigma_{\Delta,\Delta} \cup K^\sigma_{(\Delta-1),\Delta}$ such that each cycle of order 4 in $K^\sigma_{\Delta,\Delta}$ and $K^\sigma_{(\Delta-1),\Delta}$ is balanced.

Then by Lemmas 1 and 5,

$$r(\Gamma) = \frac{n-2\Delta+1}{2\Delta} r(K^\sigma_{\Delta,\Delta}) + r(K^\sigma_{(\Delta-1),\Delta}) = \frac{n+1}{\Delta}.$$

*(Necessity)* $r(\Gamma) = \frac{n+1}{\Delta}$, and $2\Delta \nmid n$.

**Case 1:** (6) and (8) turn into equalities and (7) is strict, that is $\sum_{x\in\Gamma_1} d_{\Gamma_1}(x) \geqslant r+1$. Since $r(\Gamma) = \frac{n+1}{\Delta}$, we have $\sum_{x\in\Gamma_1} d_{\Gamma_1}(x) = r+1$. That is $\Gamma_1 = \frac{r-3}{2} K^\sigma_2 \cup P^\sigma_3$. By Lemmas 1 and 3, we have $r(\Gamma_1) = r - 1$, a contradiction.

**Case 2:** (7) and (8) turn into equalities and (6) is strict, we have $|E_2| = n - r + 1$ since $r(\Gamma) = \frac{n+1}{\Delta}$. So, we can get that there exists a unique vertex $u$ in $V(\Gamma_2)$ such that $d_{\Gamma_1}(u) = 2$ and any other vertex $v$ in $V(\Gamma_2)$ have $d_{\Gamma_1}(v) = 1$. Assume that $x_1u, x_2u \in E_2$. Note that $d_{\Gamma_1}(x) = 1$ and $\Gamma_1 = \frac{r}{2} K^\sigma_2$ since (7) holds.

*Subcase 2.1:* $x_1x_2 \in E(\Gamma_1)$.

Combining with the fact that $d_{\Gamma_1}(x) = 1$, $d_\Gamma(x) = \Delta$, for each vertex $x$ in $V(\Gamma_1)$, we denote

$$N_\Gamma(x_1) = \{y_1, y_2, \ldots, y_{\Delta-2}, x_2, u\},$$
$$N_\Gamma(x_2) = \{z_1, z_2, \ldots, z_{\Delta-2}, , x_1, u\},$$

so, we have

$$y_i \neq z_j, 1 \leqslant i, j \leqslant \Delta - 2.$$

If $\Delta = 2$, then the graph induced by $x_1, x_2, u$ is $C^\sigma_3$. Let $x_3, x_4$ be two adjacent vertices distinct from $x_1, x_2$ in $\Gamma_1$. Since $d_{\Gamma_2}(x_3) = d_{\Gamma_2}(x_4) = \Delta - 1 = 1$. Denote by $m_1, m_2$ be two vertices in $\Gamma_2$ such that $x_3m_1, x_4m_2 \in E_2$, we say that $m_1m_2 \in E(\Gamma_2)$. Otherwise, let $\Gamma_3$ be the signed graph induced by $\Gamma_1 \cup \{m_1, m_2\}$. By Lemma 3,

$$r(\Gamma_3) = r(\Gamma_3 - x_3 - m_1 - x_4 - m_2) + 4 = r(\Gamma_1) + 2 > r,$$

a contradiction. So we have the graph induced by $x_3, x_4, m_1, m_2$ is $C^\sigma_4$. Then $\Gamma = \frac{n-3}{4} C^\sigma_4 \cup C^\sigma_3$. By Lemma 2,

$$r(\Gamma) = \frac{n-3}{4} r(C^\sigma_4) + 3.$$

If $C_4^\sigma$ is balanced, then $r(\Gamma)=\frac{n-3}{2}+3=\frac{n+3}{2}>\frac{n+1}{2}$, a contradiction.

If $C_4^\sigma$ is unbalanced, then $r(\Gamma)=n-3+3=n>\frac{n+1}{2}$, a contradiction.

If $\Delta\geqslant 3$, then the vertices $y_1,y_2,\dots,y_{\Delta-2}$ and $z_1,z_2,\dots,z_{\Delta-2}$ are all adjacent to $u$. Otherwise, suppose there exists a vertex $y_i$ $(i=1,2,\dots,\Delta-2)$ is not adjacent to $u$. Let $\Gamma_4$ be the signed graph induced by $\Gamma_1\cup\{y_i,u\}$. By Lemma 3,

$$r(\Gamma_4)=r(\Gamma_4-x_1-x_2-u-y_i)+4=r(\Gamma_1)+2>r,$$

a contradiction. So,

$$d_\Gamma(u)\geqslant 2+2(\Delta-2)=\Delta+\Delta-2>\Delta,$$

also a contradiction.

*Subcase 2.2:* $x_1x_2\notin E(\Gamma_1)$.

Since $d_{\Gamma_1}(x)=1$ for any vertex $x$ in $\Gamma_1$ and $\Gamma_1=\frac{r}{2}K_2^\sigma$, let $x_1x_3,x_2x_4\in E(\Gamma_1)$. We say that for each $v\in N_{\Gamma_2}(x_3)\cup N_{\Gamma_2}(x_4)$, $uv\in E(\Gamma)$. Otherwise, let $v\in N_{\Gamma_2}(x_3)$ and $uv\notin E(\Gamma)$. Let $\Gamma_5$ be the signed graph induced by $\Gamma_1\cup\{u,v\}$. By Lemma 3,

$$r(\Gamma_5)=r(\Gamma_5-x_1-x_3-u-v)+4=r(\Gamma_1)+2>r,$$

a contradiction. Since $N_\Gamma(x_3)\cap N_\Gamma(x_4)=\varnothing$, so we have $d_\Gamma(u)\geqslant 2\Delta>\Delta$, a contradiction.

**Case 3:** (6) and (7) turn into equalities and (8) is strict, we have $\sum_{x\in\Gamma_1}d_\Gamma(x)=r\Delta-1$ since $r(\Gamma)=\frac{n+1}{\Delta}$. In this case, we say that there exists an unique vertex $x_1$ in $\Gamma_1$ such that $d_\Gamma(x_1)=\Delta-1$ and other vertices have degree $\Delta$ in $\Gamma$. Since (7) turn into equality, we can get that $\Gamma_1=\frac{r}{2}K_2^\sigma$. Let $x_1x_2\in E(\Gamma_1)$ and

$$N_\Gamma(x_1)=\{y_1,y_2,\dots,y_{\Delta-2},x_2\},$$
$$N_\Gamma(x_2)=\{z_1,z_2,\dots,z_{\Delta-1},x_1\}.$$

Since (6) turns into equality, we can get that $N_\Gamma(x_1)\cap N_\Gamma(x_2)=\varnothing$. Similar to the method in Case 2, we can get that the graph induced by the vertices

$$y_1,y_2,\dots,y_{\Delta-2},x_2,z_1,z_2,\dots,z_{\Delta-1},x_1$$

is $K_{(\Delta-1),\Delta}^\sigma$. For any edge $x_3x_4$ in $E(\Gamma_1)$ distinct from $x_1x_2$, the graph induced by the $\{x_3,x_4\}$, $N_\Gamma(x_3)\cup N_\Gamma(x_4)$ is $K_{\Delta,\Delta}^\sigma$. So, $\Gamma=\frac{n-2\Delta+1}{2\Delta}K_{\Delta,\Delta}^\sigma\cup K_{(\Delta-1),\Delta}^\sigma$.

Hence,

$$r(\Gamma)=\frac{n+1}{\Delta}=\frac{n-2\Delta+1}{2\Delta}r(K_{\Delta,\Delta}^\sigma)+r(K_{(\Delta-1),\Delta}^\sigma).$$

By Lemma 1, $r(K_{\Delta,\Delta}^\sigma)\geqslant 2$, $r(K_{(\Delta-1),\Delta}^\sigma)\geqslant 2$.

We can get that

$$\frac{n-2\Delta+1}{2\Delta}r(K_{\Delta,\Delta}^\sigma)+r(K_{(\Delta-1),\Delta}^\sigma)\geqslant\frac{n+1}{\Delta}.$$

Hence,

$$r(K_{\Delta,\Delta}^\sigma)=r(K_{(\Delta-1),\Delta}^\sigma)=2.$$

By Lemma 5, we have all the cycles of order 4 in $K_{\Delta,\Delta}^\sigma$ and $K_{(\Delta-1),\Delta}^\sigma$ are balanced. □

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## REFERENCES

1. Collatz L, Sinogowitz U (1957) Spektren endlicher grafen. *Abh Math Sem Univ Hamburg* **21**, 63–77.
2. Chang S, Chang A, Zheng Y (2020) The leaf-free graphs with nullity $2c(G)-1$. *Discrete Appl Math* **277**, 44–54.
3. Ma X, Wong D, Tian F (2016) Nullity of a graph in terms of the dimension of cycle space and the number of pendant vertices. *Discrete Appl Math* **215**, 171–176.
4. Feng Z, Huang J, Li S, Luo X (2019) Relationship between the rank and the matching number of a graph. *Appl Math Comput* **354**, 411–421.
5. Li X, Guo JM (2019) No graph with nullity $\eta(G)=|V(G)|-2m(G)+2c(G)-1$. *Discrete Appl Math* **268**, 130–136.
6. Ma X, Fang X (2020) An improved lower bound for the nullity of a graph in terms of matching number. *Linear Multilinear Algebra* **68**, 1983–1989.
7. Rula S, Chang A, Zheng Y (2016) The extremal graphs with respect to their nullity. *J Inequal Appl* **2016**, ID 17.
8. Wang L (2021) Nullity of a graph in terms of path cover number. *Linear Multilinear Algebra* **69**, 1902–1908.
9. Song Y, Song X, Zhang M (2016) An upper bound for the nullity of a bipartite graph in terms of its maximum degree. *Linear Multilinear Algebra* **64**, 1107–1112.
10. Zhou Q, Wong D, Sun D (2018) An upper bound of the nullity of a graph in terms of order and maximum degree. *Linear Algebra Appl* **555**, 314–320.
11. Wei W, Li S, Ma H (2021) Bounds on the nullity, the $H$-rank and the Hermitian energy of a mixed graph. *Linear Multilinear Algebra* **69**, 2469–2490.
12. Fan Y-Z, Wang Y, Wang Y (2013) A note on the nullity of unicyclic signed graphs. *Linear Algebra Appl* **438**, 1193–1200.

13. Fan Y, Du W, Dong C (2014) The nullity of bicyclic signed graphs. *Linear Multilinear Algebra* **62**, 242–251.
14. Lu Y, Wang L, Zhou Q (2018) The rank of a signed graph in terms of the rank of its underlying graph. *Linear Algebra Appl* **538**, 166–186.
15. He S, Hao RX, Lai HJ (2019) Bounds for the matching number and cyclomatic number of a signed graph in terms of rank. *Linear Algebra Appl* **572**, 273–291.
16. Li X, Xia W (2020) Adjacency rank and independence number of a signed graph. *Bull Malays Math Sci Soc* **43**, 993–1007.
17. Hou Y, Li J, Pan Y (2003) On the Laplacian eigenvalues of signed graphs. *Linear Multilinear Algebra* **51**, 21–30.
18. Liu Y, You L (2014) Further results on the nullity of signed graphs. *J Appl Math* **2014**, ID 483735.
19. Yu G, Feng L, Qu H (2016) Signed graphs with small positive index of inertia. *Electron J Linear Algebra* **31**, 232–243.
20. Li X, Yu G (2015) The skew-rank of oriented graphs. *Sci Sin Math* **45**, 93–104. [in Chinese]
21. Wong D, Ma X, Tian F (2016) Relation between the skew-rank of an oriented graph and the rank of its underlying graph. *Eur J Comb* **54**, 76–86.
22. Ma X, Wong D, Tian F (2016) Skew-rank of an oriented graph in terms of matching number. *Linear Algebra Appl* **495**, 242–255.
23. Lu Y, Wang L, Zhou Q (2015) Bicyclic oriented graphs with skew-rank 6. *Appl Math Comput* **270**, 899–908.
24. Qu H, Yu G (2015) Bicyclic oriented graphs with skew-rank 2 or 4. *Appl Math Comput* **258**, 182–191.
25. Qu H, Yu G, Feng L (2015) More on the minimum skew-rank of graphs. *Oper Matrices* **9**, 311–324.
26. Huang J, Li S, Wang H (2018) Relation between the skew-rank of an oriented graph and the independence number of its underlying graph. *J Comb Optim* **36**, 65–80.
27. Yu G, Qu H, Tu J (2015) Inertia of complex unit gain graphs. *Appl Math Comput* **265**, 619–629.
28. Lu Y, Wang L, Xiao P (2017) Complex unit gain bicyclic graphs with rank 2, 3 or 4. *Linear Algebra Appl* **523**, 169–186.
29. Lu Y, Wang L, Zhou Q (2019) The rank of a complex unit gain graph in terms of the rank of its underlying graph. *J Comb Optim* **38**, 570–588.
30. He S, Hao R, Dong F (2020) The rank of a complex unit gain graph in terms of the matching number. *Linear Algebra Appl* **589**, 158–185.