RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2021.092

ScienceAsia 47 (2021): **773–778**

Some new upper bounds for moduli of eigenvalues of iterative matrices

Jun He*, **Yanmin Liu**, **Guangjun Xu**

School of mathematics, Zunyi Normal College, Zunyi, Guizhou, 563006 China

*Corresponding author, e-mail: hejunfan1@163.com

Received 9 Feb 2021
Accepted 8 Aug 2021

ABSTRACT: Based on the matrix splitting $M = P - Q$, some upper bounds for the maximum of moduli of eigenvalues of the iteration matrix $P^{-1}Q$ are obtained when P is an strictly diagonally dominant (SDD) matrix or a doubly strictly diagonally dominant (DSDD) matrix. In this paper, some new upper bounds are introduced, which is applicable to a Dashnic-Zusmanovich (DZ) matrix P, and proved to be better than those in Huang and Gao [Int J Comput Math **80** (2003):799–803] and Li et al [Appl Math Comput **173** (2006):977–984] in certain cases.

KEYWORDS: iteration matrix, eigenvalues, spectral radius

MSC2010: 65F50

INTRODUCTION

Let $M = (m_{ts})$ be an $n \times n$ nonsingular complex matrix and consider the following large system of linear equations

$$Mx = b, \tag{1}$$

which plays an important role in the numerical solution of elliptic partial differential equations [1, 2]. Based on the matrix splitting $M = P - Q$, some basic iterative methods, such as Jacobi, Gauss-Seidel and successive overrelaxation iterative methods, are introduced to solve the system of linear equations (1) [3]. The matrix splitting $M = P - Q$ leads to the stationary iteration scheme

$$x_{k+1} = P^{-1}Qx_k + c. \tag{2}$$

The matrix $P^{-1}Q$ in (2) is called the iteration matrix. It is well known that the iteration scheme (2) converges for any initial vector x_0 if $\rho(P^{-1}Q) < 1$, where $\rho(P^{-1}Q)$ denotes the spectral radius of the matrix $P^{-1}Q$, namely, maximum of moduli of eigenvalues of $P^{-1}Q$.

Let $\mathbb{C}^{n\times n}$ be the set of all complex matrices, and denote $[n] := \{1, 2, \dots, n\}$. A matrix $P = (p_{ts}) \in \mathbb{C}^{n\times n}$ is called an SDD (strictly diagonally dominant) matrix if for each $t \in [n]$,

$$|p_{tt}| > r_t(P),$$

where $r_t(P) = \sum_{s\neq t} |p_{ts}|$. A matrix $P = (p_{ts}) \in \mathbb{C}^{n\times n}$ is called a DSDD (doubly strictly diagonally dominant) matrix if for distinct $t, s \in [n]$, the following inequality holds:

$$|p_{tt}||p_{ss}| > r_t(P)r_s(P).$$

A matrix $P = (p_{ts}) \in \mathbb{C}^{n\times n}$ is called a DZ (Dashnic-Zusmanovich) matrix if there exists an index $t \in [n]$, for all $s \neq t$, $s \in [n]$, the following inequality holds:

$$|p_{tt}|(|p_{ss}| - r_s^t(P)) > r_t(P)|p_{st}|,$$

where $r_s^t(P) = r_s(P) - |p_{st}|$. It has been shown in [4] that the above three classes of matrices are nonsingular and have the following inclusions:

$$\{SDD\} \subseteq \{DSDD\} \subseteq \{DZ\}.$$

One practical application of the eigenvalues of $P^{-1}Q$ is that one can identify the convergence property of the iteration scheme $x_{k+1} = P^{-1}Qx_k + c$ by the maximum of moduli of eigenvalues of $P^{-1}Q$, for more details, see [5–8]. In [5, 7], the authors established the following result for an SDD matrix P.

Theorem 1 ([5, 7]) *Let $P = (p_{ts}) \in \mathbb{C}^{n\times n}$ be an SDD matrix, $Q = (q_{ts}) \in \mathbb{C}^{n\times n}$. Then*

$$|\lambda(P^{-1}Q)| \leqslant \omega_1 = \max_{t\in[n]} \frac{|q_{tt}| + r_t(Q)}{|p_{tt}| - r_t(P)}.$$

If further $P = (p_{ts}) \in \mathbb{C}^{n\times n}$ is a DSDD matrix, the result in Theorem 1 can be improved as follows.

Theorem 2 ([8]) *Let $P=(p_{ts})\in\mathbb{C}^{n\times n}$ be a DSDD matrix, $Q=(q_{ts})\in\mathbb{C}^{n\times n}$. Then*

$$|\lambda(P^{-1}Q)|\leqslant\omega_2=\max_{t,s\in[n],s\neq t}\frac{B+\sqrt{B^2-4AC}}{2A},$$

where

$$\begin{aligned}A&=|p_{tt}p_{ss}|-r_t(P)r_s(P),\\B&=|p_{tt}q_{ss}|+|q_{tt}p_{ss}|+r_t(P)r_s(Q)+r_t(Q)r_s(P),\\C&=|q_{tt}q_{ss}|-r_t(Q)r_s(Q).\end{aligned}$$

Recently, Li et al [9] presented the following result, for a DSDD matrix $P=(p_{ts})\in\mathbb{C}^{n\times n}$.

Theorem 3 ([9]) *Let $P=(p_{ts})\in\mathbb{C}^{n\times n}$ be a DSDD matrix, $Q=(q_{ts})\in\mathbb{C}^{n\times n}$. Then*

$$|\lambda(P^{-1}Q)|\leqslant\omega_3=\max_{t,s\in[n],s\neq t}\frac{B+\sqrt{B^2-4AC}}{2A},$$

where

$$\begin{aligned}A&=|p_{tt}p_{ss}|-r_t(P)r_s(P),\\B'&=|p_{tt}q_{ss}+q_{tt}p_{ss}|+r_t(P)r_s(Q)+r_t(Q)r_s(P),\\C'&=-[|q_{tt}q_{ss}|+r_t(Q)r_s(Q)].\end{aligned}$$

For an SDD matrix $P=(p_{ts})\in\mathbb{C}^{n\times n}$, Theorem 1 can be used to obtain the upper bound for $|\lambda(P^{-1}Q)|$. For a DSDD matrix P, Theorems 2 and 3 can be used to obtain the upper bounds for $|\lambda(P^{-1}Q)|$. In this paper, some new bounds for a DZ matrix $P=(p_{ts})$ are introduced.

MAIN RESULTS

Before presenting our main results, we need the following lemmas.

Lemma 1 ([4]) *If $P=(p_{ts})\in\mathbb{C}^{n\times n}$ is a DZ matrix, then P is nonsingular.*

Lemma 2 ([4]) *All eigenvalues of a matrix $P=(p_{ts})\in\mathbb{C}^{n\times n}$, $n\geqslant 2$, belong to the set*

$$\Theta(P)=\bigcap_{t\in[n]}\bigcup_{s\in[n],t\neq s}\Theta_{ts}(P),$$

where

$$\Theta_{ts}(P)=\{\lambda\in\mathbb{C}:|\lambda-p_{tt}|(|\lambda-p_{ss}|-r_s^t(P))\}\leqslant r_t(P)|p_{st}|.$$

Our first main result for a DZ matrix P is stated as follows.

Theorem 4 *Let $P=(p_{ts})\in\mathbb{C}^{n\times n}$ be a DZ matrix, $Q=(q_{ts})\in\mathbb{C}^{n\times n}$. Then*

$$|\lambda(P^{-1}Q)|\leqslant\omega_4=\min_{t\in[n]}\max_{s\in[n],s\neq t}\frac{H+\sqrt{H^2-4GK}}{2G},$$

where

$$\begin{aligned}G&=|p_{tt}|(|p_{ss}|-r_s^t(P))-r_t(P)|p_{st}|,\\H&=|p_{tt}|(|q_{ss}|+r_s^t(Q))+|p_{ss}||q_{tt}|\\&\quad+r_t(P)|q_{st}|+r_t(Q)|p_{st}|,\\K&=|q_{tt}|(|q_{ss}|+r_s^t(Q))-r_t(Q)|q_{st}|.\end{aligned}$$

Proof: Since $P=(p_{ts})\in\mathbb{C}^{n\times n}$ is a DZ matrix, by Lemma 1, $P=(p_{ts})$ is nonsingular. Assume $\lambda\in\sigma(P^{-1}Q)$ ($\sigma(P^{-1}Q)$ is the set of all the eigenvalues of $P^{-1}Q$), then

$$\det(\lambda I-P^{-1}Q)=0,$$

which means

$$\det(\lambda P-Q)=0.$$

By Lemma 2, for the index $t\in[n]$, there is an index $s\in[n]$, $s\neq t$, if

$$|\lambda p_{tt}-q_{tt}|(|\lambda p_{ss}-q_{ss}|-r_s^t(\lambda P-Q))>r_t(\lambda P-Q)|\lambda p_{st}-q_{st}|,\quad(3)$$

then $\lambda\notin\sigma(P^{-1}Q)$. Therefore, if

$$\begin{aligned}&(|\lambda||p_{tt}|-|q_{tt}|)(|\lambda||p_{ss}|-|q_{ss}|)\\&\quad-(|\lambda||p_{tt}|+|q_{tt}|)(|\lambda|r_s^t(P)+r_s^t(Q))\\&\quad>(|\lambda|r_t(P)+r_t(Q))(|\lambda||p_{st}|+|q_{st}|),\end{aligned}\quad(4)$$

then, $\lambda\notin\sigma(P^{-1}Q)$. Therefore, if $\lambda\in\sigma(P^{-1}Q)$, we have

$$\begin{aligned}&(|\lambda||p_{tt}|-|q_{tt}|)(|\lambda||p_{ss}|-|q_{ss}|)\\&\quad-(|\lambda||p_{tt}|+|q_{tt}|)(|\lambda|r_s^t(P)+r_s^t(Q))\\&\quad\leqslant(|\lambda|r_t(P)+r_s(Q))(|\lambda||p_{st}|+|q_{st}|),\end{aligned}\quad(5)$$

that is,

$$G|\lambda|^2-H|\lambda|+K\leqslant 0,\quad(6)$$

where

$$\begin{aligned}G&=|p_{tt}|(|p_{ss}|-r_s^t(P))-r_t(P)|p_{st}|,\\H&=|p_{tt}|(|q_{ss}|+r_s^t(Q))+|p_{ss}||q_{tt}|\\&\quad+r_t(P)|q_{st}|+r_t(Q)|p_{st}|,\\K&=|q_{tt}|(|q_{ss}|+r_s^t(Q))-r_t(Q)|q_{st}|.\end{aligned}$$

Since $P=(p_{ts})\in\mathbb{C}^{n\times n}$ is a DZ matrix, then $|p_{tt}|(|p_{ss}|-r_s^t(P))-r_t(P)|p_{st}|>0$, then for the index $t\in[n]$, there is an index $s\in[n]$, $s\neq t$,

$$|\lambda(P^{-1}Q)|\leqslant \max_{s\in[n],s\neq t}\frac{H+\sqrt{H^2-4GK}}{2G},$$

By the arbitrariness of t, we have

$$|\lambda(P^{-1}Q)|\leqslant \min_{t\in[n]}\max_{s\in[n],s\neq t}\frac{H+\sqrt{H^2-4GK}}{2G}.$$

□

Remark 1 In Theorem 4 of [9], when $P=(p_{ts})\in\mathbb{C}^{n\times n}$ is a DSDD matrix, $Q=(q_{ts})\in\mathbb{C}^{n\times n}$, the authors obtained the following result:

$$|\lambda(P^{-1}Q)|\leqslant \max_{t,s\in[n],s\neq t}\{\min\{\tau_1,\tau_2\}\},$$

where $\tau_1=\frac{B+\sqrt{B^2-4AC}}{2A}$, $\tau_2=\frac{E+\sqrt{E^2-4DF}}{2D}$, and

$$\begin{aligned}D&=|p_{tt}p_{ss}|+r_t(P)r_s(P),\\ E&=|p_{tt}q_{ss}+q_{tt}p_{ss}|-r_t(P)r_s(Q)-r_t(Q)r_s(P),\\ F&=|q_{tt}q_{ss}|+r_t(Q)r_s(Q).\end{aligned}$$

If $\Delta=E^2-4DF<0$, the authors take $\frac{E+\sqrt{E^2-4DF}}{2D}=\infty$. There are some errors in the calculation process of Theorem 4 in [9]. First, from inequality (7) in the proof of Theorem 4 in [9], we have

$$\begin{aligned}\big[|p_{tt}p_{ss}|+r_t(P)r_s(P)\big]|\lambda|^2-\big[|p_{tt}q_{ss}|&\\ +p_{ss}q_{tt}|-r_t(P)r_s(Q)-r_s(P)r_t(Q)\big]|\lambda|&\\ +\big[q_{tt}q_{ss}+r_t(Q)r_s(Q)\big]\geqslant 0,&\end{aligned}$$

Obviously, by the above inequality, E should be defined as follows:

$$E=|p_{tt}q_{ss}|+|q_{tt}p_{ss}|-r_t(P)r_s(Q)-r_t(Q)r_s(P).$$

Second, from inequality (7) in the proof of Theorem 4 in [9], we have

$$D|\lambda|^2-E|\lambda|+F\geqslant 0,$$

if $\Delta\geqslant 0$, then

$$|\lambda|\leqslant\frac{E-\sqrt{E^2-4DF}}{2D},\ \text{or}\ |\lambda|\geqslant\frac{E+\sqrt{E^2-4DF}}{2D},$$

which means, the result of Theorem 4 in [9] is incorrect.

Another upper bound for moduli of eigenvalues $|\lambda(P^{-1}Q)|$ is incorporated in the following theorem.

Theorem 5 *Let $P=(p_{ts})\in\mathbb{C}^{n\times n}$ be a DZ matrix, $Q=(q_{ts})\in\mathbb{C}^{n\times n}$. Then*

$$|\lambda(P^{-1}Q)|\leqslant\omega_5=\min_{t\in[n]}\max_{s\in[n],s\neq t}\frac{H'+\sqrt{H'^2-4GK'}}{2G},$$

where

$$\begin{aligned}G&=|p_{tt}|(|p_{ss}|-r_s^t(P))-r_t(P)|p_{st}|,\\ H'&=|p_{tt}q_{ss}+p_{ss}q_{tt}|+|p_{tt}|r_s^t(Q)\\ &\quad+|q_{tt}|r_s^t(P)+r_t(P)|q_{st}|+r_t(Q)|p_{st}|,\\ K'&=-|q_{tt}|(|q_{ss}|+r_s^t(Q))-r_t(Q)|q_{st}|.\end{aligned}$$

Proof: Since $P=(p_{ts})\in\mathbb{C}^{n\times n}$ is a DZ matrix, by Lemma 1, $p=(p_{ts})$ is nonsingular. Assume $\lambda\in\sigma(P^{-1}Q)$. Then

$$\det(\lambda I-P^{-1}Q)=0,$$

which implies

$$\det(\lambda P-Q)=0.$$

By Lemma 2, for the index $t\in[n]$, there is an index $s\in[n]$, $s\neq t$, if

$$\begin{aligned}|\lambda p_{tt}-q_{tt}|(|\lambda p_{ss}-q_{ss}|-r_s^t(\lambda P-Q))&\\ >r_t(\lambda P-Q)|\lambda p_{st}-q_{st}|,&\end{aligned}\tag{7}$$

then $\lambda\notin\sigma(P^{-1}Q)$. Therefore, if

$$\begin{aligned}\big(|\lambda|^2|p_{tt}p_{ss}|-|p_{tt}q_{ss}+p_{ss}q_{tt}||\lambda|-|q_{tt}q_{ss}|\big)&\\ -(|\lambda||p_{tt}|+|q_{tt}|)(|\lambda|r_s^t(P)+r_s^t(Q))&\\ >(|\lambda|r_t(P)+r_t(Q))(|\lambda||p_{st}|+|q_{st}|),&\end{aligned}\tag{8}$$

then $\lambda\notin\sigma(P^{-1}Q)$. Therefore, if $\lambda\in\sigma(P^{-1}Q)$, we have

$$\begin{aligned}\big(|\lambda|^2|p_{tt}p_{ss}|-|p_{tt}q_{ss}+p_{ss}q_{tt}||\lambda|-|q_{tt}q_{ss}|\big)&\\ -(|\lambda||p_{tt}|+|q_{tt}|)(|\lambda|r_s^t(P)+r_s^t(Q))&\\ \leqslant(|\lambda|r_t(P)+r_t(Q))(|\lambda||p_{st}|+|q_{st}|),&\end{aligned}\tag{9}$$

that is,

$$G|\lambda|^2-H'|\lambda|+K'\leqslant 0,\tag{10}$$

where

$$\begin{aligned}G&=|p_{tt}|(|p_{ss}|-r_s^t(P))-r_t(P)|p_{st}|,\\ H'&=|p_{tt}q_{ss}+p_{ss}q_{tt}|+|p_{tt}|r_s^t(Q)\\ &\quad+|q_{tt}|r_s^t(P)+r_t(P)|q_{st}|+r_t(Q)|p_{st}|,\\ K'&=-|q_{tt}|(|q_{ss}|+r_s^t(Q))-r_t(Q)|q_{st}|.\end{aligned}$$

Since $P=(p_{ts})\in\mathbb{C}^{n\times n}$ is a DZ matrix, $G=|p_{tt}|(|p_{ss}|-r_s^t(P))-r_t(P)|p_{st}|>0$, and $K'\leqslant 0$. Note that for the index $t\in[n]$, there is an index $t\neq s$, $t\in[n]$,

$$|\lambda(P^{-1}Q)|\leqslant \max_{s\in[n],s\neq t}\frac{H'+\sqrt{H'^2-4GK'}}{2G},$$

By the arbitrariness of t, we have

$$|\lambda(P^{-1}Q)|\leqslant \min_{t\in[n]}\max_{s\in[n],s\neq t}\frac{H'+\sqrt{H'^2-4GK'}}{2G}.$$

□

In [9], Li et al presented $\omega_2\leqslant\omega_1$ when $P=\mathrm{diag}(M)$. In the following, we give some comparison theorems. First, we give the relationships between ω_2 and ω_1 without the condition $P=\mathrm{diag}(M)$.

Theorem 6 *Let $P=(p_{ts})\in\mathbb{C}^{n\times n}$ be an SDD matrix, $Q=(q_{ts})\in\mathbb{C}^{n\times n}$. Then*

$$\omega_2\leqslant\omega_1.$$

Proof: If $|\lambda|\leqslant\omega_2$, from the proof of Theorem 3 in [9], there are distinct indices $t,s\in[n]$, such that

$$\begin{aligned}&(|p_{tt}||\lambda|-|q_{tt}|)(|p_{ss}||\lambda|-|q_{ss}|)\\&\qquad\leqslant(|\lambda|r_t(P)+r_t(Q))(|\lambda|r_s(P)+r_s(Q)).\end{aligned}$$

If $(|\lambda|r_t(P)+r_t(Q))(|\lambda|r_s(P)+r_s(Q))=0$, then

$$(|p_{tt}||\lambda|-|q_{tt}|)(|p_{ss}||\lambda|-|q_{ss}|))\leqslant 0,$$

which implies that, one factor in the left side of above inequality is negative. Without loss of generality, we assume $|p_{tt}||\lambda|-|q_{tt}|\leqslant 0\leqslant|\lambda|r_t(P)+r_t(Q)$, then

$$|\lambda|\leqslant\omega_1=\max_{t\in[n]}\frac{|q_{tt}|+r_t(Q)}{|p_{tt}|-r_t(P)}.$$

If $(|\lambda|r_t(P)+r_t(Q))(|\lambda|r_s(P)+r_s(Q))>0$, then

$$\frac{|p_{tt}||\lambda|-|q_{tt}|}{|\lambda|r_t(P)+r_t(Q)}\frac{|p_{ss}||\lambda|-|q_{ss}|}{|\lambda|r_s(P)+r_s(Q)}\leqslant 1.$$

Thus,

$$\frac{|p_{tt}||\lambda|-|q_{tt}|}{|\lambda|r_t(P)+r_t(Q)}\leqslant 1,$$

or

$$\frac{|p_{ss}||\lambda|-|q_{ss}|}{|\lambda|r_s(P)+r_s(Q)}\leqslant 1.$$

Therefore,

$$|\lambda|\leqslant\omega_1=\max_{t\in[n]}\frac{|q_{tt}|+r_t(Q)}{|p_{tt}|-r_t(P)}.$$

□

Second, the relationships among ω_2, ω_3, ω_4 and ω_5 are also examined.

Theorem 7 *Let $P=(p_{ts})\in\mathbb{C}^{n\times n}$ be a DSDD matrix, $Q=(q_{ts})\in\mathbb{C}^{n\times n}$. Assume that $|p_{tt}||\lambda|+|q_{tt}|\leqslant|\lambda|r_t(P)+r_t(Q)$ for all $t\in[n]$, $\lambda\in\sigma(P^{-1}Q)$. Then*

$$\omega_5\leqslant\omega_3,\quad \omega_4\leqslant\omega_2.$$

Proof: If $|\lambda|\leqslant\omega_4$, and there is an index $t\in[n]$ with $|p_{tt}||\lambda|+|q_{tt}|\leqslant|\lambda|r_t(P)+r_t(Q)$. From the proof of Theorem 4, for the index $t\in[n]$, there is an index $s\in[n]$, $s\neq t$, we have

$$\begin{aligned}&(|\lambda||p_{tt}|-|q_{tt}|)(|\lambda||p_{ss}|-|q_{ss}|)\\&\quad-(|\lambda||p_{tt}|+|q_{tt}|)(|\lambda|r_s^t(P)+r_s^t(Q))\\&\qquad\leqslant(|\lambda|r_t(P)+r_t(Q))(|\lambda||p_{st}|+|q_{st}|).\end{aligned}$$

Then

$$\begin{aligned}&(|\lambda||p_{tt}|-|q_{tt}|)(|\lambda||p_{ss}|-|q_{ss}|)\\&\quad\leqslant(|\lambda|r_t(P)+r_t(Q))(|\lambda||p_{st}|+|q_{st}|)\\&\qquad+(|\lambda||p_{tt}|+|q_{tt}|)(|\lambda|r_s^t(P)+r_s^t(Q))\\&\quad\leqslant(|\lambda|r_t(P)+r_t(Q))(|\lambda||p_{st}|+|q_{st}|)\\&\qquad+(|\lambda|r_t(P)+r_t(Q))(|\lambda|r_s^t(P)+r_s^t(Q))\\&\quad\leqslant(|\lambda|r_t(P)+r_t(Q))(|\lambda|r_s(P)+r_s(Q)).\end{aligned}$$

Therefore, $|\lambda|\leqslant\omega_2$.

If $|\lambda|\leqslant\omega_5$, and $|p_{tt}||\lambda|+|q_{tt}|\leqslant|\lambda|r_t(P)+r_t(Q)$. From the proof of Theorem 5, for the index $t\in[n]$, there is an index $s\in[n]$, $s\neq t$, such that

$$\begin{aligned}&(|\lambda|^2|p_{tt}p_{ss}|-|p_{tt}q_{ss}+p_{ss}q_{tt}||\lambda|-|q_{tt}q_{ss}|)\\&\quad-(|\lambda||p_{tt}|+|q_{tt}|)(|\lambda|r_s^t(P)+r_s^t(Q))\\&\qquad\leqslant(|\lambda|r_t(P)+r_t(Q))(|\lambda||p_{st}|+|q_{st}|).\end{aligned}$$

Then

$$\begin{aligned}&|\lambda|^2|p_{tt}p_{ss}|-|p_{tt}q_{ss}+p_{ss}q_{tt}||\lambda|-|q_{tt}q_{ss}|\\&\quad\leqslant(|\lambda|r_t(P)+r_t(Q))(|\lambda||p_{st}|+|q_{st}|)\\&\qquad+(|\lambda||p_{tt}|+|q_{tt}|)(|\lambda|r_s^t(P)+r_s^t(Q))\\&\quad\leqslant(|\lambda|r_t(P)+r_t(Q))(|\lambda||p_{st}|+|q_{st}|)\\&\qquad+(|\lambda|r_t(P)+r_t(Q))(|\lambda|r_s^t(P)+r_s^t(Q))\\&\quad\leqslant(|\lambda|r_t(P)+r_t(Q))(|\lambda|r_s(P)+r_s(Q)).\end{aligned}$$

Therefore, $|\lambda|\leqslant\omega_3$. □

Remark 2 If P is an SDD matrix, by the condition "$|p_{tt}||\lambda|+|q_{tt}|\leqslant|\lambda|r_t(P)+r_t(Q)$ for all $t\in[n]$", we have

$$|\lambda|\leqslant\frac{r_t(Q)-|q_{tt}|}{|p_{tt}|-r_t(P)}.$$

If P is a DSDD matrix and not an SDD matrix, if $|p_{tt}|>r_t(P)$ for some $t\in[n]$, by the condition

"$|p_{tt}||\lambda|+|q_{tt}| \leqslant |\lambda| r_t(P)+r_t(Q)$ for all $t \in [n]$", we have

$$|\lambda| \leqslant \frac{r_t(Q)-|q_{tt}|}{|p_{tt}|-r_t(P)}.$$

If $|p_{tt}| \leqslant r_t(P)$, then we can take $|p_{tt}| \leqslant r_t(P)$, $|q_{tt}| \leqslant r_t(Q)$.

Remark 3 The relationship between ω_4 and ω_5 is not obvious. But we can find that, if

$$\operatorname{sign}(p_{tt}q_{ss})\operatorname{sign}(p_{ss}q_{tt})<0$$

for distinct $t,s \in [n]$, then $\omega_4 \leqslant \omega_5$, where $\operatorname{sign}(x)$ is the sign function.

NUMERICAL EXAMPLES

In this section, some numerical examples are given to illustrate the efficiency of our proposed upper bounds.

Example 1 Let

$$P=\begin{bmatrix}7&2&4\\1&7&1\\1&1&9\end{bmatrix},\quad Q=\begin{bmatrix}1&0&3\\1&0&2\\1&2&0\end{bmatrix},$$

Obviously, P is an SDD matrix. By a direct computation, we have

$$\rho(P^{-1}Q)=0.2985 \leqslant \min_{t\in[n]}\frac{r_t(Q)-|q_{tt}|}{|p_{tt}|-r_t(P)}=0.4286,$$

which means that, the matrix splitting in Example 1 satisfies the condition in Theorem 7.

Example 2 Let

$$P=\begin{bmatrix}-7&2&1\\1&7&1\\1&1&9\end{bmatrix},\quad Q=\begin{bmatrix}1&0&3\\0&2&1\\1&4&1\end{bmatrix},$$

Obviously, P is an SDD matrix. By a direct computation, we have

$$\rho(P^{-1}Q)=0.4175 > \min_{t\in[n]}\frac{r_t(Q)-|q_{tt}|}{|p_{tt}|-r_t(P)}=-0.2,$$

which means that, the matrix splitting in Example 2 does not satisfy the condition in Theorem 7.

Example 3 Let

$$P=\begin{bmatrix}-7&2&1\\1&1.5&1\\1&2&2.5\end{bmatrix},\quad Q=\begin{bmatrix}1&0&0.1\\0&1&1\\0&0&1\end{bmatrix},$$

Obviously, P is not a DSDD matrix. Then, Theorems 1–3 can not be applied in Example 3, but P is a DZ matrix. By a direct computation, we have

$$\rho(P^{-1}Q)=0.6993.$$

Example 4 Let

$$p=\begin{bmatrix}-7&2&6\\0&8&1\\1&1&9\end{bmatrix},\quad Q=\begin{bmatrix}1&0&3\\0&2&1\\1&0&1\end{bmatrix},$$

Obviously, P is not an SDD matrix. Then Theorem 1 can not be applied in Example 4, but P is a DSDD matrix. If $t=1$, $|p_{11}| \leqslant r_1(P)$, $|q_{11}| \leqslant r_1(Q)$. If $t=2,3$, by a direct computation, we have

$$\rho(P^{-1}Q)=0.2437 > \min_{t\in[n]}\frac{r_t(Q)-|q_{tt}|}{|p_{tt}|-r_t(P)}=-0.1429,$$

which means that, the matrix splitting in Example 4 does not satisfy the condition in Theorem 7.

Example 5 Let

$$p=\begin{bmatrix}-7&2&6\\0&8&1\\1&1&9\end{bmatrix},\quad Q=\begin{bmatrix}1&0&3\\1&0&2\\1&2&0\end{bmatrix},$$

Obviously, P is not an SDD matrix. Then Theorem 1 can not be applied in Example 5, but P is a DSDD matrix. If $t=1$, $|p_{11}| \leqslant r_1(P)$, $|q_{11}| \leqslant r_1(Q)$. If $t=2,3$, by a direct computation, we have

$$\rho(P^{-1}Q)=0.3011 \leqslant \min_{t\in[2,3]}\frac{r_t(Q)-|q_{tt}|}{|p_{tt}|-r_t(P)}=0.4286,$$

which means that, the matrix splitting in Example 5 satisfies the condition in Theorem 7.

Table 1 summarizes for Examples 1–5 the performance of the upper bounds proposed in this paper comparing with these of [5, 7, 9]. From Table 1, we can find that, if the matrix splitting satisfies the condition in Theorem 7, then, the result in Theorem 5 is always better than the result in Theorem 2, and the result in Theorem 4 is always better than the result in Theorem 3, which are illustrated in Examples 1 and 5. If P is a DZ matrix, the upper bound can only be obtained by Theorems 4 and 5.

CONCLUSION

It is well known that {SDD} ⊆ {DSDD} ⊆ {DZ}. In this paper, two upper bounds for $|\lambda(P^{-1}Q)|$ (or $\rho(P^{-1}Q)$) are investigated, which is applicable to a DZ matrix P. We also compare our results with some existing results under certain conditions.

Table 1 Numerical comparison between our upper bounds and those of [5, 7, 9].

	Example 1	Example 2	Example 3	Example 4	Example 5
$\rho(P^{-1}Q)$	0.2985	0.4087	0.6993	0.2437	0.3011
Theorem 1	4 (>1)	1	–	–	–
Theorem 2	1.0660 (>1)	0.9170	–	0.7166	1.0179 (>1)
Theorem 3	1.0660 (>1)	0.7794	–	0.5452	1.0179 (>1)
Theorem 4	0.9784 (<1)	1	1.1506 (>1)	0.6477	0.8515 (<1)
Theorem 5	0.9784 (<1)	0.8089	0.8410 (<1)	0.4693	0.8515 (<1)

Acknowledgements: This work is supported by NSF of China (71461027, 11661084), Innovative talent team in Guizhou Province (Qian Ke He Pingtai Rencai[2016]5619), New academic talents and innovative exploration fostering project(Qian Ke He Pingtai Rencai[2017]5727-21), Guizhou Province Natural Science Foundation in China (Qian Jiao He KY[2020]094), Science and Technology Foundation of Guizhou Province (Qian Ke He Ji Chu ZK[2021]Yi Ban 014).

REFERENCES

1. Horn R, Johnson C (1985) *Matrix Analysis*, Cambridge University Press, Cambridge, UK.
2. Berman A, Plemmons R (1994) *Nonnegative Matrices in Mathematical Sciences*, SIAM, Philadelphia, PA.
3. Varga R (2000) *Matrix Iterative Analysis*, Springer, Berlin, Heidelberg.
4. Cvetković L (2006) H-matrix theory vs. eigenvalue localization. *Numer Algorithms* **42**, 229–245.
5. Wang X (1994) The upper bound of the spectral radius of $M^{-1}N$ and convergence of some iterative methods. *Int J Comput Math* **53**, 203–217.
6. Wang X (1997) Convergence theory for the general GAOR type iterative method and the MSOR iterative method applied to H-matrices. *Linear Algebra Appl* **250**, 1–19.
7. Hu J (1986) The upper and lower bounds of the eigenvalues M^1N. *Math Numer Sin* **8**, 41–46. [in Chinese]
8. Huang T, Gao Z (2003) A new upper bound for moduli of eigenvalues of iterative matrices. *Int J Comput Math* **80**, 799–803.
9. Li H, Huang T, Li H (2006) An improvement on a new upper bound for moduli of eigenvalues of iterative matrices. *Appl Math Comput* **173**, 977–984.