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Meromorphic solutions of certain types of complex functional equations

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ABSTRACT: In this paper, we investigate the properties of meromorphic solutions on complex functional equations of Malmquist type of the form

$$\sum_{\{J\}} \alpha_J(z)\left(\prod_{j\in J} w\left(q^j z\right)\right) = \frac{a_0(z)+a_1(z)(w\circ p)+\cdots+a_s(z)(w\circ p)^s}{b_0(z)+b_1(z)(w\circ p)+\cdots+b_t(z)(w\circ p)^t},$$

where $\{J\}$ is a collection of all non-empty subsets of $\{1,2,\ldots,n\}$, $q\in\mathbb{C}$, $|q|>1$, and all coefficients are small functions relative to $w(z)$ such that $a_s(z)b_t(z)\not\equiv 0$, $p(z)=d_kz^k+\cdots+d_1z+d_0$ is a polynomial with constant coefficients $d_k(\neq 0),\ldots,d_1,d_0$ and of degree k. Furthermore, we prove that the meromorphic solutions having Borel exceptional zeros and poles appear in special situations. Some other q-difference versions of complex difference equations of Malmquist type are also presented.

KEYWORDS: difference equation, functional equation, meromorphic solution, Nevanlinna theory, growth

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INTRODUCTION

Recently, Ablowitz et al [1] applied Nevanlinna theory to investigate the properties on complex difference equations reminiscent of the classical Malmquist theorem in complex differential equations. A typical example of their results tells us that if a complex difference equation

$$w(z+1)+w(z-1)=R(z,w) \tag{1}$$

with $R(z,w)$ rational in both arguments admits a transcendental meromorphic solution of finite order, then $\deg_w R(z,w)\leqslant 2$. Heittokangas et al [2] improved and extended the above results, see Propositions 8 and 9, and showed that solutions having Borel exceptional zeros and poles seem to appear in special situations only. Zhang and Huang [3] focused on Theorem 13 in [2] to present exact form of difference equations by proving some results on deficiencies of the meromorphic solutions. Laine et al [4] generalized the key lemma that $w(z)$ has to be infinite order, provided that $\deg_w R(z,w)\leqslant 2$ and that a certain growth condition for the counting function of distinct poles of $w(z)$ holds (see [5]) to higher order difference equations of more general type (see [4]), and presented related complex functional equations. The properties on the meromorphic solutions of complex functional difference equations composited with polynomials are also investigated in [6].

Bergweiler et al [7] considered nonlinear q-difference equation

$$\sum_{j=0}^{n} a_j(z)w\left(q^j z\right)=Q(z), \tag{2}$$

where $0<|q|<1$, $a_j(z)(j=0,1,\ldots,n)$ and $Q(z)$ are rational functions with $a_0(z)\not\equiv 0$, $a_n(z)\equiv 1$. They gave sufficient conditions for the existence of meromorphic solutions of (2), and also pointed out that all meromorphic solutions of (2) satisfy $T(r,w)=O((\log r)^2)$. This implies that all meromorphic solutions of (2) are of zero order of growth.

If $|q|>1$, Gundersen et al [8] showed that the order of growth of generalized Schröder q-

difference equation

$$w(qz) = R(z, w(z)) \tag{3}$$

is equal to $\log \deg_w(R)/\log|q|$, while Zheng and Chen [9] showed that the lower order $\mu(w)$ of solutions of (4) below is not less than $\log d_0/(n \log|q|)$ if $|q| > 1$. Now, we recall their results.

Theorem 1 ([8]) *Suppose that $w(z)$ is a transcendental meromorphic solution of an equation of the form*

$$w(qz) = R(z, w(z)) = \frac{a_0(z) + a_1(z)w(z) + \cdots + a_s(z)w(z)^s}{b_0(z) + b_1(z)w(z) + \cdots + b_t(z)w(z)^t},$$

where $q \in \mathbb{C}$, $|q| > 1$, $R(z, w(z))$ is irreducible in $w(z)$ with meromorphic coefficients $a_u(z)(u = 0, 1, \dots, s)$ and $b_v(z)(v = 0, 1, \dots, t)$ such that $a_s(z)b_t(z) \not\equiv 0$. Then

$$\sigma(w) = \frac{\log \deg_w(R)}{\log|q|}.$$

Theorem 2 ([9]) *Suppose that $w(z)$ is a transcendental meromorphic solution of equation*

$$\sum_{j=1}^{n} a_j(z) w\left(q^j z\right) = R(z, w(z)) = \frac{P(z, w(z))}{Q(z, w(z))}, \tag{4}$$

where $q \in \mathbb{C}$, $|q| > 1$, the coefficients $a_j(z)$ are rational functions and $P(z, w(z))$, $Q(z, w(z))$ are relatively prime polynomials in $w(z)$ over the field of rational functions satisfying $s = \det_w(P), t = \deg_w(Q)$ and $d_0 = s - t \geqslant 2$. If $w(z)$ has infinitely many poles, then for sufficiently large r,

$$n(r, w) \geqslant K {d_0}^{\log r / n \log|q|}$$

holds for some constant K.

A meromorphic function means meromorphic in the whole complex plane $\mathbb{C}$. For a meromorphic function $w(z)$, let $\sigma(w)$ be the order of growth and $\mu(w)$ be the lower order of $w(z)$. Further, let $\lambda(w)$ (respectively, $\lambda(1/w)$) be the exponent of convergence of the zeros (respectively, poles) of $w(z)$. We also assume that the reader is familiar with the standard symbols and fundamental results such as $m(r, w)$, $N(r, w)$, $\overline{N}(r, w)$ and $T(r, w)$, etc., of Nevanlinna theory, see e.g. [10, 11]. We now recall that a meromorphic function $a(z)$ is said to be a small function relative to $w(z)$ if $T(r, a) = S(r, w)$, where $S(r, w)$ is used to denote any quantity satisfying $S(r, w) = o(\{T(r, y)\}$ as $r \to \infty$, possibly outside of a set of finite logarithmic measure, furthermore, possibly outside of a set of logarithmic density 0, i.e. outside of a set E such that $\lim_{r\to\infty} \int_{[1,r]\cap E} \frac{dt}{t} / \log r = 0$. Moreover, suppose that $R(z, w(z))$ is rational in $w(z)$ with small functions relative to $w(z)$ as its coefficients. We use the notation $d = \deg_w R(z, w(z))$ for the degree of $R(z, w(z))$ with respect to $w(z)$. In the follows, we always assume that $R(z, w(z))$ is irreducible in $w(z)$.

The present paper mainly deal with functional equations of the general form

$$\sum_{\{J\}} \alpha_J(z) \left(\prod_{j\in J} w\left(q^j z\right) \right) = R(z, w \circ p) = \frac{P(z, w \circ p)}{Q(z, w \circ p)} \tag{5}$$

where $\{J\}$ is a collection of all non-empty subsets of $\{1, 2, \dots, n\}$, $q \in \mathbb{C}$, $|q| > 1$, $P(z, w)$ and $Q(z, w)$ are relatively prime polynomials in $w(z)$, and all coefficients in (5) are small functions relative to $w(z)$, $p(z) = d_k z^k + \cdots + d_1 z + d_0$ is a polynomial with constant coefficients $d_k (\neq 0), \dots, d_1, d_0$ and of degree k. We permit different expressions on both sides of equation (5).

MEROMORPHIC SOLUTIONS WITH CERTAIN GROWTH CONDITION FOR COUNTING FUNCTION OF DISTINCT POLES

Halburd and Korhonen [5] showed that the existence of sufficiently many meromorphic solutions of finite order is enough to single out a discrete form of the second Painlevé equation from a more general class (1). A key lemma in their reasoning is to show that $w(z)$ has to be of infinite order, provided that $\deg_w R(z, w) \leqslant 2$ and that a certain growth condition for the counting function of distinct poles $w(z)$ holds. Laine et al [4] extended it into a more general type. Zheng and Chen [9] proved a q-difference counterpart of the above results. In this section, we proceed to extend Theorem 4 in [9] into a more general type again.

Theorem 3 *Suppose that $w(z)$ is a transcendental meromorphic solution of (5), where $\{J\}$ is a collection of all non-empty subsets of $\{1, 2, \dots, n\}$, $q \in \mathbb{C}$, $|q| > 1$, $P(z, w(z))$ and $Q(z, w(z))$ are relatively prime polynomials in $w(z)$, and all coefficients in (5) are small functions relative to $w(z)$. Moreover, we assume that $t = \deg_w(Q) > 0$, $b_t(z) \equiv 1$, and*

$$n = \max\{s, t\} := \max\{\deg_f(P), \deg_f(Q)\}.$$

If there exists $\alpha \in [0, n)$ such that, for all sufficiently *large r,*

$$\sum_{j=1}^{n}\overline{N}\left(r,w\left(q^{j}z\right)\right)\leqslant\alpha\overline{N}(r,w(z)),\qquad(6)$$

then the order of growth $\sigma(w)>0$ *and*

$$Q(z,w(z))=(w(z)-s(z))^{t},$$

where $s(z)$ *is a small function relative to* $w(z)$*.*

At this point, we first need to recall the following lemmas. Weisseborn obtained the following result.

Lemma 1 ([12]) *Let* $f(z)$ *be a meromorphic function and* ϕ *be given by*

$$\phi=w^{n}+a_{n-1}w^{n-1}+\cdots+a_{0},$$
$$T(r,a_{j})=S(r,w),\qquad j=0,1,\ldots,n-1.$$

Then either

$$\phi\equiv\left(w+\frac{a_{n-1}}{n}\right)^{n},$$

or

$$T(r,w)\leqslant\overline{N}\left(r,\frac{1}{\phi}\right)+\overline{N}(r,w)+S(r,w).$$

Lemma 2 ([4]) *Let* $f(z)$ *be a nonconstant meromorphic function and let* $P(z,w)$*,* $Q(z,w)$ *be two polynomials in* $w(z)$ *with meromorphic coefficients small relative to* $w(z)$*. If* $P(z,w)$ *and* $Q(z,w)$ *have no common factors of positive degree in* $w(z)$ *over the field of small functions relative to* $w(z)$*, then*

$$\overline{N}\left(r,\frac{1}{Q(z,w)}\right)\leqslant\overline{N}\left(r,\frac{P(z,w)}{Q(z,w)}\right)+S(r,w).$$

Lemma 3 ([13]) *Given distinct meromorphic functions* $w_1,\ldots,w_n$*, let* $\{J\}$ *denote the collection of all non-empty subsets of* $\{1,2,\ldots,n\}$*, and suppose that* $\alpha_J\in\mathbb{C}$ *for each* $J\in\{J\}$*. Then*

$$T\left(r,\sum_{\{J\}}\alpha_J\Big(\prod_{j\in J}w_j\Big)\right)\leqslant\sum_{k=1}^{n}T(r,w_k)+O(1)$$

Lemma 4 ([14]) *If* $T:\mathbb{R}^+\to\mathbb{R}^+$ *is a piecewise continuous increasing function such that*

$$\lim_{r\to\infty}\frac{\log T(r)}{\log r}=0,$$

then the set

$$E:=\{r: T(C_1 r)\geqslant C_2T(r)\}$$

has logarithmic density 0 for all $C_1>1$ *and* $C_2>1$*.*

Remark 1 By using similar method of Thereom 1.1 and Theorem 1.3 in [15] and q-difference version of lemma on logarithmic derivatives [16, 17], we deduce from Lemma 4 that, for $|q|>1$,

$$T(r,w(qz))=T(r,w(z))+S(r,w)\quad\text{and}$$
$$\overline{N}(r,w(qz))=\overline{N}(r,w(z))+S(r,w)$$

on a set of logarithmic density 1.

Proof of Theorem 3: Assume that the second alternative of the assertion is incorrect. Then, we deduce from Lemmas 1–3, (5) and (6) that

$$\begin{aligned}T(r,w)&\leqslant\overline{N}\left(r,\frac{1}{Q(z,w)}\right)+\overline{N}(r,w(z))+S(r,w)\\&\leqslant\overline{N}\left(r,\frac{P(z,w)}{Q(z,w)}\right)+\overline{N}(r,w(z))+S(r,w)\\&=\overline{N}\left(r,\sum_{\{J\}}\alpha_J(z)\Big(\prod_{j\in J}w\big(q^{j}z\big)\Big)\right)+\overline{N}(r,w(z))+S(r,w)\\&\leqslant\sum_{j=1}^{n}\overline{N}\left(r,w\left(q^{j}z\right)\right)+\overline{N}(r,w(z))+S(r,w)\\&\leqslant\alpha\overline{N}(r,w(z))+\overline{N}(r,w(z))+S(r,w).\end{aligned}$$

Therefore,

$$T(r,w)-\overline{N}(r,w(z))\leqslant\alpha\overline{N}(r,w(z))+S(r,w).$$

Now, assume in contrary to the assertion that $\sigma(w)=0$, we get from Remark 1 that for all $j=1,2,\ldots,n$, $S\left(r,w\left(q^{j}\right)\right)=S(r,w)$ and

$$\begin{aligned}T(r,w(q^{j}z))-\overline{N}(r,w(q^{j}z))&\leqslant\alpha\overline{N}\left(r,w(q^{j}z)\right)+S(r,w)\\&=\alpha\overline{N}(r,w(z))+S(r,w)\qquad(7)\end{aligned}$$

on a set of logarithmic density 1.

We also conclude from Remark 1, Lemma 3, (6) and (7) that

$$\begin{aligned}nT(r,w)&=T\left(r,\sum_{\{J\}}\alpha_J(z)\Big(\prod_{j\in J}w\left(q^{j}z\right)\Big)\right)+S(r,w)\\&\leqslant\sum_{j=1}^{n}T\left(r,w\left(q^{j}z\right)\right)+S(r,w)\\&=\sum_{j=1}^{n}\left[T\left(r,w\left(q^{j}z\right)\right)-\overline{N}\left(r,w\left(q^{j}z\right)\right)\right]\\&\quad+\sum_{j=1}^{n}\overline{N}\left(r,w\left(q^{j}z\right)\right)+S(r,w)\\&\leqslant\sum_{j=1}^{n}\alpha\overline{N}(r,w(z))+\alpha\overline{N}(r,w(z))+S(r,w)\\&=(n+1)\alpha\overline{N}(r,w(z))+S(r,w)\qquad(8)\end{aligned}$$

on a set of logarithmic density 1. Thus,

$$T(r,w)-\overline{N}(r,w(z)) \leqslant \frac{n+1}{n}\alpha\overline{N}(r,w(z))-\overline{N}(r,w(z))+S(r,w) \quad (9)$$

on a set of logarithmic density 1.

Moreover, we obtain from (6), (8), (9) and Remark 1 that

$$\begin{aligned} nT(r,w) &\leqslant \sum_{j=1}^{n}\Big[T\left(r,w\left(q^j z\right)\right)-\overline{N}\left(r,w\left(q^j z\right)\right)\Big] \\ &\quad +\sum_{j=1}^{n}\overline{N}\left(r,w\left(q^j z\right)\right)+S(r,w) \\ &\leqslant \sum_{j=1}^{n}\Big[\frac{n+1}{n}\alpha\overline{N}\left(r,w\left(q^j z\right)\right)-\overline{N}\left(r,w\left(q^j z\right)\right)\Big] \\ &\quad +\alpha\overline{N}(r,w(z))+S(r,w) \\ &= (n+2)\alpha\overline{N}(r,w(z))-n\overline{N}(r,w(z))+S(r,w) \end{aligned}$$

on a set of logarithmic density 1. Thus,

$$T(r,w)-\overline{N}(r,w(z)) \leqslant \frac{n+2}{n}\alpha\overline{N}(r,w(z))-2\overline{N}(r,w(z))+S(r,w)$$

on a set of logarithmic density 1. By repeating this process for m times, we deduce that

$$T(r,w)-\overline{N}(r,w(z)) \leqslant \frac{n+m}{n}\alpha\overline{N}(r,w(z))-m\overline{N}(r,w(z))+S(r,w) \quad (10)$$

on a set of logarithmic density 1. Since $\alpha\in[0,n)$, we immediately see from (10) that, for sufficiently large m,

$$\overline{N}(r,w(z)) \leqslant \frac{n+m}{n(m-1)}\alpha\overline{N}(r,w(z)) < \overline{N}(r,w(z)),$$

on a set of logarithmic density 1, a contradiction.

On the other hand, if the second alternative of the assertion is valid, then we must have $\sigma(w)>0$. Otherwise, by Remark 1 and the q-version of Mohon'ko lemma, we again obtain a contradiction. □

MEROMORPHIC SOLUTIONS WITH FINITELY MANY POLES

Theorem 3 shows that either the order of growth $\sigma(w)>0$, and

$$Q(z,w(z)) = (w(z)-s(z))^t,$$

provided that $\deg_w Q(z,w)>0$ and that a certain growth condition for the counting function of distinct poles $w(z)$ holds. However, if $w(z)$ just has finitely many zeros, we further obtain the following theorem.

Theorem 4 *Suppose that $w(z)$ is a transcendental meromorphic solution of (5), where $\{J\}$ is a collection of all non-empty subsets of $\{1,2,\dots,n\}$, $q\in\mathbb{C}$, $|q|>1$, $P(z,w(z))$ and $Q(z,w(z))$ are relatively prime polynomials in $w(z)$, and all coefficients in (5) are rational functions. Moreover, we assume that $t=\deg_w(Q)>0$ and $b_t(z)\equiv 1$. If $w(z)$ has finitely many poles only, then*

(1) $Q(z,w(z))=(w(z)-s(z))^t$, where $s(z)$ is a rational function;

(2) $w(z)$ must be of the form

$$w(z)=r(z)\,\mathrm{e}^{g(z)}+s(z), \quad (11)$$

where $s(z)$ is a rational function, $r(z)$ is a small function relative to $w(z)$, $g(z)$ is a transcendental entire function satisfying a q-difference equation of the form

$$k_0 g(z)+k_1 g(qz)+\cdots+k_n g(q^n z)=\tau,$$

where $\tau\in\mathbb{C}$, and k_j, $j\in\{0,1,\dots,n\}$ are integers and not identically zeros.

We firstly recall the following lemmas.

Lemma 5 ([18]) *Suppose that $w_1(z),w_2(z),\dots,w_n(z)$ are meromorphic functions and that $g_1(z),g_2(z),\dots,g_n(z)$ are entire functions satisfying the following conditions.*

(i) $\sum_{j=1}^{n} w_j(z)\,\mathrm{e}^{g_j(z)}\equiv 0$;

(ii) $g_j(z)-g_k(z)$ are not constants for $1\leqslant j<k\leqslant n$;

(iii) for $1\leqslant j\leqslant n$, $1\leqslant h<k\leqslant n$,

$$T(r,w_j)=o\{T(r,\mathrm{e}^{g_h-g_k})\} \quad (r\to\infty, r\notin E),$$

where $E\subset(1,\infty)$ is of finite linear measure or finite logarithmic measure.

Then $w_j(z)\equiv 0$ $(j=1,2,\cdots,n)$.

Proof of Theorem 4: (1) Since $P(z,w(z))$ and $Q(z,w(z))$ are relatively prime polynomials in $w(z)$ with coefficients are rational functions, it follows from Lemma 2 that $P(z,w(z))$ and $Q(z,w(z))$ have finitely many common zeros only. Thus, from (5), Lemma 3 and the assumption that $w(z)$ has finitely many poles, we conclude that

$$N\left(r,\frac{1}{Q(z,w(z))}\right)\leqslant N\left(r,\frac{P(z,w(z))}{Q(z,w(z))}\right)+O(\log r)$$
$$=N\Big(r,\sum_{\{J\}}\alpha_J(z)\Big(\prod_{j\in J}w\left(q^jz\right)\Big)\Big)+O(\log r)$$
$$\leqslant\sum_{j=1}^{n}N(r,w(q^jz))+O(\log r)=O(\log r).\quad(12)$$

Thus, we deduce from Lemma 1 that

$$Q(z,w(z))=(w(z)-s(z))^t,$$

where $s(z)$ is a rational function.

(2) Since $w(z)$ is transcendental with finitely many poles only, so is $Q(z,w(z))=(w(z)-s(z))^t$. We also note that $Q(z,w(z))=(w(z)-s(z))^t$ has finitely many zeros only from (12). Thus, there exists a rational function $h(z)$ and a nonconstant entire function $k(z)$ such that

$$w(z)-s(z)=\beta h(z)^{1/t}\,\mathrm{e}^{k(z)/t},$$

where β is the t-th root of unity. Denoting $g(z)=k(z)/t$, and noting that $r(z):=\beta h(z)^{1/t}$ is small function relative to $w(z)$, we get the desired form (11).

Now, substituting (11) into (5), and noting that $Q(z,f(z))=(f(z)-s(z))^t$, we conclude an equation of form

$$h(z)\alpha_M(z)\Big(\prod_{j\in M}r(q^jz)\Big)\exp\Big(tg(z)+\sum_{j\in M}g(q^jz)\Big)$$
$$+\sum_{J\in\{K\}}H_J(z)\exp\Big(tg(z)+\sum_{j\in J}g(q^jz)\Big)$$
$$=\sum_{j=0}^{s}p_j^*(z)\exp\left(jg(z)\right),\quad(13)$$

where the cardinality of the set M is maximal among the sets in the collection $\{J\}$ such that $\alpha_M(z)\neq 0$, $\{K\}$ is a collection of non-empty subsets of $\{1,2,\dots,n\}$ such that $M\notin\{K\}$, $H_J(z)$ is rational function for every J, $p_j^*(z)(j=0,1,\dots,s)$ are rational functions with $p_s^*(z)\not\equiv 0$. Therefore, we deduce from Lemma 5 that there must exist at least two exponents in (13) that cancel each other to a constant $\tau\in\mathbb{C}$ such that

$$\sum_{j=1}^{n}g\left(q^jz\right)=\sum_{j\in K}g\left(q^jz\right)+\tau,\quad\text{or}$$
$$\sum_{j=1}^{n}g\left(q^jz\right)=(j_0-t)g(z)+\tau.$$

These mean that there are at most $n+1$ integers $k_0,k_1,\dots,k_n$, which are not identically zeros such that

$$k_0g(z)+k_1g(qz)+\cdots+k_ng(q^nz)=\tau.$$

In the follows, we prove that $g(z)$ is a transcendental entire function. Assume that $g(z)=c_kz^k+c_{k-1}z^{k-1}+\cdots+c_1z+c_0$ is a nonconstant polynomial with degree k. Then for every $j\in\{1,2,\dots,n\}$, we may write

$$g\left(q^jz\right)=a_kq^{jk}g(z)+g_j(z),\quad(14)$$

where $g_j(z)(j=1,2,\dots,n)$ are polynomials in z with degree no greater than $k-1$. Substituting (11) and (14) into (5) again, we conclude that

$$h(z)\,\mathrm{e}^{tg(z)}\sum_{\{J\}}\alpha_J(z)\prod_{j\in J}\Big(r(q^jz)\,\mathrm{e}^{g_j(z)}\,\mathrm{e}^{a_kq^{jk}g(z)}+s(q^jz)\Big)$$
$$=\sum_{j=0}^{p}a_j(z)\left(r(z)\,\mathrm{e}^{g(z)}+s(z)\right)^j.$$

Since polynomials $P(z,w(z))$ and $Q(z,w(z))$ are relatively prime, there is no common factor of positive degree in $w(z)$ for $P(z,w(z))$ and $Q(z,w(z))$. But, we deduce from Lemma 5 that $\sum_{j=0}^{p}a_j(z)s(z)^j\equiv 0$, a contradiction. □

MEROMORPHIC SOLUTIONS WITH FEW POLES AND ZEROS

Gundersen et al [8] proved the reduction theorems for functional equation of the form

$$w(qz)=R(z,w(z))$$
$$=\frac{a_0(z)+a_1(z)w(z)+\cdots+a_s(z)w(z)^s}{b_0(z)+b_1(z)w(z)+\cdots+b_t(z)w(z)^t},$$

which admits meromorphic solutions with relatively few distinct zeros and poles only, see Theorem 5.2 in [8]. As an application of Tumura-Clunie theorem, Rieppo extended the above result and proved the reduction theorems for certain functional equation that admit meromorphic solutions with relatively few distinct poles only [19]. The reasoning relies on the combination of Nevanlinna theory and algebraic field theory.

We now proceed to consider the reduction theorems for functional equations of form

$$\prod_{i=0}^{n}w\left(q^iz\right)^{\lambda_i}=R(z,w\circ p)$$
$$=\frac{a_0(z)+a_1(z)(w\circ p)+\cdots+a_s(z)(w\circ p)^s}{b_0(z)+b_1(z)(w\circ p)+\cdots+b_t(z)(w\circ p)^t},\quad(15)$$

where $p(z) = d_k z^k + \cdots + d_1 z + d_0$ is a polynomial with constant coefficients $d_k(\neq 0), \ldots, d_1, d_0$ and of degree k, I is a finite set of multi-indexes $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$, and all coefficients in (15) are small meromorphic functions relative to $w(z)$ such that $a_s(z) b_t(z) \not\equiv 0$. The following results tell us that solutions having Borel exceptional zeros and poles appear in special situations only.

Theorem 5 *Let $q \in \mathbb{C}$, $|q| > 1$, and $w(z)$ be a transcendental meromorphic solution of (15). If*

$$\max\left\{\lambda(w), \lambda\left(\frac{1}{w}\right)\right\} < \sigma(w), \tag{16}$$

then (15) is either of the form

$$\prod_{i=0}^{n} w\left(q^i z\right)^{\lambda_i} = \alpha \frac{a_s(z)}{b_0(z)} (w \circ p)^{n+1} \quad \text{or}$$

$$\prod_{i=0}^{n} w\left(q^i z\right)^{\lambda_i} = \alpha \frac{a_0(z)}{b_t(z)} \frac{1}{(w \circ p)^{n+1}}, \tag{17}$$

where α is some nonzero constant.

We now give Example 1 and Example 2 to show Theorem 5 is sharp. Example 3 shows that condition (16) is necessary and cannot be replaced by

$$\min\left\{\lambda(w), \lambda\left(\frac{1}{w}\right)\right\} < \sigma(w).$$

Example 1 Let $w(z) = e^z$. Then $\lambda(w) = \lambda(1/w) = 0 < 1 = \sigma(w)$ and $w(z)$ solves equation

$$w(z)^4 w(2z)^2 = w(4z)^2 \text{ or } w(z)^4 w(-2z)^4 = \frac{1}{w(2z)^2},$$

which is the form of (17).

Example 2 Let $w(z) = e^z$. Then $\lambda(w) = \lambda(1/w) = 0 < 1 = \sigma(w)$ and $w(z)$ solves equation

$$w(z) w(-2z) w(4z) = w(z)^3 \text{ or } w(z) w(-3z) = \frac{1}{w(z)^2},$$

which is the form of (17).

Example 3 $w(z) = \cos z$ solves the equation

$$w(2z) w(4z) = 2w(2z)^3 - w(2z).$$

Clearly, $\lambda(1/w) = 0 < 1 = \lambda(w) = \sigma(w)$.

However, if we consider the inverse problem of Theorem 5, we can use similar techniques of Theorem 5.3 in [8] and Theorem 2 in [20] to get the following result.

Theorem 6 *Let $q \in \mathbb{C}$, $|q| > 1$, $c(z)$ be nontrivial meromorphic functions, and $w(z)$ be a transcendental meromorphic solution of equation*

$$\prod_{i=0}^{n} w\left(q^i z\right)^{\lambda_i} = c(z) w(z)^m, \quad m \in \mathbb{Z} \setminus \{0\}. \tag{18}$$

If $\sigma(c) < \sigma(w)$, then

$$\max\left\{\overline{\lambda}(w), \overline{\lambda}\left(\frac{1}{w}\right)\right\} < \sigma(w).$$

We now proceed to prepare some Lemmas.

Lemma 6 ([21]) *Let $w(z)$ be a transcendental meromorphic function, $p(z) = d_k z^k + \cdots + d_1 z + d_0$ $(d_k \neq 0)$ be a polynomial of degree k. Given $0 < \delta < |d_k|$, denote $\nu := |d_k| + \delta$ and $\mu := |d_k| - \delta$. Then, given $\varepsilon > 0$ and $a \in \mathbb{C} \cup \{\infty\}$, we have for all $r \geqslant r_0 > 0$,*

$$kn\left(\mu r^k, a, w\right) \leqslant n(r, a, w \circ p) \leqslant kn\left(\nu r^k, a, w\right),$$

$$kN\left(\mu r^k, a, w\right) + O(\log r) \leqslant N(r, a, w \circ p) \leqslant kn\left(\nu r^k, a, w\right) + O(\log r),$$

$$(1-\varepsilon) T\left(\mu r^k, w\right) \leqslant T(w \circ p) \leqslant (1+\varepsilon) T\left(\nu r^k, w\right).$$

Lemma 7 ([22]) *Suppose that $w(z)$ is a transcendental meromorphic solution of equation*

$$\sum_{j=0}^{n} a_j(z) w\left(q^j z\right) = Q(z),$$

where $q \in \mathbb{C}$, $|q| \neq 0, 1$, and all coefficients $a_0, \ldots, a_n, Q$ are meromorphic and of finite order $\leqslant \rho$. Then $\sigma(w) \leqslant \rho$.

Proof of Theorem 5: Let τ be the multiplicity of pole of $w(z)$ at the origin, and let $q(z)$ be a canonical product of $w(z)$ formed by the nonzero poles of $w(z)$. Since $\max\{\lambda(w), \lambda(1/w)\} < \sigma(w)$, then $h(z) = z^\tau q(z)$ is an entire function such that

$$\sigma(h) = \lambda\left(\frac{1}{w}\right) < \sigma(w) \tag{19}$$

and $g(z) = h(z) w(z)$ is a transcendental entire function with

$$\begin{aligned} T(r, g) &= T(r, w) + S(r, w), \\ \sigma(g) = \sigma(w), &\quad \lambda(g) = \lambda(w). \end{aligned} \tag{20}$$

We now conclude from the last assertion of Lemma 6, (19) and (20) that

$$\sigma(h \circ p) = k\sigma(h) = k\lambda\left(\frac{1}{w}\right) < k\sigma(g) = \sigma(g \circ p).$$

Therefore,

$$T(r, h\circ p) = S(r, g\circ p). \tag{21}$$

Substituting $w(z) = g(z)/h(z)$ into (15), we conclude that

$$\frac{(h\circ p)^{s-t}}{\prod_{i=0}^{n} h\left(q^i z\right)^{\lambda_i}} \prod_{i=0}^{n} g\left(q^i z\right)^{\lambda_i} = \frac{a_0(z)(h\circ p)^s + \cdots + a_s(z)(g\circ p)^s}{b_0(z)(h\circ p)^t + \cdots + b_t(z)(g\circ p)^t}. \tag{22}$$

Obviously, it follows from (19), (20) and (21) that

$$\begin{cases} T\left(r, \prod_{i=0}^{n} h\left(q^i z\right)^{\lambda_i}\right) = S(r, g\circ p), \\ T(r, (h\circ p)^{s-t}) = S(r, g\circ p), \\ T\left(r, a_u(z)(h\circ p)^{s-u}\right) = S(r, g\circ p), \quad u = 0, 1, \ldots, s, \\ T\left(r, b_v(z)(h\circ p)^{t-v}\right) = S(r, g\circ p), \quad v = 0, 1, \ldots, t. \end{cases} \tag{23}$$

Denoting $A(z) = (h\circ p)^{s-t} / \prod_{i=0}^{n} h\left(q^i z\right)^{\lambda_i}$, we get from (23) that

$$T(r, A) = S(r, g\circ p). \tag{24}$$

Since zeros and poles are Borel exceptional values of $w(z)$ by (16), we may apply a result due to Whittaker, see Satz 13.4 in [23], to deduce that $w(z)$ is of regular growth. Thus, we use (23) again to get

$$T\Big(r, \frac{w'}{w}\Big) = \overline{N}(r, w) + \overline{N}\Big(r, \frac{1}{w}\Big) + S(r, w) = S(r, g\circ p). \tag{25}$$

Similarly, if we set $B(z) = A(z)\big(\prod_{i=0}^{n} g\left(q^i z\right)^{\lambda_i}\big)$, we also deduce from the lemma of the logarithmic derivative, (16), (20) and (24) that

$$T\Big(r, \frac{B'}{B}\Big) = T\Big(r, \frac{A'}{A} + \sum_{i=0}^{n} \lambda_i q^i \frac{g'\left(q^i z\right)}{g\left(q^i z\right)}\Big) = S(r, g\circ p). \tag{26}$$

Denote $F(z) = g\circ p$,

$$P(z, F) = \frac{a_0(z)}{a_s(z)}(h\circ p)^s + \frac{a_1(z)}{a_s(z)}(h\circ p)^{s-1} F(z) + \cdots + F(z)^s,$$

and

$$Q(z, F) = \frac{b_0(z)}{b_t(z)}(h\circ p)^t + \frac{b_1(z)}{b_t(z)}(h\circ p)^{t-1} F(z) + \cdots + F(z)^t.$$

Therefore, we deduce from (20) and (21) that the coefficients of $P(z, F)$ and $Q(z, F)$ are small functions relative to $g\circ p$. Thus, (22) can be written in the form

$$\frac{b_t(z)}{a_s(z)} B(z) = \frac{P(z, F)}{Q(z, F)} = u(z, F). \tag{27}$$

By denoting

$$\psi(z) = \frac{F'(z)}{F(z)} \quad \text{and} \quad U(z) = \frac{u'(z, F)}{u(z, F)},$$

we get $T(r, U) = S(r, w)$ from (26) and (27). We also conclude from the lemma of logarithmic derivative, Lemma 6, (16), (20) and (21) that

$$\begin{aligned} T(r, \psi) &= T\Big(r, \frac{F'}{F}\Big) = m\Big(r, \frac{F'}{F}\Big) + N\Big(r, \frac{F'}{F}\Big) \\ &\leqslant \overline{N}(r, F) + \overline{N}\Big(r, \frac{1}{F}\Big) + S(r, F) \\ &= \overline{N}(r, g\circ p) + \overline{N}\Big(r, \frac{1}{g\circ p}\Big) + S(r, g\circ p) \\ &\leqslant N\Big(r, \frac{1}{g\circ p}\Big) + S(r, g\circ p) \\ &\leqslant N\Big(\nu r^k, \frac{1}{g}\Big) + S(r, g\circ p) = S(r, g\circ p), \end{aligned}$$

where ν is defined as Lemma 6. Since

$$\frac{P'Q - PQ'}{Q^2} = u' = Uu = \frac{UP}{Q},$$

we conclude that

$$P'Q - PQ' = UPQ. \tag{28}$$

Now, writing $F' = \psi F$ in (28), regarding then (28) as an algebraic equation in F with coefficients of growth $S(r, F)$ (in fact $S(r, w)$), and comparing the leading coefficients, we deduce that

$$(s - t)\psi = U.$$

By integrating both sides of the above equality, we conclude that

$$u(z, F) = \alpha F(z)^{s-t}, \tag{29}$$

for some $\alpha \in \mathbb{C}\backslash\{0\}$. Therefore, by combing the representations of F, B, A, g with (29), we conclude that

$$\prod_{i=0}^{n} w\left(q^i z\right)^{\lambda_i} = \alpha \frac{a_s(z)}{b_t(z)} (w\circ p)^{s-t}. \tag{30}$$

If $st \neq 0$, we deduce from (15) and (30) that

$$\alpha \frac{a_s(z)}{b_t(z)} (w\circ p)^{s-t} = R(z, w\circ p) = \frac{a_0(z) + a_1(z)(w\circ p) + \cdots + a_s(z)(w\circ p)^s}{b_0(z) + b_1(z)(w\circ p) + \cdots + b_t(z)(w\circ p)^t}.$$

From this, we get that $R(z, w\circ p)$ is not irreducible in $w\circ p$, a contradiction. Thus, $s=0$ or $t=0$. Therefore, we deduce from (30) that

$$\prod_{i=0}^{n} w\left(q^i z\right)^{\lambda_i} = \alpha \frac{a_s(z)}{b_0(z)}(w\circ p)^s \quad \text{or}$$
$$\prod_{i=0}^{n} w\left(q^i z\right)^{\lambda_i} = \alpha \frac{a_0(z)}{b_t(z)}\frac{1}{(w\circ p)^t}. \quad (31)$$

Applying Valiron-Mohon'ko theorem [24] to (31), we obtain $s=n+1$ or $t=n+1$. Thus, the desired forms (17) are obtained. □

Proof of Theorem 6: Denote $y(z)=w'(z)/w(z)$. We conclude from (18) that

$$\sum_{i=1}^{n}\lambda_i q^i y\left(q^i z\right) + (\lambda_0 - m)y(z) = \frac{c'(z)}{c(z)}.$$

Thus, we deduce from Lemma 7 that

$$\sigma(y) \leqslant \sigma\Big(\frac{c'(z)}{c(z)}\Big) < \sigma(w).$$

Therefore,

$$\max\left\{\overline{\lambda}(w), \overline{\lambda}\Big(\frac{1}{w}\Big)\right\} = \lambda\Big(\frac{w'}{w}\Big) \leqslant \sigma(y) < \sigma(w).$$

□

GROWTH OF MEROMORPHIC SOLUTIONS

At this point, we briefly introduce some notations used below. Let I be a finite set of multi-indexes $\lambda=(\lambda_0,\lambda_1,\dots,\lambda_n)$. A q-difference monomial of a meromorphic function $w(z)$ is defined as

$$\prod_{i=0}^{n} w\left(q^i z\right)^{\lambda_i},$$

and a q-difference polynomial $H_q(z,w(z))$ of a meromorphic function $w(z)$, a finite sum of q-difference monomials, is defined as

$$H_q(z,w(z)) = \sum_{\lambda\in I}\alpha_\lambda(z)\prod_{i=0}^{n} w\left(q^i z\right)^{\lambda_i}, \quad (32)$$

where the coefficients $\alpha_\lambda(z)$ are small functions relative to $w(z)$. The degree of the q-difference polynomial (32) is defined by

$$\deg_w(H_q) = \max_{\lambda\in I}\left\{\sum_{i=0}^{n}\lambda_i\right\}.$$

For instance, the degree of the q-difference polynomial $w^2(z)w(qz)w\left(q^2z\right)+w(z)w\left(q^3z\right)$ is four.

In follows, we consider the growth of meromorphic solutions of some functional difference equations.

Theorem 7 *Let $q\in\mathbb{C}$, $|q|>1$, and $w(z)$ be a transcendental meromorphic solution of equation*

$$\sum_{\lambda\in I}\alpha_\lambda(z)\prod_{i=0}^{n} w\left(q^i z\right)^{\lambda_i} = R(z,w(z))$$
$$= \frac{a_0(z)+a_1(z)w(z)+\cdots+a_s(z)w(z)^s}{b_0(z)+b_1(z)w(z)+\cdots+b_t(z)w(z)^t}, \quad (33)$$

where I is a finite set of multi-indexes $\lambda=(\lambda_0,\lambda_1,\dots,\lambda_n)$, and all coefficients in (33) are small meromorphic functions relative to $w(z)$ such that $a_s(z)b_t(z)\not\equiv 0$. If $d=\max\{s,t\}>(n+1)\deg_w(H_q)$, then $\sigma(w)>0$.

Example 4 $w(z)=e^z/z$ solves the functional equation

$$f(z)^2 f(-2z)f(4z)+f(-8z)^2 = \frac{1-8z^{18}f(z)^{20}}{64z^{18}f(z)^{16}}$$

of type (33). Here, $q=-2$, $d=20>16=(3+1)4=(n+1)\deg_{H_q}$ and $\sigma(f)=1>0$.

In fact, the following Example 5 shows that the assertion of Theorem 7 may occur if $d=\deg_f(H_q)$. But we can not find a proper method to prove it.

Example 5 $w(z)=\cos z$ solves functional equation

$$w(2z)w\left(2^2z\right) = 16w(z)^6 - 24w(z)^4 + 10w(z)^2 - 1$$

of type (33). Here, $q=2$, $d=6=(2+1)2=(n+1)\deg_f(H_q)$, and $\sigma(f)=1>0$.

Theorem 8 *Let $q\in\mathbb{C}$, $|q|>1$, and $w(z)$ be a transcendental meromorphic solution of equation*

$$\sum_{\lambda\in I}\alpha_\lambda(z)\prod_{i=0}^{n} w\left(q^i z\right)^{\lambda_i} = R(z,w\circ p)$$
$$= \frac{a_0(z)+a_1(z)(w\circ p)+\cdots+a_s(z)(w\circ p)^s}{b_0(z)+b_1(z)(w\circ p)+\cdots+b_t(z)(w\circ p)^t}, \quad (34)$$

where $p(z)=d_kz^k+\cdots+d_1z+d_0$ is a polynomial with constant coefficients $d_k(\neq 0),\dots,d_1,d_0$ and of degree $k\geqslant 2$, I is a finite set of multi-indexes $\lambda=(\lambda_0,\lambda_1,\dots,\lambda_n)$, and all coefficients in (34) are small meromorphic functions relative to $f(z)$ such that $a_s(z)b_t(z)\not\equiv 0$. Moreover, we assume that $kd=$

$k\max\{s,t\}\leqslant(n+1)\deg_f(H_q)$, where $\deg_w(H_q)$ is the degree of q-difference polynomial (32). Then

$$T(r,w)=O((\log r)^{\alpha+\varepsilon}),$$

where $\alpha=\left(\log(n+1)+\log\deg_w(H_q)-\log d\right)/\log k$.

We now prepare some lemmas. By denoting $w_i=w\left(q^iz\right)$ $(i=0,1,\dots,n)$, it is easy to prove the following result from Lemma 3.

Lemma 8 *Let $q\in\mathbb{C}$, $|q|>1$, and $w(z)$ be a meromorphic function. Then the characteristic function of q-difference polynomial (32) satisfies*

$$T\left(r,\sum_{\lambda\in I}\alpha_\lambda(z)\prod_{i=0}^{n}w\left(q^iz\right)^{\lambda_i}\right)\leqslant(n+1)\deg_w(H_q)T\left(|q|^nr,w\right)+S(r,w).$$

Lemma 9 ([24, 25]) *Let $g(r)$ and $h(r)$ be monotone nondecreasing functions on $[0,\infty)$ such that $g(r)\leqslant h(r)$ for all $r\notin E\cup[0,1]$, where $E\subset(1,\infty)$ is a set of finite logarithmic measure. Let $\alpha>1$ be a given constant. Then there exists an $r_0=r_0(\alpha)>0$ such that $g(r)\leqslant h(\alpha r)$ for all $r\geqslant r_0$.*

Lemma 10 ([26]) *Let $\psi(r)$ be a function of $r(r\geqslant r_0)$, positive and bounded in every finite interval.*

(i) Suppose that $\psi(\mu r^m)\leqslant A\psi(r)+B(r\geqslant r_0)$, where $\mu(\mu>0)$, $m(m>1)$, $A(A\geqslant1)$, B are constants. Then $\psi(r)=O\left((\log r)^\alpha\right)$ with $\alpha=\log A/\log m$, unless $A=1$ and $B>0$; and if $A=1$ and $B>0$, then for any $\varepsilon>0$, $\psi(r)=O\left((\log r)^\varepsilon\right)$.

(ii) Suppose that (with the notation of (i)) $\psi(\mu r^m)\geqslant A\psi(r)(r\geqslant r_0)$. Then for all sufficiently large values of r, $\psi(r)\geqslant K(\log r)^\alpha$ with $\alpha=\log A/\log m$ for some positive constant K.

Proof of Theorem 7: Assume in contrary to the assertion that $w(z)$ is meromorphic with $\sigma(w)=0$. For any $\varepsilon(0<\varepsilon<(d-(n+1)\deg_w(H_q))/(d+(n+1)\deg_w(H_q))$, we may apply Valiron-Mohon'ko lemma, Lemma 8, (32) and (33) to conclude that,

$$\begin{aligned}d(1-\varepsilon)T(r,w)&\leqslant dT(r,w)+S(r,w)\\&=T\left(r,\frac{a_0(z)+a_1(z)w(z)+\cdots+a_s(z)w(z)^s}{b_0(z)+b_1(z)w(z)+\cdots+b_t(z)w(z)^t}\right)\\&=T(r,H_q(z,w(z)))\\&\leqslant(n+1)\deg_w(H_q)T(|q|^nr,w)+S(r,w)\\&\leqslant(n+1)\deg_w(H_q)(1+\varepsilon)T(|q|^nr,w),\end{aligned}$$

on a set of logarithmic density 1. So, we get

$$\begin{aligned}T(r,w)&\leqslant\frac{(n+1)\deg_w(H_q)}{d}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)T(|q|^nr,w)\\&:=\gamma T(|q|^nr,w)\end{aligned}$$

on a set of logarithmic density 1, where

$$\gamma:=\frac{(n+1)\deg_w(H_q)}{d}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)<1,$$

since $\varepsilon(0<\varepsilon<(d-(n+1)\deg_w(H_q))/(d+(n+1)\deg_w(H_q))$ and the assumption that $d>(n+1)\deg_w(H_q)$. Thus, we deduce from Lemma 4 that $\sigma(w)>0$, a contradiction. □

Proof of Theorem 8: For any $\varepsilon(0<\varepsilon<1)$, we may apply Valiron-Mohon'ko lemma, Lemma 6, Lemma 8, (32) and (34) to conclude that

$$\begin{aligned}d(1-\varepsilon)T(\mu r^k,w)&\leqslant dT(r,w\circ p)+S(r,w\circ p)\\&=T\left(r,\frac{a_0(z)+a_1(z)(w\circ p)+\cdots+a_s(z)(w\circ p)^s}{b_0(z)+b_1(z)(w\circ p)+\cdots+b_t(z)(w\circ p)^t}\right)\\&\quad+S(r,w)\\&=T\left(r,\sum_{\lambda\in I}\alpha_\lambda(z)\prod_{i=0}^{n}w(q^iz)^{\lambda_i}\right)+S(r,w)\\&\leqslant(n+1)\deg_w(H_q)T(|q|^nr,w)+S(r,w)\\&\leqslant(n+1)\deg_w(H_q)(1+\varepsilon)T(|q|^nr,w)\end{aligned}$$

holds for all sufficiently large r, possibly outside of an exceptional set of finite logarithmic measure, where μ is defined as Lemma 6. Now, we may apply Lemma 9 to deal with the exceptional set, and conclude that, for every $\eta>1$, there exists an $r_0>0$ such that

$$d(1-\varepsilon)T\left(\mu r^k,w\right)\leqslant(n+1)\deg_w(H_q)(1+\varepsilon)T(\eta|q|^nr,w)\quad(35)$$

holds for all $r\geqslant r_0$. Denoting $\tau=\eta|q|^nr$. Then (35) can be written in the form

$$T\left(\frac{\mu}{\eta^k|q|^{nk}}\tau^k,w\right)\leqslant\frac{(n+1)\deg_w(H_q)(1+\varepsilon)}{d(1-\varepsilon)}T(\tau,w).$$

Since $dk\leqslant(n+1)\deg_w(H_q)$, we get $(n+1)\deg_w(H_q)(1+\varepsilon)/d(1-\varepsilon)>1$ for all $0<\varepsilon<1$. Thus, we now apply Lemma 10 (i) to conclude that

$$T(r,w)=O\left((\log r)^{\alpha+\varepsilon}\right),$$

and

$$\alpha = \frac{\log \frac{(n+1)\deg_w(H_q)(1+\varepsilon)}{d(1-\varepsilon)}}{\log k} = \frac{\log(n+1) + \log \deg_w(H_q) - \log d}{\log k} + o(1).$$

□

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