RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2021.040
ScienceAsia 47 (2021): **382–387**

Some generalizations of numerical radius inequalities for Hilbert space operators

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Received 2 Dec 2020
Accepted 16 Mar 2021

ABSTRACT: In this article, we generalize several upper and lower bounds of the numerical radius inequalities for Hilbert space operators. In particular, we show that if $A \in B(\mathscr{H})$ with the Cartesian decomposition $A = B + iC$ and f is an increasing concave function, then $f(\omega(A)) \geqslant \frac{1}{2}\|f(|B+C|)+f(|B-C|)\|$. This is a complementary result of El-Haddad and Kittaneh [Studia Math **182** (2007):133–140].

KEYWORDS: numerical radius, function, operator norm, Cartesian decomposition

MSC2010: 47A63 47A30

INTRODUCTION

Let $B(\mathscr{H})$ denote the C*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with an inner product $\langle\cdot\rangle$. For $A \in B(\mathscr{H})$, let $\|A\|$ denote the usual operator norm of A. The numerical range of A is defined by $W(A) = \{\langle Ax, x\rangle : x \in \mathscr{H}, \|x\| = 1\}$. The numerical radius of A is defined by $\omega(A) = \sup\{|\lambda| : \lambda \in W(A)\}$. We note that if $A \in B(\mathscr{H})$ and if f is a non-negative increasing function on $[0,\infty)$, then $\|f(|A|)\| = f(\|A\|)$. Recall that $A \in B(\mathscr{H})$ is said to be hyponormal if $A^*A - AA^* \geqslant 0$, or equivalently if $\|A^*x\| \leqslant \|Ax\|$.

It is well known that $\omega(\cdot)$ defines a norm on $B(\mathscr{H})$. In fact, for any $A \in B(\mathscr{H})$,

$$\frac{1}{2}\|A\| \leqslant \omega(A) \leqslant \|A\|, \tag{1}$$

which indicates the usual operator norm and the numerical radius norm are equivalent. For more information about numerical radius inequalities, readers are referred to [1, 2].

Before proceeding, we give the definition of geometrical convexity. First we note that all functions in this article satisfy the following condition unless otherwise specified: J is a subinterval of $(0,\infty)$ and $f : J \to (0,\infty)$. We say that f is geometrically convex if $f(a^{1-t}b^t) \leqslant f^{1-t}(a)f^t(b)$ for all $t \in [0,1]$. Recent studies on numerical radius inequalities involving convex and concave functions can be found in [3].

For positive real numbers a and b, the classical Young inequality says that if $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}.$$

In particular, when $p = q = 2$, this is the scalar arithmetic-geometric mean inequality.

A refinement of the scalar arithmetic-geometric mean inequality is presented in [4] as follows:

$$\left(1 + \frac{(\ln a - \ln b)^2}{8}\right)\sqrt{ab} \leqslant \frac{a+b}{2}. \tag{2}$$

Kittaneh [5, 6] had shown the following inequalities which improved the inequalities in (1) by using several norm inequalities and ingenious techniques:

$$\omega(A) \leqslant \frac{1}{2}\left(\|A\| + \left\|A^2\right\|^{1/2}\right), \tag{3}$$

and

$$\frac{1}{4}\left\||A|^2 + |A^*|^2\right\| \leqslant \omega^2(A) \leqslant \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|. \tag{4}$$

In [3], Omidvar et al presented the following inequalities which are improvements and generalizations of (3) and (4) for hyponormal operators, respectively. Let $A \in B(\mathscr{H})$ be a hyponormal operator, then for all $1 \leqslant r \leqslant 2$,

$$\omega^r(A) \leqslant \frac{1}{2\left(1 + \frac{\xi_{|A|}^2}{8}\right)^r}\||A|^r + |A^*|^r\|, \tag{5}$$

and

$$\omega^r(A) \leqslant \frac{1}{2\left(1 + \frac{\xi_{|A|}^2}{8}\right)^r}\left(\|A\|^r + \left\||A|^{\frac{r}{2}}|A^*|^{\frac{r}{2}}\right\|\right),$$

where $\xi^2_{|A|} = \inf\limits_{\|x\|=1} \left\{ \frac{\langle (|A|-|A^*|)x,x\rangle}{\langle (|A|+|A^*|)x,x\rangle} \right\}$.

In [7], Burqan and Abu-Rahma proved that if $A, B, C \in B(\mathscr{H})$ and $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geqslant 0$, then

$$\omega^r(B) \leqslant \frac{1}{2} \|A^r + C^r\| \quad \text{for} \quad r \geqslant 1, \tag{6}$$

which gave an estimate for the numerical radius of the off-diagonal block operator matrices. For more information about numerical radius inequalities for block operator matrices and off-diagonal operator matrices, readers are referred to [8, 9]. In the same paper, they also obtained a generalization of inequality (4) for two matrices as follows. Let $A, B \in B(\mathscr{H})$ and $0 < \alpha < 1$, then, for $r \geqslant 1$,

$$\omega^r(A+B) \leqslant \frac{1}{2} \Big\| (|A^*|^{2\alpha} + |B^*|^{2\alpha})^r + (|A|^{2(1-\alpha)} + |B|^{2(1-\alpha)})^r \Big\|. \tag{7}$$

In [10], El-Haddad and Kittaneh gave generalizations of inequalities (3) and (4) as follows. Let $A \in B(\mathscr{H})$ with the Cartesian decomposition $A = B + iC$ and let $0 < r \leqslant 2$. Then

$$\omega^r(A) \leqslant \| |B|^r + |C|^r \| . \tag{8}$$

They also showed that if $r \geqslant 2$, then

$$\omega^r(A) \leqslant 2^{r/2-1} \| |B|^r + |C|^r \| , \tag{9}$$

and

$$2^{-r/2-1} \| |B+C|^r + |B-C|^r \| \leqslant \omega^r(A) \leqslant \frac{1}{2} \| |B+C|^r + |B-C|^r \| . \tag{10}$$

In this paper, we first give a different proof of inequality (5) for $r \geqslant 2$, then we give some generalizations of several upper and lower bounds of the numerical radius inequalities for Hilbert space operators involving inequalities (6)–(10) for geometrically convex functions and concave functions.

MAIN RESULTS

We begin this section with some lemmas which will be necessary to prove our main results.

Lemma 1 ([11]) *If $A \in B(\mathscr{H})$, then*

$$| \langle Ax, y \rangle | \leqslant | \langle |A|x, y \rangle |^{\frac{1}{2}} | \langle |A^*|x, y \rangle |^{\frac{1}{2}}$$

for all $x, y \in \mathscr{H}$.

Lemma 2 ([12]) *If $A \in B(\mathscr{H})$ and f and g be nonnegative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ for all $t \in [0, \infty)$, then*

$$| \langle Ax, y \rangle | \, \|f(|A|)x\| \, \|g(|A^*|)y\|$$

for all $x, y \in \mathscr{H}$.

Lemma 3 (McCarthy inequality [12]) *Let A be a positive operator in $B(\mathscr{H})$. For every unit vector $x \in \mathscr{H}$ and a given positive real number r,*

(a) $\langle Ax, x\rangle^r \leqslant \langle A^r x, x\rangle$ *for $r \geqslant 1$,*

(b) $\langle A^r x, x\rangle \leqslant \langle Ax, x\rangle^r$ *for $0 < r \leqslant 1$.*

Lemma 4 ([3]) *For each $\alpha \geqslant 1$, we have*

$$\frac{\alpha-1}{\alpha+1} \leqslant \ln \alpha.$$

Lemma 5 *Let $A \in B(\mathscr{H})$ be a hyponormal operator and let f and g be nonnegative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$\omega^r(A) \leqslant \frac{1}{2\big(1+\frac{\xi^2_{|A|}}{8}\big)^r} \Big\| \frac{1}{p} \big(f^{2p}(|A|^{\frac{r}{2}}) + f^{2p}(|A^*|^{\frac{r}{2}}) \big) + \frac{1}{q} \big(g^{2q}(|A|^{\frac{r}{2}}) + g^{2q}(|A^*|^{\frac{r}{2}}) \big) \Big\|,$$

where $r \geqslant 2$, $p, q \geqslant 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $\xi_{|A|} = \inf\limits_{\|x\|=1} \left\{ \frac{\langle (|A|-|A^*|)x,x\rangle}{\langle (|A|+|A^*|)x,x\rangle} \right\}$.

Proof: Since A is a hyponormal operator we have $1 \leqslant \frac{\langle |A|x,x\rangle}{\langle |A^*|x,x\rangle}$ for each $x \in \mathscr{H}$. On choosing $\alpha = \frac{\langle |A|x,x\rangle}{\langle |A^*|x,x\rangle}$ in Lemma 4 we get

$$0 \leqslant \frac{\langle (|A|-|A^*|)x, x\rangle}{\langle (|A|+|A^*|)x, x\rangle} \leqslant \ln \frac{\langle |A|x, x\rangle}{\langle |A^*|x, x\rangle}.$$

Whence

$$\inf_{\|x\|=1} \frac{\langle (|A|-|A^*|)x, x\rangle}{\langle (|A|+|A^*|)x, x\rangle} \leqslant \ln \frac{\langle |A|x, x\rangle}{\langle |A^*|x, x\rangle}. \tag{11}$$

We denote the expression on the left side of (11) by $\xi_{|A|}$. In inequality (2), by taking $a = \langle |A|x, x\rangle$ and $b = \langle |A^*|x, x\rangle$ and taking into account that $\xi_{|A|} \leqslant \ln \frac{\langle |A|x,x\rangle}{\langle |A^*|x,x\rangle}$, we infer that

$$\sqrt{\langle |A|x, x\rangle \, \langle |A^*|x, x\rangle} \leqslant \frac{1}{2\big(1+\frac{\xi^2_{|A|}}{8}\big)} \langle (|A|+|A^*|)x, x\rangle .$$

By Lemma 1, we get

$$|\langle Ax,x\rangle| \leqslant \frac{1}{2\big(1+\frac{\xi_{|A|}^2}{8}\big)} \langle (|A|+|A^*|)x,x\rangle .$$

Now by taking $\|x\|=1$, we have

$$\begin{aligned}
|\langle Ax,x\rangle|^r &\leqslant \frac{1}{2^r\big(1+\frac{\xi_{|A|}^2}{8}\big)^r} \langle (|A|+|A^*|)x,x\rangle^r \\
&\leqslant \frac{1}{2\big(1+\frac{\xi_{|A|}^2}{8}\big)^r}(\langle |A|x,x\rangle^r+\langle |A^*|x,x\rangle^r) \\
&\leqslant \frac{1}{2\big(1+\frac{\xi_{|A|}^2}{8}\big)^r}\Big(\big\langle |A|^{\frac{r}{2}}x,x\big\rangle^2+\big\langle |A^*|^{\frac{r}{2}}x,x\big\rangle^2\Big) \\
&\leqslant \frac{1}{2\big(1+\frac{\xi_{|A|}^2}{8}\big)^r}\Big(\big\langle f^2(|A|^{\frac{r}{2}})x,x\big\rangle\big\langle g^2(|A|^{\frac{r}{2}})x,x\big\rangle \\
&\quad +\big\langle f^2(|A^*|^{\frac{r}{2}})x,x\big\rangle\big\langle g^2(|A^*|^{\frac{r}{2}})x,x\big\rangle\Big) \quad \text{(Lemma 2)} \\
&\leqslant \frac{1}{2\big(1+\frac{\xi_{|A|}^2}{8}\big)^r}\Big(\tfrac{1}{p}\big\langle f^2(|A|^{\frac{r}{2}})x,x\big\rangle^p+\tfrac{1}{q}\big\langle g^2(|A|^{\frac{r}{2}})x,x\big\rangle^q \\
&\quad +\tfrac{1}{p}\big\langle f^2(|A^*|^{\frac{r}{2}})x,x\big\rangle^p+\tfrac{1}{q}\big\langle g^2(|A^*|^{\frac{r}{2}})x,x\big\rangle^q\Big) \\
&\leqslant \frac{1}{2\big(1+\frac{\xi_{|A|}^2}{8}\big)^r}\Big(\tfrac{1}{p}\big\langle f^{2p}(|A|^{\frac{r}{2}})x,x\big\rangle+\tfrac{1}{q}\big\langle g^{2q}(|A|^{\frac{r}{2}})x,x\big\rangle \\
&\quad +\tfrac{1}{p}\big\langle f^{2p}(|A^*|^{\frac{r}{2}})x,x\big\rangle+\tfrac{1}{q}\big\langle g^{2q}(|A^*|^{\frac{r}{2}})x,x\big\rangle\Big) \\
&= \frac{1}{2\big(1+\frac{\xi_{|A|}^2}{8}\big)^r}\Big(\tfrac{1}{p}\big(f^{2p}(|A|^{\frac{r}{2}})+f^{2p}(|A^*|^{\frac{r}{2}})\big) \\
&\quad +\tfrac{1}{q}\big\langle\big(g^{2q}(|A|^{\frac{r}{2}})+g^{2q}(|A^*|^{\frac{r}{2}})\big)\big)x,x\big\rangle .
\end{aligned}$$

Now the result follows by taking the supremum over all unit vectors in $\mathscr{H}$. □

Theorem 1 *Let $A\in B(\mathscr{H})$ is a hyponormal operator. Then, for all $r\geqslant 1$,*

$$\omega^r(A) \leqslant \frac{1}{2\big(1+\frac{\xi_{|A|}^2}{8}\big)^r} \||A|^r+|A^*|^r\| ,$$

where $\xi_{|A|}=\inf_{\|x\|=1}\Big\{\frac{\langle(|A|-|A^*|)x,x\rangle}{\langle(|A|+|A^*|)x,x\rangle}\Big\}$.

Proof: The case $1\leqslant r\leqslant 2$ in Theorem 1 follows from the result of Omidvar et al. The case $r\geqslant 2$ is a direct result of Lemma 5 by setting $p=q=2$ and $f(t)=g(t)=t^{\frac{1}{2}}$. □

Theorem 2 *Let $A\in B(\mathscr{H})$ is a hyponormal operator. Then, for all $r\geqslant 1$,*

$$\omega^r(A) \leqslant \frac{1}{2\big(1+\frac{\xi_{|A|}^2}{8}\big)^r}\big(\|A\|^r+\big\||A|^{\frac{r}{2}}|A^*|^{\frac{r}{2}}\big\|\big),$$

where $\xi_{|A|}=\inf_{\|x\|=1}\Big\{\frac{\langle(|A|-|A^*|)x,x\rangle}{\langle(|A|+|A^*|)x,x\rangle}\Big\}$.

Proof: Straightforward. □

Lemma 6 ([13]) *Let $A,B,C\in B(\mathscr{H})$ such that $\begin{bmatrix} A & B^* \\ B & C\end{bmatrix}\geqslant 0$. Then $|\langle Bx,y\rangle|^2\leqslant\langle Ax,x\rangle\langle Cy,y\rangle$ for all $x,y\in\mathscr{H}$.*

Theorem 3 *Let $A,B,C\in B(\mathscr{H})$ be such that $\begin{bmatrix} A & B^* \\ B & C\end{bmatrix}\geqslant 0$ and f be an increasing geometrically convex function. If in addition f is convex, then*

$$f(\omega(B))\leqslant\frac{1}{2}\|f(A)+f(C)\| .$$

Proof: For any unit vector $x\in\mathscr{H}$, we have the following chain of inequalities

$$\begin{aligned}
f(|\langle Bx,x\rangle|) &\leqslant f(\langle Ax,x\rangle^{\frac{1}{2}}\langle Cx,x\rangle^{\frac{1}{2}}) \quad \text{(Lemma 6)} \\
&\leqslant \sqrt{f(\langle Ax,x\rangle)f(\langle Cx,x\rangle)} \\
&\leqslant \sqrt{\langle f(A)x,x\rangle\langle f(C)x,x\rangle} \\
&\leqslant \frac{1}{2}\langle (f(A)+f(C))x,x\rangle .
\end{aligned}$$

Hence,

$$\begin{aligned}
f(\omega(B)) &= f(\sup_{\|x\|=1}|\langle Bx,x\rangle|) \\
&= \sup_{\|x\|=1} f(|\langle Bx,x\rangle|) \\
&\leqslant \sup_{\|x\|=1}\frac{1}{2}\langle (f(A)+f(C))x,x\rangle \\
&= \frac{1}{2}\|(f(A)+f(C))\| ,
\end{aligned}$$

as required. □

Remark 1 It is easy to verify that the function $f(t)=t^r(r\geqslant 1)$ satisfies the assumptions of Theorem 3, thus (6) is a special case of Theorem 3.

Lemma 7 ([12]) *Let $A,B,C\in B(\mathscr{H})$ such that A and B are positive, $BC=CA$, and let f and g be nonnegative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$. If $\begin{bmatrix} A & B^* \\ B & C\end{bmatrix}\geqslant 0$, then $\begin{bmatrix} f^2(A) & B^* \\ B & g^2(C)\end{bmatrix}\geqslant 0$.*

Theorem 4 *Let $A_i,B_i,X_i\in B(\mathscr{H})$ $(i=1,\dots,n)$, and let f_i and g_i $(i=1,\dots,n)$ be nonnegative functions on $[0,\infty)$ which are continuous and satisfying the* *relation $f_i(t)g_i(t)=t$ for all $t\in[0,\infty)$. Then for any positive integer m, it holds*

$$\omega^r\Big(\sum_{i=1}^n A_iX_i|X_i|^{m-1}B_i^*\Big)\leqslant\frac{1}{2}\Big\|\Big(\sum_{i=1}^n A_if_i^2(|X_i^*|^m)A_i^*\Big)^r+\Big(\sum_{i=1}^n B_ig_i^2(|X_i|^m)B_i^*\Big)^r\Big\|,$$

where $r\geqslant 1$.

Proof: Note that for any $X_i\in B(\mathscr{H})$ it admits a polar decomposition $X_i=U_i|X_i|$. Since an operator A on $\mathscr{H}$ is positive if and only if the operator $\begin{bmatrix}A & A\\ A & A\end{bmatrix}$ on $\mathscr{H}\oplus\mathscr{H}$ is positive, by simple computations, we have

$$\begin{bmatrix}|X_i^*|^m & |X_i|^mU_i^*\\ U_i|X_i|^m & U_i|X_i|^mU_i^*\end{bmatrix}=\begin{bmatrix}U_i & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}|X_i|^m & |X_i|^m\\ |X_i|^m & |X_i|^m\end{bmatrix}\begin{bmatrix}U_i^* & 0\\ 0 & I\end{bmatrix}\geqslant 0,$$

which indicates

$$\begin{bmatrix}|X_i^*|^m & U_i|X_i|^m\\ |X_i|^mU_i^* & |X_i^*|^m\end{bmatrix}=\begin{bmatrix}U_i|X_i|^mU_i^* & U_i|X_i|^m\\ |X_i|^mU_i^* & |X_i^*|^m\end{bmatrix}\geqslant 0.$$

Therefore

$$\begin{bmatrix}|X_i^*|^m & X_i|X_i|^{m-1}\\ |X_i|^{m-1}X_i^* & |X_i|^m\end{bmatrix}=\begin{bmatrix}U_i|X_i|^mU_i^* & U_i|X_i|^m\\ |X_i|^mU_i^* & |X_i|^m\end{bmatrix}\geqslant 0.$$

For the special case $m=1$, we set $|X_i|^0=I$. To apply Lemma 7, note that $|X_i|^m|X_i|^{m-1}X_i^*=|X_i|^{2m}U^*=|X_i|^{m-1}|X_i|U^*U|X_i|^mU^*=|X_i|^{m-1}X_i^*|X_i^*|^m$. Thus $\begin{bmatrix}f_i^2(|X_i^*|^m) & X_i|X_i|^{m-1}\\ |X_i|^{m-1}X_i^* & g_i^2(|X_i|^m)\end{bmatrix}\geqslant 0$. Pre-post multiply the above matrix by $\begin{bmatrix}A & 0\\ 0 & B\end{bmatrix}$ and $\begin{bmatrix}A^* & 0\\ 0 & B^*\end{bmatrix}$, respectively, we have $\begin{bmatrix}Af_i^2(|X_i^*|^m)A^* & AX_i|X_i|^{m-1}B^*\\ B|X_i|^{m-1}X_i^*A^* & Bg_i^2(|X_i|^m)B^*\end{bmatrix}\geqslant 0$. Summing up the previous matrices for $i=1,2,\dots,n$, we have

$$\begin{bmatrix}\sum_{i=1}^n Af_i^2(|X_i^*|^m)A^* & \sum_{i=1}^n AX_i|X_i|^{m-1}B^*\\ \sum_{i=1}^n B|X_i|^{m-1}X_i^*A^* & \sum_{i=1}^n Bg_i^2(|X_i|^m)B^*\end{bmatrix}\geqslant 0.$$

By applying Theorem 3 to the above matrix and letting $f(t)=t^r(r\geqslant 1)$, we thus obtain the result. □

Remark 2 In Theorem 4, if we take $m=n=1$, $0\leqslant\alpha\leqslant 1$, $f(t)=t^\alpha$, $g(t)=t^{1-\alpha}$, $A_1=B_1=I$, and $X_1=A$, we get Theorem 1 in [10].

Remark 3 In Theorem 4, if we take $m=1$, $n=2$, $0\leqslant\alpha\leqslant 1$, $f(t)=t^\alpha$, $g(t)=t^{1-\alpha}$, $A_i=B_i=I(i=1,2)$, $X_1=A$, and $X_2=B$, we get (7).

Remark 4 In Theorem 4, if we take $m=n=1$, $0\leqslant\alpha\leqslant 1$, $f(t)=t^\alpha$, $g(t)=t^{1-\alpha}$, $A_1=B^*$, $B_1=A$, and $X_1=I$, we get Theorem 1 in [14].

Theorem 5 *Let $A\in B(\mathscr{H})$ with the Cartesian decomposition $A=B+iC$ and f be an increasing concave function. Then*

$$f(\omega^2(A))\leqslant\big\|f(|B|^2)+f(|C|^2)\big\|.$$

Proof: Since $A=B+iC$ is the Cartesian decomposition of A, we have $|\langle Ax,x\rangle|^2=\langle Bx,x\rangle^2+\langle Cx,x\rangle^2$ for every unit vector x. Therefore

$$\begin{aligned}f(|\langle Ax,x\rangle|^2)&=f(\langle Bx,x\rangle^2+\langle Cx,x\rangle^2)\\&\leqslant f(\langle|B|x,x\rangle^2+\langle|C|x,x\rangle^2)\\&\leqslant f\big(\langle|B|^2x,x\rangle+\langle|C|^2x,x\rangle\big)\\&=f\big(\langle(|B|^2+|C|^2)x,x\rangle\big).\end{aligned}$$

Since $\|f(A+B)\|\leqslant\|f(A)+f(B)\|$ for positive operator A,B and every nonnegative concave function f on $[0,\infty)$, it follows that

$$\begin{aligned}f(\omega^2(A))&=f(\sup_{\|x\|=1}|\langle Ax,x\rangle|^2)\\&=\sup_{\|x\|=1}f(|\langle Ax,x\rangle|^2)\\&\leqslant\sup_{\|x\|=1}f\big(\langle(|B|^2+|C|^2)x,x\rangle\big)\\&=f\big(\||B|^2+|C|^2\|\big)\\&=\big\|f(|B|^2+|C|^2)\big\|\\&\leqslant\big\|f(|B|^2)+f(|C|^2)\big\|,\end{aligned}$$

completing the proof. □

Remark 5 Since the function $f(t)=t^r(0<r\leqslant 1)$ satisfies the assumptions of Theorem 5, it is clear that inequality (8) is a special case of Theorem 5.

Theorem 6 *Let $A\in B(\mathscr{H})$ with the Cartesian decomposition $A=B+iC$ and f be an increasing geometrically convex function. If in addition f is convex and $f(1)=1$, then*

$$f\Big(\frac{\omega(A)}{\sqrt{2}}\Big)\leqslant\sqrt{\frac{\|f(|B|^2)+f(|C|^2)\|}{2}}.$$

Proof: For every unit vector $x \in \mathscr{H}$, we have

$$\begin{aligned} f\left(\frac{|\langle Ax,x\rangle|}{\sqrt{2}}\right) &= f\left(\left(\frac{\langle Bx,x\rangle^2+\langle Cx,x\rangle^2}{2}\right)^{\frac{1}{2}}\right) \\ &\leqslant f^{\frac{1}{2}}\left(\frac{\langle Bx,x\rangle^2+\langle Cx,x\rangle^2}{2}\right)f^{\frac{1}{2}}(1) \\ &\leqslant \sqrt{\frac{f(\langle Bx,x\rangle^2)+f(\langle Cx,x\rangle^2)}{2}} \\ &\leqslant \sqrt{\frac{f(\langle |B|x,x\rangle^2)+f(\langle |C|x,x\rangle^2)}{2}} \\ &\leqslant \sqrt{\frac{f(\langle |B|^2x,x\rangle)+f(\langle |C|^2x,x\rangle)}{2}} \\ &\leqslant \sqrt{\frac{\langle f(|B|^2)x,x\rangle+\langle f(|C|^2)x,x\rangle}{2}} \\ &= \sqrt{\frac{\langle (f(|B|^2)+f(|C|^2))x,x\rangle}{2}}. \end{aligned}$$

Hence

$$\begin{aligned} f\left(\frac{\omega(A)}{\sqrt{2}}\right) &= f\left(\sup_{\|x\|=1}\frac{|\langle Ax,x\rangle|}{\sqrt{2}}\right) \\ &= \sup_{\|x\|=1} f\left(\frac{|\langle Ax,x\rangle|}{\sqrt{2}}\right) \\ &\leqslant \sup_{\|x\|=1}\sqrt{\frac{\langle (f(|B|^2)+f(|C|^2))x,x\rangle}{2}} \\ &= \sqrt{\frac{\sup_{\|x\|=1}\langle (f(|B|^2)+f(|C|^2))x,x\rangle}{2}} \\ &= \sqrt{\frac{\|f(|B|^2)+f(|C|^2)\|}{2}}, \end{aligned}$$

as required. □

Remark 6 Since the function $f(t)=t^r (r \geqslant 1)$ satisfies the assumptions of Theorem 6, it is clear that inequality (9) is a special case of Theorem 6.

Theorem 7 *Let $A \in B(\mathscr{H})$ with the Cartesian decomposition $A = B + iC$ and f be an increasing geometrically convex function. If in addition f is convex and $f(1)=1$, then*

$$f(\omega(A)) \leqslant \sqrt{\frac{\|f(|B+C|^2)+f(|B-C|^2)\|}{2}}.$$

Proof: Since for any two real numbers a and b, we have $a^2+b^2=\frac{(a+b)^2+(a-b)^2}{2}$. It follows that

$$\begin{aligned} f(|\langle Ax,x\rangle|) &= f((\langle Bx,x\rangle^2+\langle Cx,x\rangle^2)^{\frac{1}{2}}) \\ &= f\left(\left(\frac{\langle (B+C)x,x\rangle^2+\langle (B-C)x,x\rangle^2}{2}\right)^{\frac{1}{2}}\right) \end{aligned}$$

for any unit vector x, the rest of the proof follows from Theorem 6. □

Remark 7 Since the function $f(t)=t^r (r \geqslant 1)$ satisfies the assumptions of Theorem 7, it is clear that the right-hand side of inequality (10) is a special case of Theorem 7.

Theorem 8 *Let $A \in B(\mathscr{H})$ with the Cartesian decomposition $A = B + iC$ and f be an increasing concave function. Then*

$$f(\omega(A)) \geqslant \frac{1}{2}\|f(|B+C|)+f(|B-C|)\|.$$

Proof: Since for any two real numbers a and b, we have $a^2+b^2=\frac{(a+b)^2+(a-b)^2}{2}$. It follows that

$$\begin{aligned} f(|\langle Ax,x\rangle|) &= f((\langle Bx,x\rangle^2+\langle Cx,x\rangle^2)^{\frac{1}{2}}) \\ &= f\left(\left(\frac{\langle (B+C)x,x\rangle^2+\langle (B-C)x,x\rangle^2}{2}\right)^{\frac{1}{2}}\right) \\ &\geqslant f\left(\frac{|\langle (B+C)x,x\rangle|+|\langle (B-C)x,x\rangle|}{2}\right) \\ &\geqslant \frac{f(|\langle (B+C)x,x\rangle|)+f(|\langle (B-C)x,x\rangle|)}{2}. \end{aligned}$$

By taking the supremum over unit vector x, we obtain

$$f(\omega(A)) \geqslant \frac{f(\|(B+C)\|)+f(\|(B-C)\|)}{2}.$$

Thus by the triangle inequality for operator norm, we have

$$\begin{aligned} f(\omega(A)) &\geqslant \frac{f(\|(B+C)\|)+f(\|(B-C)\|)}{2} \\ &= \frac{\|f(|B+C|)\|+\|f(|B-C|)\|}{2} \\ &\geqslant \frac{\|f(|B+C|)+f(|B-C|)\|}{2}, \end{aligned}$$

which completes the proof. □

Remark 8 Since the function $f(t)=t^r (0 < r \leqslant 1)$ satisfies the assumptions of Theorem 8, we have $\omega^r(A) \geqslant \frac{1}{2}\||B+C|^r+|B-C|^r\|$ for $0 < r \leqslant 1$, which can be viewed as a complement of the left-hand side part of inequality (10). To show that $\omega(A) \geqslant \frac{1}{2}\||B+C|+|B-C|\|$ is sharp, consider $A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, then $\omega(A)=1$ and $\||B+C|+|B-C|\|=2$.

Theorem 9 *Let $A \in B(\mathscr{H})$ with the Cartesian decomposition $A = B + iC$ and f be an increasing concave function. Then*

$$f\left(\frac{\omega(A)}{\sqrt{2}}\right) \geqslant \frac{1}{2}\|f(|B|)+f(|C|)\|.$$

Proof: For every unit vector $x \in \mathscr{H}$, we have

$$f\left(\frac{|\langle Ax,x\rangle|}{\sqrt{2}}\right) = f\left(\left(\frac{\langle Bx,x\rangle^2+\langle Cx,x\rangle^2}{2}\right)^{\frac{1}{2}}\right) \geqslant f\left(\frac{|\langle Bx,x\rangle|+|\langle Cx,x\rangle|}{2}\right) \geqslant \frac{f(|\langle Bx,x\rangle|)+f(|\langle Cx,x\rangle|)}{2}.$$

By taking the supremum over x, we obtain

$$f\left(\frac{\omega(A)}{\sqrt{2}}\right) \geqslant \frac{f(\|B\|)+f(\|C\|)}{2}.$$

Thus

$$f\left(\frac{\omega(A)}{\sqrt{2}}\right) \geqslant \frac{f(\|B\|)+f(\|C\|)}{2} = \frac{\|f(|B|)\|+\|f(|C|)\|}{2} \geqslant \frac{\|f(|B|)+f(|C|)\|}{2},$$

which completes the proof. □

Remark 9 Since the function $f(t)=t^r(0<r\leqslant 1)$ satisfies the assumptions of Theorem 9, we have $\omega^r(A)\geqslant 2^{\frac{r}{2}-1}\||B|^r+|C|^r\|$ for $0<r\leqslant 1$, which can be viewed as a complement and reverse of inequalities (8) and (9). To show that $\omega(A)\geqslant \frac{1}{\sqrt{2}}\||B|+|C|\|$ is sharp, consider $A=(1+i)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, then $\omega(A)=\sqrt{2}$ and $\||B|+|C|\|=2$.

Acknowledgements: This project was funded by China Postdoctoral Science Foundation (No. 2020M681575).

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