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A sharp L^p-Hardy type inequality on the n-sphere

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ABSTRACT: We obtain an L^p-Hardy inequality on the n-sphere and give the corresponding sharp constant. Furthermore, the obtained inequalities are used to derive an uncertainty principle inequality and some corollaries. The results generalize and improve some related inequalities in recent literature.

KEYWORDS: Hardy inequality, sphere, sharp constant

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INTRODUCTION

For $n \geqslant 3$, $p \geqslant 1$, and all $f \in C_c^{\infty}(\mathbb{R}^n\backslash 0)$, the classical L^p-Hardy inequality is given by

$$\int_{\mathbb{R}^n} |\nabla f|^p dx \geqslant \left(\frac{n-p}{p}\right)^p \int_{\mathbb{R}^n} \frac{|f|^p}{|x|^p} dx,$$

where the constant $(\frac{n-p}{p})^p$ is sharp. This inequality has been studied extensively in the Euclidean spaces (see [1–3]) due to its applications in different fields such as harmonic analysis, physics, spectral theory, geometry, and partial differential equations. For this line of research, we refer to [4–6] and the references therein.

In the case of Riemannian manifolds, there are also many valuable research (see [4, 7] and so on) in Hardy inequality. Carron [8] studied the weighted L^2-Hardy inequalities under several geometric assumptions. More specifically, he proved that

$$\int_M \rho^\alpha |\nabla f|^2 dV \geqslant \left(\frac{C+\alpha-1}{2}\right)^2 \int_M \rho^\alpha \frac{f^2}{\rho^2} dV,$$

for any function $f \in C_c^{\infty}(M\backslash \rho^{-1}\{0\})$ and $C+\alpha-1 > 0$, where the weight function ρ must satisfy both $|\nabla\rho| = 1$ and $\Delta\rho \geqslant C/\rho$. In particular, Kombe-Özaydin [9] extended Carron's results to the general case for $1 \leqslant p < C+1+\alpha$ and derived that

$$\int_M \rho^\alpha |\nabla f|^p dV \geqslant \left(\frac{C+1+\alpha-p}{p}\right)^p \int_M \rho^\alpha \frac{|f|^p}{\rho^p} dV.$$

For more generalizations see [5, 10, 11].

There are several important known results (see [12–15]) which reveal an importance of the scale invariance of the classical Hardy inequality in a ball. For more details, we further refer to [16–19] and the references therein.

As is well known, the n-sphere is of constant curvature and possess the delicate symmetry. However, there are only a few Hardy inequalities obtained on the n-sphere so far. Recently, Dai-Xu [20] and Xiao [21] discussed this issue and established some L^2-Hardy inequalities. By introducing the tangent function, Yin [22] acquired the following

$$\frac{n-2}{2}\int_{\mathbb{S}^n} f^2 dV + \int_{\mathbb{S}^n} |\nabla f|^2 dV \geqslant \frac{(n-2)^2}{4}\int_{\mathbb{S}^n} \frac{f^2}{\tan^2 d(p,x)} dV, \quad (1)$$

where p is a fixed point in $\mathbb{S}^n$, and the constant $\frac{(n-2)^2}{4}$ is sharp. Based on the results above, Abolarinwa-Apata [1], Abolarinwa-Rauf-Yin [23], and Sun-Pan [24] further gave some L^p-Hardy inequalities on the sphere.

MAIN RESULT

In this paper, we present a more general version of L^p Hardy inequalities on the unit n-sphere and show that the associated constant is the best possible, which is a generalization of those in [23] and [22]. Applications of the obtained inequality yield an uncertainty principle inequality and some corollaries. The main theorem is stated as follows:

Theorem 1 *Let $\mathbb{S}^n$ be the standard n-sphere with sectional curvature 1. Then for any function $f \in$*

$C^{\infty}(\mathbb{S}^n)$*, we have*

$$\int_{\mathbb{S}^n} (\sin^{\alpha} r)|\nabla f|^p dV + \left(\frac{n+\alpha-p}{p}\right)^{p-1} \int_{\mathbb{S}^n} \frac{|f|^p}{(\sin r)^{p-\alpha-2}} dV \geqslant \left(\frac{n+\alpha-p}{p}\right)^{p} \int_{\mathbb{S}^n} \frac{(\sin^{\alpha} r)|f|^p}{|\tan r|^p} dV,$$

where r is the distance function on the sphere, $1 \leqslant p < n+\alpha$ and the constant $(\frac{n+\alpha-p}{p})^p$ is sharp.

Comparing to the related inequalities, the inequality above has some important features. We give some remarks as follows.

Remark 1 The obtained inequality has a more complicated structure than that in [22]. But when $\alpha = 0$ and $p = 2$, Theorem 1 returns to the corresponding one (see (1)). By choosing different α and p, we can obtain some other interesting inequalities (see the next section). Besides, the dimension n can be chosen freely as long as $1 \leqslant p < n+\alpha$. Especially, in the case $n = 1$, the inequality holds on the circle with curvature $\kappa = 1$ and radius $R = 1/\kappa = 1$. Especially, the gradient and the Laplacian of a function are read as $\nabla f = f'$ and $\Delta f = f''$.

Remark 2 The L^p-Hardy inequalities obtained in [1, 23, 24] are divided into two cases: $1 \leqslant p < 2$ and $p \geqslant 2$. They declare that if $1 \leqslant p < 2$, then $\int_{\mathbb{S}^n} \frac{|f|^p}{(\sin r)^{p-2}} dV$ can not control $\int_{\mathbb{S}^n} \frac{|f|^p}{r^p} dV$ when $r \to 0$. Indeed, there is no need to do this because all integrals are convergent, and the right-hand side of the inequality is always less than the left-hand side. See the proof below for details.

Remark 3 In Euclidean spaces (respectively, Riemannian manifolds), the Laplacian of the distance function is equal to $(n-1)/|x|$ (respectively, is not less than C/ρ). So the Hardy inequality naturally contains the term $|f|^p/|x|^p$ (respectively, $|f|^p/\rho^p$). However, on the sphere $\Delta r = (n-1)\cot r$, it is natural to introduce the tangent function $\tan r$ in our inequality as in [22]. Also, the second term in the left-hand side cannot be removed because it will lead to a contradiction when f is a nonzero constant.

To prove the result, we follow the arguments in [22] (see also [23]) with some modifications. First, we construct an auxiliary function by utilizing the symmetry of the sphere, and then using the antipodal points. We will carry out the calculation in two hemispheres. Considering that the auxiliary function can only be continuous, we use an approximation of smooth functions to show the sharpness of the constant. Since the introduction of general p and α makes the calculation more complicated than that in [22], we need some techniques in the cumbersome computation below to estimate some terms in the L^p case.

Proof of Theorem 1

Let $f = \psi^{\gamma}\phi$ with $\gamma < 0$ be a smooth function in $C^{\infty}(\mathbb{S}^n)$. Then we have

$$|\nabla f|^p = |\phi\gamma\psi^{\gamma-1}\nabla\psi + \psi^{\gamma}\nabla\phi|^p.$$

Notice that the following inequality is valid for any $a, b \in \mathbb{R}^n$ and $p \geqslant 1$:

$$|a+b|^p - |a|^p \geqslant p|a|^{p-2}\langle a, b\rangle.$$

Therefore, one obtains

$$\begin{aligned}|\nabla f|^p &\geqslant |\phi|^p\psi^{\gamma p-p}|\gamma|^p|\nabla\psi|^p \\ &\quad + p|\gamma|^{p-2}|\phi|^{p-2}\psi^{\gamma p-2\gamma-p+2}\left\langle \phi\gamma\psi^{\gamma-1}\nabla\psi, \psi^{\gamma}\nabla\phi\right\rangle \\ &= |\phi|^p\psi^{\gamma p-p}|\gamma|^p|\nabla\psi|^p \\ &\quad + p|\gamma|^{p-2}\gamma|\phi|^{p-2}\phi\psi^{\gamma p-p+1}\langle\nabla\psi, \nabla\phi\rangle.\end{aligned}$$

Compute

$$\begin{aligned}\psi^{\alpha}|\nabla f|^p &\geqslant |\phi|^p\psi^{\gamma p-p+\alpha}|\gamma|^p|\nabla\psi|^p \\ &\quad + p|\gamma|^{p-2}\gamma|\phi|^{p-2}\phi\psi^{\gamma p-p+1+\alpha}\langle\nabla\psi, \nabla\phi\rangle \\ &= |\phi|^p\psi^{\gamma p-p+\alpha}|\gamma|^p|\nabla\psi|^p \\ &\quad + \frac{|\gamma|^{p-2}\gamma}{\gamma p-p+\alpha+2}\left\langle \nabla\psi^{\gamma p-p+\alpha+2}, \nabla\phi^p\right\rangle \\ &= |\phi|^p\psi^{\gamma p-p+\alpha}|\gamma|^p|\nabla\psi|^p \\ &\quad + \frac{|\gamma|^{p-2}\gamma}{\gamma p-p+\alpha+2}\operatorname{div}\left(\phi^p\nabla\psi^{\gamma p-p+\alpha+2}\right) \\ &\quad - \frac{|\gamma|^{p-2}\gamma}{\gamma p-p+\alpha+2}\phi^p\Delta\left(\psi^{\gamma p-p+\alpha+2}\right), \quad (2)\end{aligned}$$

and

$$\begin{aligned}\Delta(\sin r)^{-\beta} &= \operatorname{div}\left(\nabla(\sin r)^{-\beta}\right) \\ &= \operatorname{div}\left(-\beta(\sin r)^{-\beta-1}\cos r\nabla r\right) \\ &= -\beta(\sin r)^{-\beta-1}\cos r\Delta r \\ &\quad + \beta(\beta+1)(\sin r)^{-\beta-2}\cos^2 r + \beta(\sin r)^{-\beta}. \quad (3)\end{aligned}$$

Substituting $\Delta r = (n-1)\cot r$ into (3) yields

$$\begin{aligned}\Delta(\sin r)^{-\beta} &= \beta(\beta+2-n)(\sin r)^{-\beta-2} \\ &\quad + \beta(n-\beta-1)(\sin r)^{-\beta}.\end{aligned}$$

Now by taking $\beta=-(\gamma p-p+\alpha+2)$ and $\gamma=\frac{p-\alpha-n}{p}$, we derive

$$\Delta(\sin r)^{-\beta}=(n-2)(\sin r)^{2-n}.$$

Further, letting $\psi=\sin r$, it follows from (2) that

$$\begin{aligned}&(\sin^\alpha r)|\nabla f|^p\\ &\quad\geqslant(\sin r)^{\gamma p}|f|^p(\sin r)^{-n}\left(\tfrac{n+\alpha-p}{p}\right)^p|\cos r|^p\\ &\qquad+\tfrac{1}{n-2}\left(\tfrac{n+\alpha-p}{p}\right)^{p-1}\operatorname{div}\left(\phi^p\nabla\psi^{\gamma p-p+\alpha+2}\right)\\ &\qquad-\tfrac{1}{n-2}\left(\tfrac{n+\alpha-p}{p}\right)^{p-1}(\sin r)^{\gamma p}|f|^p(n-2)(\sin r)^{2-n}\\ &\quad=\left(\tfrac{n+\alpha-p}{p}\right)^p(\sin^\alpha r)|f|^p\left|\tfrac{\cos r}{\sin r}\right|^p\\ &\qquad+\tfrac{1}{n-2}\left(\tfrac{n+\alpha-p}{p}\right)^{p-1}\operatorname{div}\left(\phi^p\nabla\psi^{\gamma p-p+\alpha+2}\right)\\ &\qquad-\left(\tfrac{n+\alpha-p}{p}\right)^{p-1}(\sin r)^{\alpha-p+2}|f|^p\\ &\quad=\left(\tfrac{n+\alpha-p}{p}\right)^p\tfrac{(\sin^\alpha r)|f|^p}{|\tan r|^p}\\ &\qquad-\left(\tfrac{n+\alpha-p}{p}\right)^{p-1}\tfrac{|f|^p}{(\sin r)^{p-\alpha-2}}\\ &\qquad+\tfrac{1}{n-2}\left(\tfrac{n+\alpha-p}{p}\right)^{p-1}\operatorname{div}\left(\phi^p\nabla\psi^{\gamma p-p+\alpha+2}\right).\end{aligned}$$

Integrating both sides of the above inequality on $\mathbb{S}^n$, and applying the divergence theorem, we deduce that

$$\begin{aligned}&\int_{\mathbb{S}^n}(\sin^\alpha r)|\nabla f|^p dV\\ &\qquad+\left(\frac{n+\alpha-p}{p}\right)^{p-1}\int_{\mathbb{S}^n}\frac{|f|^p}{(\sin r)^{p-\alpha-2}}dV\\ &\qquad\qquad\geqslant\left(\frac{n+\alpha-p}{p}\right)^p\int_{\mathbb{S}^n}\frac{(\sin^\alpha r)|f|^p}{|\tan r|^p}dV.\end{aligned}$$

This completes the proof of the inequality stated in Theorem 1. Next, we show that the constant $(\frac{n+\alpha-p}{p})^p$ is sharp. Let $\zeta:R\to[0,1]$ be a smooth function such that $0\leqslant\zeta\leqslant1$ and

$$\zeta(t)=\begin{cases}1, & |t|\leqslant1;\\ 0, & |t|\geqslant2.\end{cases}$$

Let $H(t)=1-\zeta(t)$, and for sufficient small $\varepsilon>0$ we construct

$$f_\varepsilon(r)=\begin{cases}0, & r=0;\\ H\left(\frac{r}{\varepsilon}\right)(\tan r)^{\frac{p-\alpha-n}{p}}, & 0<r\leqslant\frac{\pi}{2};\\ H\left(\frac{\pi-r}{\varepsilon}\right)(\tan(\pi-r))^{\frac{p-\alpha-n}{p}}, & \frac{\pi}{2}\leqslant r<\pi;\\ 0, & r=\pi.\end{cases}$$

Observe that $f_\varepsilon(r)$ is continuous and can be approximated by smooth functions on the sphere $\mathbb{S}^n$.

For the antipodal points q and q' on $\mathbb{S}^n$, let r_q (respectively, r'_q) denote the distance function from q (respectively, q'). Then we have $r_q+r'_q=\pi$, and thus

$$\begin{aligned}\int_{\mathbb{S}^n}\frac{|f_\varepsilon|^p}{(\sin r)^{p-\alpha-2}}dV&=\int_{B_q(\frac{\pi}{2})}\frac{|f_\varepsilon|^p}{\left(\sin r_q\right)^{p-\alpha-2}}dV\\ &\qquad+\int_{B_{q'}(\frac{\pi}{2})}\frac{|f_\varepsilon|^p}{(\sin r_{q'})^{p-\alpha-2}}dV,\end{aligned}$$

where

$$\begin{aligned}&\int_{B_q(\frac{\pi}{2})}\frac{|f_\varepsilon|^p}{(\sin r_q)^{p-\alpha-2}}dV\\ &\qquad=\mathrm{Vol}(\mathbb{S}^{n-1})\int_\varepsilon^{\frac{\pi}{2}}H^p\left(\tfrac{r_q}{\varepsilon}\right)(\tan r_q)^{p-\alpha-n}\\ &\qquad\qquad\times(\sin r_q)^{2+\alpha-p}(\sin r_q)^{n-1}dr\\ &\qquad\leqslant\mathrm{Vol}(\mathbb{S}^{n-1})\int_\varepsilon^{\frac{\pi}{2}}r_q dr=\tfrac{\mathrm{Vol}(\mathbb{S}^{n-1})}{2}\left(\tfrac{\pi^2}{4}-\varepsilon^2\right)\end{aligned}$$

and

$$\begin{aligned}&\int_{B_{q'}(\frac{\pi}{2})}\frac{|f_\varepsilon|^p}{(\sin r_{q'})^{p-\alpha-2}}dV\\ &\qquad=\mathrm{Vol}(\mathbb{S}^{n-1})\int_{\frac{\pi}{2}}^{\pi-\varepsilon}H^p\left(\tfrac{\pi-r_q}{\varepsilon}\right)(\tan(\pi-r_q))^{p-\alpha-n}\\ &\qquad\qquad\times(\sin(\pi-r_q))^{2+\alpha-p}(\sin(\pi-r_q))^{n-1}dr\\ &\qquad\leqslant\mathrm{Vol}(\mathbb{S}^{n-1})\int_\varepsilon^{\frac{\pi}{2}}r_{q'}dr=\tfrac{\mathrm{Vol}(\mathbb{S}^{n-1})}{2}\left(\tfrac{\pi^2}{4}-\varepsilon^2\right).\end{aligned}$$

Combing the above two inequalities, we obtain

$$\int_{\mathbb{S}^n}\frac{|f_\varepsilon|^p}{(\sin r)^{p-\alpha-2}}dV\leqslant\mathrm{Vol}(\mathbb{S}^{n-1})\left(\tfrac{\pi^2}{4}-\varepsilon^2\right).\tag{4}$$

On the other hand, we have

$$\begin{aligned}&\int_{B_q(\frac{\pi}{2})}\frac{(\sin r_q)^\alpha|f_\varepsilon|^p}{(\tan r_q)^p}dV\\ &\quad=\mathrm{Vol}(\mathbb{S}^{n-1})\\ &\qquad\times\int_\varepsilon^{\frac{\pi}{2}}H^p\left(\tfrac{r_q}{\varepsilon}\right)(\tan r_q)^{p-\alpha-n}\tfrac{(\sin r_q)^\alpha}{(\tan r_q)^p}(\sin r_q)^{n-1}dr\\ &\quad\geqslant\mathrm{Vol}(\mathbb{S}^{n-1})\\ &\qquad\times\int_{2\varepsilon}^{\frac{\pi}{2}}H^p\left(\tfrac{r_q}{\varepsilon}\right)(\tan r_q)^{-\alpha-n}(\sin r_q)^{\alpha+n-1}dr\\ &\quad=\mathrm{Vol}(\mathbb{S}^{n-1})\int_{2\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{-\alpha-n}(\sin r_q)^{\alpha+n-1}dr\end{aligned}$$

and

$$\int_{B_{q'}(\frac{\pi}{2})} \frac{(\sin r_{q'})^{\alpha}|f_{\varepsilon}|^p}{(\tan r_{q'})^p} dV$$
$$= \mathrm{Vol}(\mathbb{S}^{n-1}) \int_{\varepsilon}^{\frac{\pi}{2}} H^p\left(\tfrac{\pi - r_q}{\varepsilon}\right)\big(\tan(\pi - r_q)\big)^{p-\alpha-n}$$
$$\times \frac{\big(\sin(\pi - r_q)\big)^{\alpha}}{\big(\tan(\pi - r_q)\big)^{p}}\big(\sin(\pi - r_q)\big)^{n-1} dr$$
$$= \mathrm{Vol}(\mathbb{S}^{n-1}) \int_{\varepsilon}^{\frac{\pi}{2}} H^p\left(\tfrac{r_{q'}}{\varepsilon}\right)(\tan r_{q'})^{p-\alpha-n}$$
$$\times \frac{(\sin r_{q'})^{\alpha}}{(\tan r_{q'})^p}(\sin r_{q'})^{n-1} dr$$
$$\geqslant \mathrm{Vol}(\mathbb{S}^{n-1}) \int_{2\varepsilon}^{\frac{\pi}{2}} H^p\left(\tfrac{r_{q'}}{\varepsilon}\right)(\tan r_{q'})^{-\alpha-n}$$
$$\times (\sin r_{q'})^{\alpha+n-1} dr$$
$$= \mathrm{Vol}(\mathbb{S}^{n-1}) \int_{2\varepsilon}^{\frac{\pi}{2}} (\tan r_{q'})^{-\alpha-n}(\sin r_{q'})^{\alpha+n-1} dr.$$

Adding the two inequalities above together gives

$$\int_{\mathbb{S}^n} \frac{(\sin^{\alpha} r)\,|f_{\varepsilon}|^p}{|\tan r|^p} dV \geqslant 2\mathrm{Vol}(\mathbb{S}^{n-1}) \times \int_{2\varepsilon}^{\frac{\pi}{2}} (\tan r_q)^{-\alpha-n}(\sin r_q)^{\alpha+n-1} dr. \quad (5)$$

Next, we are going to estimate the integral

$$\int_{\mathbb{S}^n} (\sin^{\alpha} r)\,|\nabla f_{\varepsilon}|^p dV = \int_{B_q(\frac{\pi}{2})} \big(\sin r_q\big)^{\alpha}|\nabla f_{\varepsilon}|^p dV + \int_{B_{q'}(\frac{\pi}{2})} (\sin r_{q'})^{\alpha}|\nabla f_{\varepsilon}|^p dV.$$

By a cumbersome calculation, we have

$$\left(\int_{B_q(\frac{\pi}{2})} (\sin r_q)^{\alpha}|\nabla f_{\varepsilon}|^p dV\right)^{\frac{1}{p}}$$
$$= \mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}\Bigg(\int_{\varepsilon}^{\frac{\pi}{2}} \Big| H'\left(\tfrac{r_q}{\varepsilon}\right)\tfrac{1}{\varepsilon}(\tan r_q)^{p-\alpha-n}$$
$$+\left(\tfrac{p-\alpha-n}{p}\right) H\left(\tfrac{r_q}{\varepsilon}\right)(\tan r_q)^{-\frac{n+\alpha}{p}}\sec^2 r_q\Big|^p (\sin r_q)^{n-1} dr\Bigg)^{\frac{1}{p}}$$
$$\leqslant \tfrac{\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}}{\varepsilon}\left(\int_{\varepsilon}^{\frac{\pi}{2}}\Big|H'\left(\tfrac{r_q}{\varepsilon}\right)\Big|^p(\tan r_q)^{p-\alpha-n}(\sin r_q)^{\alpha+n-1} dr\right)^{\frac{1}{p}}$$
$$+\left(\tfrac{n+\alpha-p}{p}\right)\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}} \times \left(\int_{\varepsilon}^{\frac{\pi}{2}} H^p\left(\tfrac{r_q}{\varepsilon}\right)(\tan r_q)^{-n-\alpha}(\sin r_q)^{\alpha+n-1} dr\right)^{\frac{1}{p}}$$
$$+\left(\tfrac{n+\alpha-p}{p}\right)\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}} \times \left(\int_{\varepsilon}^{\frac{\pi}{2}} H^p\left(\tfrac{r_q}{\varepsilon}\right)(\tan r_q)^{2p-n-\alpha}(\sin r_q)^{\alpha+n-1} dr\right)^{\frac{1}{p}}$$
$$= \tfrac{\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}}{\varepsilon}\left(\int_{\varepsilon}^{2\varepsilon}\Big|H'\left(\tfrac{r_q}{\varepsilon}\right)\Big|^p(\tan r_q)^{p-\alpha-n}(\sin r_q)^{\alpha+n-1} dr\right)^{\frac{1}{p}}$$
$$+\left(\tfrac{n+\alpha-p}{p}\right)\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}} \times \left(\int_{\varepsilon}^{\frac{\pi}{2}} H^p\left(\tfrac{r_q}{\varepsilon}\right)(\tan r_q)^{-n-\alpha}(\sin r_q)^{\alpha+n-1} dr\right)^{\frac{1}{p}}$$
$$+\left(\tfrac{n+\alpha-p}{p}\right)\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}} \times \left(\int_{\varepsilon}^{\frac{\pi}{2}} H^p\left(\tfrac{r_q}{\varepsilon}\right)(\tan r_q)^{2p-n-\alpha}(\sin r_q)^{\alpha+n-1} dr\right)^{\frac{1}{p}}$$
$$\leqslant \tfrac{\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}}{\varepsilon}\max_{t\in[0,2]} H'(t)\left(\int_{\varepsilon}^{2\varepsilon} r_q\, dr\right)^{\frac{1}{p}}$$
$$+\tfrac{n+\alpha-p}{p}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}\left(\int_{\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{-n-\alpha}(\sin r_q)^{\alpha+n-1} dr\right)^{\frac{1}{p}}$$
$$+\tfrac{n+\alpha-p}{p}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}\left(\int_{\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{2p-n-\alpha}(\sin r_q)^{\alpha+n-1} dr\right)^{\frac{1}{p}}$$
$$= \left(\tfrac{2^p-1}{p}\right)^{\frac{1}{p}}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}\max_{t\in[0,2]} H'(t)$$
$$+\tfrac{n+\alpha-p}{p}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}\left(\int_{\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{-n-\alpha}(\sin r_q)^{\alpha+n-1} dr\right)^{\frac{1}{p}}$$
$$+\tfrac{n+\alpha-p}{p}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}\left(\int_{\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{2p-n-\alpha}(\sin r_q)^{\alpha+n-1} dr\right)^{\frac{1}{p}}$$

and

$$\int_{B_{q'}(\frac{\pi}{2})} (\sin r_{q'})^{\alpha}|\nabla f_{\varepsilon}|^p dV$$
$$= \mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}\Bigg(\int_{\frac{\pi}{2}}^{\pi-\varepsilon}\Big|H'\left(\tfrac{\pi - r_q}{\varepsilon}\right)\left(\tfrac{-1}{\varepsilon}\right)\big(\tan(\pi - r_q)\big)^{p-\alpha-n}$$
$$-\tfrac{p-\alpha-n}{p} H\left(\tfrac{\pi - r_q}{\varepsilon}\right)\big(\tan(\pi - r_q)\big)^{-\frac{n+\alpha}{p}}$$
$$\times \sec^2(\pi - r_q)\Big|^p(\sin(\pi - r_q))^{\alpha+n-1} dr\Bigg)^{\frac{1}{p}}$$

$$= \mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}\Bigg(\int_{\varepsilon}^{\frac{\pi}{2}}\Big|H'\big(\tfrac{r_{q'}}{\varepsilon}\big)\big(\tfrac{-1}{\varepsilon}\big)(\tan r_{q'})^{p-\alpha-n} - \tfrac{p-\alpha-n}{p}H\big(\tfrac{r_{q'}}{\varepsilon}\big)(\tan r_{q'})^{-\frac{n+\alpha}{p}}\sec^2 r_{q'}\Big|^p \times (\sin r_{q'})^{\alpha+n-1}dr\Bigg)^{\frac{1}{p}}$$

$$\leqslant \Big(\tfrac{2^p-1}{p}\Big)^{\frac{1}{p}}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}\max_{t\in[0,2]}H'(t) + \tfrac{n+\alpha-p}{p}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}\Bigg(\int_{\varepsilon}^{\frac{\pi}{2}}(\tan r_{q'})^{-n-\alpha}(\sin r_{q'})^{\alpha+n-1}dr\Bigg)^{\frac{1}{p}} + \tfrac{n+\alpha-p}{p}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}\Bigg(\int_{\varepsilon}^{\frac{\pi}{2}}(\tan r_{q'})^{2p-n-\alpha}(\sin r_{q'})^{\alpha+n-1}dr\Bigg)^{\frac{1}{p}}.$$

Therefore, it is not difficult to obtain that

$$\int_{\mathbb{S}^n}\sin^\alpha r|\nabla f_\varepsilon|^p dV \leqslant 2\Bigg[\Big(\tfrac{2^p-1}{p}\Big)^{\frac{1}{p}}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}\max_{t\in[0,2]}H'(t) + \tfrac{n+\alpha-p}{p}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}\Bigg(\int_{\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{-n-\alpha}(\sin r_q)^{\alpha+n-1}dr\Bigg)^{\frac{1}{p}} + \tfrac{n+\alpha-p}{p}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}} \times \Bigg(\int_{\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{2p-n-\alpha}(\sin r_q)^{\alpha+n-1}dr\Bigg)^{\frac{1}{p}}\Bigg]^p. \quad (6)$$

Define

$$A := \inf_{f\in C^\infty(\mathbb{S}^n)\setminus\{0\}}\Bigg[\int_{\mathbb{S}^n}(\sin^\alpha r)|\nabla f|^p dV + \Big(\tfrac{n+\alpha-p}{p}\Big)^{p-1}\int_{\mathbb{S}^n}\frac{|f|^p}{(\sin r)^{p-\alpha-2}}dV\Bigg]\Bigg/\int_{\mathbb{S}^n}\frac{(\sin^\alpha r)|f|^p}{|\tan r|^p}dV.$$

Since $f_\varepsilon(r)$ can be approximated by smooth functions on the sphere $\mathbb{S}^n$, it follows from (4)–(6) that

$$A \leqslant \frac{\int_{\mathbb{S}^n}(\sin^\alpha r)|\nabla f_\varepsilon|^p dV + \Big(\frac{n+\alpha-p}{p}\Big)^{p-1}\int_{\mathbb{S}^n}\frac{|f_\varepsilon|^p}{(\sin r)^{p-\alpha-2}}dV}{\int_{\mathbb{S}^n}\frac{(\sin^\alpha r)|f_\varepsilon|^p}{|\tan r|^p}dV}$$

$$\leqslant \frac{\mathrm{Vol}(\mathbb{S}^{n-1})\Big(\frac{\pi^2}{4}-\varepsilon^2\Big)}{2\mathrm{Vol}(\mathbb{S}^{n-1})\int_{2\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{-\alpha-n}(\sin r_q)^{\alpha+n-1}dr} + 2\Bigg[\frac{\Big(\frac{2^p-1}{p}\Big)^{\frac{1}{p}}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}\max_{t\in[0,2]}H'(t)}{2^{\frac{1}{p}}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}\Big(\int_{2\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{-\alpha-n}(\sin r_q)^{\alpha+n-1}dr\Big)^{\frac{1}{p}}} + \frac{\frac{n+\alpha-p}{p}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}\Big(\int_{\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{-n-\alpha}(\sin r_q)^{\alpha+n-1}dr\Big)^{\frac{1}{p}}}{2^{\frac{1}{p}}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}\Big(\int_{2\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{-\alpha-n}(\sin r_q)^{\alpha+n-1}dr\Big)^{\frac{1}{p}}} + \frac{\frac{n+\alpha-p}{p}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}\Big(\int_{\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{2p-n-\alpha}(\sin r_q)^{\alpha+n-1}dr\Big)^{\frac{1}{p}}}{2^{\frac{1}{p}}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{p}}\Big(\int_{2\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{-\alpha-n}(\sin r_q)^{\alpha+n-1}dr\Big)^{\frac{1}{p}}}\Bigg]^p$$

$$:= I + 2(II + III + IV)^p. \quad (7)$$

Obviously, we have

$$\lim_{\varepsilon\to 0}\int_{2\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{-\alpha-n}(\sin r_q)^{\alpha+n-1}dr = \infty.$$

By L'Hôpital rule, it holds that

$$\lim_{\varepsilon\to 0}\frac{\int_{\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{-\alpha-n}(\sin r_q)^{\alpha+n-1}dr}{\int_{2\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{-\alpha-n}(\sin r_q)^{\alpha+n-1}dr} = 1$$

and

$$\lim_{\varepsilon\to 0}\frac{\int_{\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{2p-n-\alpha}(\sin r_q)^{\alpha+n-1}dr}{\int_{2\varepsilon}^{\frac{\pi}{2}}(\tan r_q)^{-\alpha-n}(\sin r_q)^{\alpha+n-1}dr} = 0.$$

This means $I = II = IV = 0$. Therefore, we get from (7) that

$$A \leqslant 2\left(\frac{\frac{n+\alpha-p}{p}}{2^{\frac{1}{p}}}\right)^p = \left(\frac{n+\alpha-p}{p}\right)^p.$$

The reverse inequality is also valid by (2). Thus the constant $(\frac{n+\alpha-p}{p})^p$ is sharp.

COROLLARIES

Choosing some special $\alpha = 0$ and $\alpha = p$ in Theorem 1, we obtain the following results.

Corollary 1 *Let $\mathbb{S}^n$ be the standard n-sphere as in Theorem 1. Then*

$$\int_{\mathbb{S}^n}|\nabla f|^p dV + \Big(\tfrac{n-p}{p}\Big)^{p-1}\int_{\mathbb{S}^n}\frac{|f|^p}{(\sin r)^{p-2}}dV \geqslant \Big(\tfrac{n-p}{p}\Big)^p\int_{\mathbb{S}^n}\frac{|f|^p}{|\tan r|^p}dV,$$

and the constant $(\frac{n-p}{p})^p$ is sharp.

Remark 4 When $\alpha = 0$ the corresponding inequality is obtained in [23], where it is divided into two cases: $1 \leqslant p < 2$ and $p \geqslant 2$ due to some technical reasons. In fact, we can combine them into a unified inequality as above, and thus Corollary 1 is in a form more concise than that in [23].

Corollary 2 *Let $\mathbb{S}^n$ be the standard n-sphere as in Theorem 1. Then*

$$\int_{\mathbb{S}^n} ((\sin r)|\nabla f|)^p dV + \left(\tfrac{n}{p}\right)^{p-1} \int_{\mathbb{S}^n} (\sin^2 r)|f|^p dV \geqslant \left(\tfrac{n}{p}\right)^p \int_{\mathbb{S}^n} (|\cos r||f|)^p dV,$$

and the constant $(\frac{n}{p})^p$ is sharp.

Remark 5 This new version of L^p-Hardy inequality is stronger than Corollary 3.2 in [22]. Indeed, if $p = 2$, it gives

$$\int_{\mathbb{S}^n} |\nabla f|^2 \sin^2 r\, dV + \frac{n}{2} \int_{\mathbb{S}^n} f^2 \sin^2 r\, dV \geqslant \frac{n^2}{4} \int_{\mathbb{S}^n} f^2 \cos^2 r\, dV,$$

which yields that

$$\int_{\mathbb{S}^n} |\nabla f|^2 dV + \frac{n}{2} \int_{\mathbb{S}^n} f^2 dV \geqslant \frac{n^2}{4} \int_{\mathbb{S}^n} f^2 \cos^2 r\, dV,$$

while in [22], the coefficient in the right-hand side is $n(n-2)/4$.

The classical uncertainty principle as introduced in quantum mechanics says that the position and the momentum of a particle can not be exactly determined at the same time, but only with an "uncertainty". There are various forms of the uncertainty principle. At present we shall apply Theorem 1 to derive a new form as follows.

Corollary 3 *Let $\mathbb{S}^n$ be the standard n-sphere as in Theorem 1. Then*

$$\left(\int_{\mathbb{S}^n} |f|^p \sin^\alpha r |\tan r|^q dV\right)^{\frac{p}{q}} \left[\int_{\mathbb{S}^n} \sin^\alpha r |\nabla f|^p dV + \left(\tfrac{n+\alpha-p}{p}\right)^{p-1} \int_{\mathbb{S}^n} \frac{|f|^p}{(\sin r)^{p-\alpha-2}} dV\right] \geqslant \left(\tfrac{n+\alpha-p}{p}\right)^p \left(\int_{\mathbb{S}^n} |f|^p \sin^\alpha r\, dV\right)^p,$$

where $1/p + 1/q = 1$.

Proof: By Hölder's inequality, we have

$$\int_{\mathbb{S}^n} |f|^p \sin^\alpha r\, dV \leqslant \left(\int_{\mathbb{S}^n} \frac{|f|^p \sin^\alpha r}{|\tan r|^p} dV\right)^{\frac{1}{p}} \left(\int_{\mathbb{S}^n} |f|^p \sin^\alpha r |\tan r|^q dV\right)^{\frac{1}{q}}.$$

A simple calculation yields

$$\int_{\mathbb{S}^n} \frac{|f|^p \sin^\alpha r}{|\tan r|^p} dV \geqslant \left(\int_{\mathbb{S}^n} |f|^p \sin^\alpha r\, dV\right)^p \left(\int_{\mathbb{S}^n} |f|^p \sin^\alpha r |\tan r|^q dV\right)^{-\frac{p}{q}}.$$

Combining it with Theorem 1, the result follows directly. □

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