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Re-characterization of $PGL(2,p)$ by its order and one class length

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ABSTRACT: The present article used the classification of simple groups to prove that group G is isomorphic to the projective linear group $PGL(2,p)$ if $|G| = p(p^2-1)$ and $p^2-1 \in N(G)$, where $N(G)$ is the collection of conjugacy class length of G and p is a prime.

KEYWORDS: finite group, projective linear group, group order, conjugacy class length

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INTRODUCTION

For a finite group G, $\pi(G)$ is a collection whose elements are prime factors of $|G|$, and $\Gamma(G)$ is the *prime graph* related to G, whose collection of vertices is $\pi(G)$. Any two elements p and q of $\pi(G)$ are adjacent when and only when there is an element of order pq in G [1]. Further, $T(G)$ is the set of connected branches of $\Gamma(G)$, and $\pi_i(G)$ is its ith connected branch. It follows that $|G|$ can be written as a product of $m_1, m_2, \dots, m_{|T(G)|}$, where the prime factors set of m_i is $\pi_i(G)$. We call these m_i's the *order components* of G, and usually let $2 \in \pi_1(G)$ when $2 \mid |G|$.

Thompson posted a conjecture in 1988 [2]: *Let G be a finite group with $Z(G) = 1$. Then G is isomorphic to L if $N(G) = N(L)$, where L is a non-abelian simple group.* Recently, this conjecture was proved completely for all finite non-abelian simple group [3–18]. Now, many mathematicians begin to study the extensive problems of the conjecture in different ways. Several years ago, Chen and his students only used group order and some class lengths to study this conjecture, and successfully characterized almost sporadic simple groups, alternative groups with prime degree, almost K_3-simple groups, K_4-simple groups and linear groups $PSL_4(4)$, $PSL_2(p)$ and so on [19–26], by which they proved the conjecture to be true for these groups. Soon, it caught the attention of Asboei et al, and they characterized alternative groups A_{p+1}, A_{p+2} (p is a prime) classical groups ${}^2D_n(2)$, ${}^2D_{n+1}(2)$ ($2^n+1>5$ is a prime) by their order and some class lengths [27, 28].

Here we continue to study this topic. By referring to the existing literature related to the study of Thompson's conjecture, we find that these literature except [9, 23], all used the simple group classification theorem. Paper [9] avoided the use of simple group classification due to Held's results in 1968 [29], while [23] avoided the simple group classification with the help of Brauer and Reynolds' results in 1958 [30]. It follows that if we want to avoid using the simple group classification theorem in the process of characterizing some finite almost simple groups, we have to resort to the literature before the simple group classification. Therefore, although the research ideas in literature [9, 23] are nice, they have certain limitations. Since most of the results related to Thompson's conjecture are proved by the simple group classification theorem, can all of them be proved by the classification theorem? In particular, the results of literature [9] can also be proved by means of the simple group classification theorem [7, 10]. Therefore, we used the classification theorem of simple groups to study this characterization problem of the projective linear groups $PGL(2,p)$ in the present paper.

Besides, let $\epsilon = \pm 1$ and n_π be π-part of a positive integer n. Other unspecified symbols can be found in references [31, 32].

LEMMAS

Here some lemmas about *Frobenius group* and *2-Frobenius group* are quoted which are necessary to the proof of main theorem, where Lemmas 1 and 2 are from [33, 34], and Lemma 3 is from [1].

Lemma 1 ([33]) *Let G be a Frobenius group of even order, M be its kernel, and N be the complement of M in G. Then $T(G)=\{\pi(M),\pi(N)\}$.*

Lemma 2 ([33, 34]) *If G is a 2-Frobenius group of even order, then there is a normal series $1\subseteq M\subseteq N\subseteq G$ satisfying that $T(G)=\{\pi(N/M),\pi(M)\cup\pi(G/N)\}$, and $|G/N|\mid|\mathrm{Out}(M/N)|$. Especially, if M is an elementary abelian p-group of order p^n and N/M is a cyclic group of order p^n-1, then $|G/N|\mid n$.*

Lemma 3 ([1]) *Let G be a finite group. If $\Gamma(G)$ has more than two connected branches, then G is isomorphic to one of the following structures:*
(i) *a Frobenius group;*
(ii) *a 2-Frobenius group;*
(iii) *simple;*
(iv) *an extension of a π_1-group by a simple graup;*
(v) *an extension of a simple graup by a π_1-solvable group;*
(vi) *an extension of a π_1-group by a simple graup, and by a π_1-group.*

CHARACTERIZATION OF $PGL(2,p)$

Theorem 1 *If a group G has order of $p(p^2-1)$ with a prime p, and p^2-1 is an element of $N(G)$, then G is isomorphic to the projective linear group $PGL(2,p)$.*

Proof: By the assumption, one has an element $v\in G$ with $|v|=p$ satisfying that $\langle v\rangle$ is self-centered in G. Then each subgroup of order p of G is self-centered by Sylow theorem, and so $\{p\}$ is a connected branch of $\Gamma(G)$; moreoever, $\{p\}\in T(G)$, which means that the number of connected branches of $\Gamma(G)$ is more than two. So we can obtain that G is one of the groups in Lemma 3. Note that $p=max(\pi(G))$. Furthermore, there exist two normal subgroups M and N such that there is a normal series $1\subseteq M\subseteq N\subseteq G$.

If G is a Frobenius group with kernel M and its complement T, then $|T|\mid(|M|-1)$. Let $p\mid|M|$. Then, by Lemma 1, T has order of $(p^2-1)/2$, and M has order of p, which means that p is equal to 1, contradicting $p\geqslant 2$. Thus $p\mid|T|$, and so $|T|=p$ and $|M|=p^2-1$, which implies $p=2$. Therefore, $G\cong PGL(2,2)$, as expected.

If N and G/M are Frobenius groups, M and N/M are their kernels, then G is a 2-Frobenius group. Then, by Lemma 2, one can get that the set $\pi_1(G)$ is equal to $\pi(M)\cup\pi(G/N)$, and $\pi_2(G)$ is equal to $\pi(N/M)$, and so $\pi_2(G)=\{p\}$. It follows that $|N/M|=p$ and $|M|=p+1$. If M is not a 2-group, then there is an odd prime f satisfying that $|M_f|<p$, and thus p and $|\mathrm{Aut}(M_f)|$ are relatively prime. It follows that there is an edge between f and p in $\Gamma(G)$, contradicting $\{p\}\in T(G)$. Hence M must be a 2-group. Further, M is an elementary commutative group. Otherwise, p and $|\mathrm{Aut}(M)|$ are coprime, which implies that 2 and p are adjacent in $\Gamma(G)$, a contradiction. Therefore M is elementary. Let $|M|=2^t$, and then by Lemma 2, $|G/N|=(2^t-2)\mid t$, which means that $t=2$ and $p=3$. It follows that $|G|=24$, and each subgroup of order 3 in G is self-centered. Checking all groups of order 24, we can get that G must be isomorphic to $PGL(2,3)$ as desired.

Now, by Lemma 3, we know that there is a normal subgroup series $1\trianglelefteq M\trianglelefteq N\trianglelefteq G$ satisfying that N/M is a non-communicative simple group, M is a nilpotent group, and $\pi(M)\cup\pi(G/N)\subseteq\pi_1(G)$. Further, $N/M\leqslant G/M\leqslant\mathrm{Aut}(N/M)$, $|G/N|\mid|\mathrm{Out}(N/M)|$, and each uneven order component of G is N/M's. It follows that N/M has an order component p, which means that N/M has at least two order components. Therefore, N/M is one of groups in Tables 1–3 of [34], and p must be bigger than 5.

According to all possible cases of N/M, we will take part in 3 steps to complete the following proof.

Step 1. It is supposed that N/M is one group of Table 1 in [34]. Let N/M be one of M_{12}, Ru, J_2, Co_1, He, M^cL, Co_3, HN, F_{22}, ${}^2A_3(2)$ and ${}^2F_4(2)'$. It follows that p is a prime number that is not more than 30, and $|N/M|_2\mid(p^2-1)$. By [31], one has that $|N/M|_2\geqslant 2^6$, which contradicts $(p^2-1)_2\leqslant 2^5$.

Let $N/M=A_{p'-1}(q')$, where the ordered pair (p',q') is not equal to any of $(3,2)$ and $(3,4)$. Then p is equal to $\frac{q'^{p'}-1}{(q'-1)(p',q'-1)}$ and $q'^{\frac{p'(p'-1)}{2}}$ is a nontrivial factor of p^2-1. Hence $q'^{\frac{p'(p'-1)}{2}}<p^2<q'^{2p'}$, and thus $(p'(p'-1))/2<2p'$, which applies $p'=3$. It follows that $q'^3(q'-1)^2(q'+1)\leqslant(p^2-1)$ and $p^2=\frac{(q'^3-1)^2}{[(q'-1)(3,q'-1)]^2}$. In view of $4q'^4>p^2\geqslant 2(q'^2-1)q'^3$, one has $q'^2-2q'-1<0$. Hence $q'=2$ when $p'=3$, a contradiction.

Let $N/M=E_6(q')$. Then $q'^{36}(q'^2-1)\mid(p^2-1)$ and $p=\left(q'^6+q'^3+1\right)/(3,q'-1)$. Therefore, we can obtain $q'^{36}<p^2<q'^{18}$, which is a contradictory inequality.

Let $N/M={}^3D_4(q')$. Then $(q'^2-1)q'^{12}\mid(p^2-1)$ and $p=q'^4-q'^2+1$, which mean that $q'^{12}<p^2<q'^8$, a contradictory inequality.

Let $N/M={}^2A_{p'}(q')$, where the ordered pair (p',q') is not equal to any of $(3,3)$ and $(5,2)$, and $q'+1$ is a factor of $(p'+1)$. Then $q'^{\frac{p'(p'+1)}{2}}$

is a nontrivial factor of p^2-1, and p is equal to $\left(q'^{p'}+1\right)/(q'+1)$. Therefore, $q'^{\frac{p'(p'+1)}{2}} < q'^{2(p'+1)}$, which implies $p'=3$. Thus $q'^6 < p^2 \leqslant q'^4$, which is a contradictory inequality.

Let $N/M = {}^2D_{p'}(3)$, where p' is more than 5, and p' is not equal to $2^{m'}+1$. Then $3^{p'(p'-1)}$ is nontrivial factor of p^2-1, and $p=\left(3^{p'}+1\right)/4$. Hence $3^{p'(p'-1)} < p^2 < 3^{2(p'+1)}$, and thus $0<p'<4$, which is impossible.

Let $N/M = {}^2D_{n'}(q')$, where the number n' is a power of 2, and more than 4. Then $q'^{n'(n'-1)}$ is a nontrivial factor of p^2-1, and $p=\left(q'^{n'}+1\right)/(2,q'+1)$. It follows that $q'^{n'(n'-1)} < p^2 < q'^{2(n'+1)}$, and thus $n'=2$, which is impossible.

Let $N/M = B_{p'}(3)$. We get $3^{p'^2}(3^{p'}+1) \mid (p^2-1)$ and $p=\left(3^{p'}-1\right)/2$. It follows that $3^{p'^2} < p^2 < 3^{2p'}$, and thus $p'<2$, which is impossible.

Let $N/M = D_{p'}(q')$, where q' is 2, 3 or 5, and p' is more than 5. Then $q'^{p'(p'-1)}$ is a nontrivial factors of p^2-1, and $p=\left(q'^{p'}-1\right)/(q'-1)$. Since $q'^{p'(p'-1)} < p^2 < q'^{2p'}$, one has that $p'(p'-1)$ is less than $2p'$, and so $p'<3$, which is a contradictory inequality.

Let $N/M = D_{p'+1}(q')$, where q' is 2 or 3. Then $q'^{p'(p'+1)}$ is a nontrivial factor of p^2-1, and $p=\left(q'^{p'}-1\right)/(2,q'-1)$. Hence $q'^{p'(p'+1)} < p^2 < q'^{2p'}$, and then $p'(p'+1)$ is less than $2p'$, a contradictory inequality.

Let $N/M = C_{n'}(q')$, where the number n' is a power of 2, and is bigger than 2. Then $q'^{n'^2}$ is a nontrivial factor of p^2-1, and p is equal to $\left(q'^{n'}+1\right)/(2,q'-1)$, which implies $q'^{n'^2} < p^2 < q'^{2(n'+1)}$. One has that $n'^2-2n'-2<0$, and so $n'=2$. Since $(q'^2-1)^2q'^4 < p'^2 < 4q'^4$, one can get that $(q'^2-1)^2<4$, a contradictory inequality.

Let $N/M = C_{p'}(q')$, where q' is 2 or 3. Then $q'^{p'^2}(q'^{p'}+1) \mid (p^2-1)$ and $p=\left(q'^{p'}-1\right)/(2,q'-1)$. Since $q'^{p'^2} < p^2 < q'^{2p'}$, one can get $p'<2$, which is impossible.

Let $N/M = {}^2D_{n'}(q')$, where $n'=2^{m'}+1 \geqslant 9$ when $q'=3$, and $n'=2^{m'}+1 \geqslant 5$ when $q'=2$. Then $q'^{n'(n'-1)}$ is a nontrivial factor of p^2-1, and $p=\left(q'^{n'-1}+1\right)/(2,q'-1)$. Since $q'^{n'(n'-1)} < p^2 < q'^{2n'}$, one has that $0<n'<3$, which is impossible.

Let $N/M = B_{n'}(q')$, where q' is uneven, and n' is a power of 2, and bigger than 4. Then $(q'^{n'}-1)q'^{n'^2}$ is a factor p^2-1, and $p=\left(q'^{n'}+1\right)/2$. In view of $q'^{n'^2} < p^2 < q'^{2(n'+1)}$, one can get that $n'^2-2n'-2<0$ such that $n'<3$, a contradictory inequality.

Let $N/M = G_2(q')$, where q' is bigger than 2, and q' is congruent with ϵ modulo 3. Then q'^6 is a nontrivial factor of p^2-1, and $p=q'^2-\epsilon q'+1$, and so $q'^6 < p^2 < q'^6$, a contradictory inequality.

Let $N/M = {}^2A_{p'-1}(q')$. Then p is equal to $\left(q'^{p'}+1\right)/(q'+1)(p',q'+1)$, and $q'^{\frac{p'(p'-1)}{2}}$ is a nontrivial factor of (p^2-1). By $q'^{\frac{p'(p'-1)}{2}} < p^2 < q'^{2(p'+1)}$, one has that $p'(p'-1)/2$ is less than $2(p'+1)$, and thus $p'=3$ or 5. When $p'=3$, one has that $q'^4 < p^2 \leqslant q'^4$, a contradictory inequality. When $p'=5$, we can obtain similarly contradictory inequality $q'^{10} < p^2 \leqslant q'^8$.

Let $N/M = F_4(q')$, where q' is an uneven number. Then q'^{24} is a nontrivial factor of p^2-1, and p is equal to $q'^4-q'^2+1$. It follows that $q'^{24} < p^2 < q'^8$, a contradictory inequality.

Let $N/M = A_{p'}(q')$, where $q'-1$ is a divisor of $p'+1$. Then $q'^{\frac{p'(p'+1)}{2}}$ is a nontrivial factor of p^2-1, and p is equal to $\left(q'^{p'}-1\right)/(q'-1)$. Therefore, one can get $q'^{\frac{p'(p'+1)}{2}} < p^2 \leqslant q'^{2p'}$, and then $p'<3$, a contradictory inequality.

Let $N/M = {}^2E_6(q')$, where q' is bigger than 2. Then $p=\left(q'^6-q'^3+1\right)/(3,q'-1)$, and q'^{36} is a nontrivial factor of p^2-1. Therefore, $q'^{36} < p^2 < q'^{12}$, a contradictory inequality.

Let $N/M = A_{n'}$, where n' is bigger than 6 and n' is p', $p'+1$ or $p'+2$, and $n'-2$ and n' are not all prime. Then p is equal to p', and $n'!/2$ is a divisor of $p(p^2-1)$. If $n'=p'$, we get $p+1>3(p-2)$, and thus $p<4$, contradicting $p>4$. For other cases of n', a contradiction always can be reached.

Step 2. Assume that N/M is one group of Table 2 in [34]. Let $N/M = M_{11}$. Then $p=11$, and then $2^4\cdot3^2\cdot5 \mid 120$, a contradiction.

Let N/M be one group of J_3 and M_{23}. Then $p=19,23$, and $|N/M|_2 \mid (p^2-1)$. By [31], one can find that $2^7 \geqslant |N/M|_2$, which contradicting $(p^2-1)_2 \leqslant 2^4$.

Let N/M be one group of Sz, $E_7(3)$, ${}^2A_5(2)$, M_{24}, $E_7(2)$, F_{23}, HS, Co_2, F_2 and F_3. Then $p=11,13,19,31,47,23,127,1093$, and $|N/M|_2 \mid (p^2-1)$. Also by [31], one can get that $|N/M|_2 \leqslant 2^{10}$, contradicting $(p^2-1)_2 \leqslant 2^8$.

Let $N/M = A_{p'}$, where each of p' and $p'-2$ is prime, and p' is a number greater than 6. Then $p'!/2 \mid (p^2-1)$ and $p=p'$. Therefore, $p+1>3(p-2)$, and thus $p<4$, contradicting $p>4$.

Let $N/M = A_1(q')$, where q' is a number bigger than 3, and q' is congruent with ϵ modulo 4. Then $p=q'$ and $(q'+\epsilon)/2$.

(a) If $p=(q'+1)/2$, one has that $q'(q'-1)$ is a factor of (p^2-1), and $q'=2p-1$. Hence $2(2p-1) \leqslant$

$(p+1)$, which means that $p \leqslant 1$, contradicting $p \geqslant 2$.

(b) If $p=(q'-1)/2$, one has that $q'(q'+1)$ is a factor of (p^2-1), and $q'=2p+1$. Therefore, $2(2p+1) \leqslant (p-1)$, contradicting $p \geqslant 5$.

(c) If $p=q'$, one has that N/M is the special projective linear group $PSL_2(p)$. Hence $PSL_2(p) \leqslant G/M \leqslant PGL_2(p)$, and thus G/M is isomorphic to one of $PSL_2(p)$ and $PGL(2,p)$. Let G/M be isomorphic to $PSL_2(p)$. It follows that $|M|=2$, which means $M \subseteq Z(G)$, contradicting $Z(G)=1$. Thus, G/M is isomorphic to $PGL(2,p)$, and then $M=1$, which implies that $G \cong PGL_2(p)$, as expected.

Let $N/M={}^2D_{p'}(3)$, where m' is a number more than 2, and p' is equal to $2^{m'}+1$. Then $p=\left(3^{p'-1}+1\right)/2$ or $\left(3^{p'}+1\right)/4$.

(d) If $p=\left(3^{p'-1}+1\right)/2$, then $3^{p'(p'-1)}$ is a nontrivial factor of p^2-1. By $3^{p'(p'-1)}<p^2<3^{2p'}$, one can get that $p'(p'-1)<2p'$, a contradictory inequality.

(e) If $p=\left(3^{p'}+1\right)/4$, then $(3^{p'-1}+1)3^{p'(p'-1)} \mid (p^2-1)$. Since $3^{p'(p'-1)}<p^2<3^{2(p'+1)}$, one has that $p'=3$ and $p=7$, which is impossible.

Let $N/M=G_2(q')$, where q' is divisible by 3. Then q'^6 is a nontrivial factor of p^2-1, and $p=q'^2-\epsilon q'+1$, and then we can get a contradictory inequality $q'^6<p^2<q'^6$.

Let $N/M=A_1(q')$, q' is an even integer greater than 4. Then $q'(q'-\epsilon)$ is a divisor of p^2-1, and p is equal to $q'+\epsilon$. Thus $(p-2\epsilon) \mid (p+\epsilon)$, and so $p=5$ and $q'=4$, contradicting $q'>4$.

Let $N/M={}^2G_2(q')$, where q' is bigger than 3, and is an odd power of 3. Then $q'^3(q'^2-1)$ is a nontrivial factor of p^2-1, and $p=q'-\epsilon\sqrt{3q'}+1$. In view of $q'^3(q'^2-1)<p^2<q'^4$, we have $q'>q'^2-1$, a contradictory inequality.

Let $N/M={}^2F_4(q')$, where q' is bigger than 2, and is an odd power of 2. Then p is equal to $q'^2+\epsilon\sqrt{2q'^3}+q'+\epsilon\sqrt{2q'}+1$, and $q'^{12}(q'-1)$ is a divisor of p^2-1, which implies $q'^{12}<p^2<q'^8$, a contradictory inequality.

Let $N/M=F_4(q')$, where q' is an even integer greater than 2. Then p is one of q'^4+1 and $q'^4-q'^2+1$. If p is equal to q'^4+1, we have that q'^{24} is a nontrivial factor of (p^2-1), which means that $q'^{24}<p^2<q'^{10}$, a contradictory inequality. Hence p is equal to $q'^4-q'^2+1$, and so $(q'^4+1)q'^{24} \mid (p^2-1)$, which implies $q'^{24}<p^2<q'^8$, a contradictory inequality.

Step 3. It is supposed that N/M is one of the groups of Table 3 in [34].

Let $N/M=J_1$. Then $p=19$, and $11 \mid (p^2-1)$, which contradicts $p^2-1=720$.

Let $N/M=J_4$. Then $37 \mid (p^2-1)$, and $p=43$, contradicting $p^2-1=2^3\cdot 3\cdot 7\cdot 11$.

Let $N/M=ON$. Then 19 is a divisor of p^2-1, and p is equal to 31, contradicting $p^2-1=2^6\cdot 15$.

Let $N/M={}^2B_2(q')$, where q' is bigger than 2, and is an odd power of 2. Then p is one of $q'-1$ and $q'+\epsilon\sqrt{2q'}+1$.

(f) If $p=q'-1$, then q'^2 is a nontrivial factor of p^2-1, which means that $q'^2<p^2<q'^2$, a contradictory inequality.

(g) If $p=q'-\sqrt{2q'}+1$, then $q'^2(q'+\sqrt{2q'}+1)$ is a factor of p^2-1, and so $q'^3<p^2<q'^2$, a contradictory inequality.

(h) If $p=q'+\sqrt{2q'}+1$, then q'^2 is a nontrivial factor of p^2-1. Let $q'=2^{2d'+1}$ with d' a positive integer. We can get that $p=2^{2d'+1}+2^{d'+1}+1$, and $p^2-1=2^{d'+2}(2^{d'}+1)(2^{2d'}+2^{d'}+1)$. In view of $|N/M|_2 \mid (p^2-1)_2$, one has $2^{2(2d'+1)} \mid 2^{d'+2}$, and thus $2(2d'+1) \leqslant d'+2$, which is a contradictory inequality.

Let $N/M=E_8(q')$, where q' is congruent to 0, 1 or 4 modulo 5. Then p is one of $q^8-q'^6+q'^4-q'^2+1$, $q'^8+q'^7-q'^5-q'^4-q'^2+q'+1$, $q'^8-q'^4+1$, and $q'^8-q'^7+q'^5-q'^4+q'^2-q'+1$. Similarly, a contradiction always can be obtained for other cases.

Let N/M be one group of $A_2(4)$, ${}^2E_6(2)$, M_{22}, Ly, F'_{24} and F_1. Then $|N/M|_2 \mid (p^2-1)$, and $p=7,11,19,29,67,71$. Therefore, we can get $|N/M|_2 \geqslant 2^6$ by [31], but $(p^2-1)_2 \leqslant 2^4$, a contradiction.

Let $N/M=E_8(q')$, where q' is congruent to 2 or 3 modulo 5. Then p is equal to one of $q'^8+q'^7-q'^5-q'^4-q'^2+q'+1$, $q'^8-q'^4+1$, and $q'^8-q'^7+q'^5-q'^4+q'^2-q'+1$. If $p=q'^8-q'^4+1$, then $p^2<q'^{16}$, contradicting $q'^{120}<p^2$. By a similar method, a contradiction always can be reached for other cases of p.

Thus, one has that G must be isomorphic to the group $PGL(2,p)$ with a prime p as desired. □

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