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On groups with consecutive three smallest character degrees

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ABSTRACT: Huppert determined the structure of groups whose degrees are consecutive, then Qian improved that Huppert's result and showed the structure of finite groups whose degrees of nonlinear characters are consecutive. In this paper, we considered the case when the first three smallest degrees of nonlinear irreducible characters of an almost simple group G are consecutive. Furthermore, Huppert's conjecture is proved valid for those groups.

KEYWORDS: almost simple group, character degree graph, solvable

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INTRODUCTION

All groups considered here are finite. Let $\mathrm{Irr}(G)$ be the set of all complex irreducible characters of a group G, and let $\mathrm{cd}(G)$ be the set of character degrees of a group G. Suppose that n and $n+1$, $n>1$, are two consecutive integers standing for the two greatest sizes of the conjugacy classes of group G. Then $G/Z(G)=(C_{p^a-1}, E_{p^a})$ is a Frobenius group with kernel E_{p^a} and complement C_{p^a-1}, $n+1=p^a$, and the pre-images in G of the kernels of $G/Z(G)$ are abelian. Meanwhile it determined that the groups with conjugacy classes of lengths 1, n or $n+1$ are considered[1]. Several authors have determined the structure of finite groups whose conjugacy class sizes are consecutive[2,3]. Dual to the sizes of the conjugacy classes of a group G, what is the influence that character degrees of a group G may have on the structure of a finite group? The influence of the set $\mathrm{cd}(G)$, $|\mathrm{cd}(G)|\geqslant 4$, on the structure of finite groups has been investigated by a number of authors[6–11].

For a group G, if $\mathrm{cd}(G)=\{s_0,s_1,\dots,s_t\}$ with $1=s_0\leqslant s_1<\cdots<s_t$, then make $d_i(G)=s_i$ stand for the ith smallest degree of G for all $1\leqslant i\leqslant t$. When there is no confusion, we may simply write $d_i=s_i$. If $t=0$, then G is abelian; if $t=1$, then, for $\mathrm{cd}(G)=\{1,m\}$, either G has an abelian normal subgroup of index m, or $m=p^e$ for a prime p and G is the direct product of a p-group and an abelian group[4]. If $t=2$, then $|\mathrm{cd}(G)|=3$, G is solvable[5], and $G'''=1$.

Huppert[5] proved the following result.

Theorem 1 (Ref. 5) *Suppose* $\mathrm{cd}(G)=\{1,2,\dots,k-1,k\}$. *Then G is solvable if and only if $k\leqslant 4$; if $k>4$, then $k=6$ and $G=HZ(G)$ with $H\cong SL(2,5)$.*

Furthermore, Qian[12] proved the following results.

Theorem 2 *(1) If G is a solvable group whose all nonlinear character degrees are consecutive integers, then one of the following are true:*
(i) $|\mathrm{cd}(G)|\leqslant 2$;
(ii) $\mathrm{cd}(G)=\{1,p^m-1,p^m\}$ *or* $\{1,p^m,p^m+1\}$;
(iii) $\mathrm{cd}(G)=\{1,2,3,4\}$.

(2) An insolvable group G whose nonlinear character degrees are consecutive integers if and only if $G/Z(G)\cong PGL(2,q)$ for some prime power $q\geqslant 4$.

Hypothesis (C): The three smallest degrees of nonlinear irreducible characters of a group G are consecutive.

Many authors have employed the degrees of irreducible characters to determined the simple groups and some solvable groups. Then what will occur if a group G satisfies Hypothesis (C)?

Recall that a group A is almost simple if there is a simple group S such that $S\leqslant A\leqslant \mathrm{Aut}(S)$. In this paper we only consider the influence of Hypothesis (C) on the structure of almost simple finite groups.

Theorem 3 *Let G be an almost simple group and let L be a nonabelian simple group. Assume that G and L satisfy Hypothesis (C), that is, $d_i(G)=d_i(L)$ for all $i\in\{1,2,3\}$. Then $L=L_2(2^m)$ with $m\geqslant 2$, and G has* *a normal subgroup K such that G/K is isomorphic to L. If $|G| = |L|$, then $G \cong L$.*

Remark 1 Unfortunately, we cannot determine the structure of solvable groups since there are many examples proven to satisfy the condition of Hypothesis (C). For instance, $m = d_1 \in \mathrm{cd}(G)$, $m+1 = d_1 \in \mathrm{cd}(H)$, and $m+2 = d_1 \in \mathrm{cd}(P)$, where G, H and P are solvable groups, then $m, m+1, m+2 \in \mathrm{cd}(G\times H\times P)$.

Remark 2 Even if G is an insolvable group, we also cannot ascertain that G is a product of L by an abelian group. For example, if $d_1, d_2, d_3 \in \mathrm{cd}(L)$ and M is a nonabelian simple group such that $d_1(M) > d_3(L)$, then $d_1, d_2, d_3 \in \mathrm{cd}(L \times M)$. Thus in Theorem 3, the condition "$d_i(G) = d_i(L)$ for all $i \in \{1,2,3\}$, where L is a nonabelian simple group", is essential and cannot be removed.

Remark 3 The condition "G is almost simple" cannot be replaced by "G is insolvable". For instance, for $n \geqslant 9$, $d_1(A_n) = n-1$, $d_1(A_{n+1}) = n$, and $d_1(A_{n+2}) = n+1$, then by Lemma 2, $n-1, n, n+1 \in \mathrm{cd}(A_n \times A_{n+1} \times A_{n+2})$.

Corollary 1 *Let G be a nonabelian simple group with Hypothesis (C). Then G is isomorphic to $L_2(2^m)$ with $m \geqslant 2$.*

Proof: By Theorem 3, $G/A \cong L_2(2^m)$ with $m \geqslant 2$. Since G is simple, then $A = 1$ and $G \cong L_2(2^m)$ with $m \geqslant 2$. □

In this paper, we also use the properties of character degree graph $\Gamma(G)$ in the proofs of some results. So, we will introduce the notion of character degree graph $\Gamma(G)$. Let

$$\rho(G) = \{p \in \pi(G) \mid p \text{ divides } \chi(1),\ \chi \in \mathrm{Irr}(G)\}.$$

Recall that the graph $\Gamma(G)$ is called *character degree graph*[13] whose vertices are members of $\rho(G)$ and two vertices p and q are joined by an edge if pq divides some character degree of G, written as $p \sim q$.

FINITE SIMPLE GROUP

Lemma 1 *Let $q-1, q, q+1 \in \mathrm{cd}(G)$ for a positive integer $q > 1$. Then the order $|G|$ of G is divisible by $\frac{1}{2}q(q^2-1)$.*

Proof: Let $1 < m \in \mathrm{cd}(G)$. Then by Theorem 3.11 of Ref. 4, there is a character $\chi \in \mathrm{Irr}(G)$ such that $m = \chi(1)$ and $m \mid |G|$. Since $(q-1, q+1) = 2$ for odd q, and $(q-1, q+1) = 1$ for even q, then $\frac{1}{2}q(q^2-1)$ divides the order $|G|$ of G. □

Table 1 The first three smallest degrees of sporadic simple groups.

G	$d_1(G)$	$d_2(G)$	$d_3(G)$
M_{11}	10	11	16
M_{12}	11	16	45
J_1	56	76	77
M_{22}	21	45	55
J_2	14	21	36
M_{23}	22	45	230
HS	22	77	154
J_3	85	323	24
M_{24}	23	45	231
McL	22	231	252
He	51	153	680
Ru	378	406	783
Suz	143	364	780
ON	10 944	13 376	25 916
Co_3	23	253	275
Co_2	23	253	275
Fi_{22}	78	429	1001
HN	133	760	3344
Ly	2480	45 694	48 174
M_{12}	11	16	45
Th	248	4123	27 000
Fi_{23}	782	3588	5083
Co_1	276	299	1771
J_4	1333	299 367	887 778
Fi'_{24}	8671	57 477	249 458
B	4371	96 255	1 139 374
M	196 883	21 296 876	842 609 326
${}^2F_4(2)'$	26	27	78

Lemma 2 *Let $G = A_n$ with $n \geqslant 9$. Then*
(1) $d_1(G) = n-1$;
(2) $d_2(G) = \frac{1}{2}n(n-3)$;
(3) $d_3(G) = \frac{1}{2}(n-1)(n-2)$.

Proof: This results are taken from Ref. 14. □

Lemma 3 *Let G be a sporadic simple group. Then $d_i(G)$ with $i \in \{1,2,3\}$ are as listed in Table 1.*

Proof: The results were obtained from ATLAS[15]. □

Lemma 4 *Let G be a simple group of Lie type satisfying Hypothesis (C). Then G is isomorphic to $L_2(2^m)$ with $m \geqslant 2$ or $L_2(5)$.*

In the following proof of this Lemma, the results of Refs. 16, 17 will often be used without explicit reference.

Proof: Let G be a simple group of Lie type. Then the following cases will be considered.

Case 1. Linear groups $L_n(q)$.

Let $n = 2$:

(i) If $q=2^m\geqslant 4$, then $\mathrm{cd}(G)=\{1,2^m-1,2^m,2^m+1\}$, and so $d_1(G)=2^m-1$, $d_2(G)=2^m$, and $d_3(G)=2^m+1$. Hence we have that $L_2(2^m)$ with $m\geqslant 2$, the desired result.
(ii) If $q=p^m\geqslant 5$ is odd, then $\mathrm{cd}(G)=\{1,q-1,q,q+1,(q+\varepsilon)/2\}$ with $\varepsilon=(-1)^{(q-1)/2}$. If $q\equiv 1 \pmod 4$, then $\mathrm{cd}(G)=\{1,(q+1)/2,q-1,q,q+1\}$, and so $d_1(G)=(q+1)/2$, $d_2(G)=q-1$, and $d_3(G)=q$. Thus $(q+1)/2+1=q-1$, and so $q=5$. If $q\equiv -1 \pmod 4$, then $\mathrm{cd}(G)=\{1,(q-1)/2,q-1,q,q+1\}$, and so $d_1(G)=(q-1)/2$, $d_2(G)=q-1$, and $d_3(G)=q$. Thus $(q-1)/2+1=q-1$ and $q=3\ngeqslant 5$, a contradiction.

Let $n=3$:

If $q=3$, then $\mathrm{cd}(G)=\{1,12,13,16,26,27,39\}$; if $q=4$, then $\mathrm{cd}(G)=\{1,20,35,45,63,64\}$ [15]. If $q\geqslant 5$, then by Ref. 18, $\mathrm{cd}(G)=\{1,q^3,\ q(q+1),\ (q-1)^2(q^2+q+1),\ (q-1)(q^2+q+1),\ q^2+q+1,\ (q+1)(q^2+q+1),\ \frac{1}{3}(q+1)(q^2+q+1)\}$, where the last degree appears only if $q\equiv 1 \pmod 3$, and so $d_1(G)=q(q+1)$, $d_2(G)=q^2+q+1$, and $d_3(G)=q^3-1$ when $q\not\equiv 1 \pmod 3$, and $d_1(G)=q(q+1)$, $d_2(G)=\frac{1}{3}(q+1)(q^2+q+1)$ and $d_3(G)=q^3-1$ when $q\equiv 1 \pmod 3$. If $q\not\equiv 1 \pmod 3$, then $q^2+q+2=q^3-1$ by the hypothesis, and so this equation has no solution in $\mathbb{N}$, since $q\geqslant 5$. If $q\equiv 1 \pmod 3$, then $q(q+1)+1=\frac{1}{3}(q+1)(q^2+q+1)$, i.e., $q^3-q^2-q-2=0$, and so we obtain no answer in $\mathbb{N}$. So we rule out this case.

Let $n=4$: Then $d_1(G)=q(q^2+q+1)$, $d_2(G)=(q+1)(q^2+1)$, $d_3(G)=(q^2+q+1)(q-1)^2$. By the hypothesis, $d_3(G)-1=d_2(G)$, and so $(q^2+q+1)(q-1)^2-1=(q+1)(q^2+1)$. It is easy to show that the equation has no solution in $\mathbb{N}$.

Let $n=5$: Then $d_1(G)=q(q+1)(q^2+1)$, $d_2(G)=q^4+q^3+q^2+q+1$, $d_3(G)=q^2(q^4+q^3+q^2+q+1)$. Then the equation $q^4+q^3+q^2+q+1+1=q^2(q^4+q^3+q^2+q+1)$ has no solution in $\mathbb{N}$.

Let $n\geqslant 6$: Since $d_1(G)+1=d_2(G)$ by the hypothesis, we obtain that

$$1=\frac{q^n-1}{q-1}\left(\frac{q^{n-1}-1}{q^2-1}-1\right).$$

Note that $n\geqslant 6$ and $q\geqslant 2$, $(q^n-1)/(q-1)\geqslant 1$ and $(q^{n-1}-1)/(q^2-1)-1\geqslant 1$, so in this case, there is no solution in $\mathbb{N}$.

Case 2. Unitary groups $U_n(q^2)$ with $n\geqslant 3$ and $q\geqslant 3$.

If $n=3$, then by Ref. 18, $\mathrm{cd}(G)=\{1,q^3,(q-1)(q+1)^2,q(q-1),q^2-q+1,(q-1)(q^2-q+1),q(q^2-q+1),(q+1)(q^2-q+1),\frac{1}{3}(q-1)(q^2-q+1)\}$, where the last degree appears only if $q\equiv -1 \pmod 3$ and $d_1(G)=q(q-1)$, $d_2(G)=(q^2-q+1)$, $d_3(G)=(q-1)(q^2-q+1)$. Thus, by the hypothesis, $d_2(G)+1=d_3(G)$, i.e., $(q-2)(q^2-q+1)=1$. It is easy to see that this equation has no solution in $\mathbb{N}$ since $q\geqslant 2$; so, this case is ruled out.

If $n=4$, then by Ref. 16, $d_1(G)=(q-1)(q^2+1)$, $d_2(G)=q(q^2-q+1)$, $d_3(G)=(q^2+1)(q^2-q+1)$, and so $1=(q^2-q+1)^2$. Thus there is no solution in this equation in $\mathbb{N}$.

If $n=5$, then by Ref. 16, $d_1(G)=q(q-1)(q^2+1)$, $d_2(G)=q^4-q^3+q^2-q+1$, $d_3(G)=q^2(q^4-q^3+q^2-q+1)$. It is easy to show that the equation $d_2(G)+1=d_3(G)$ has no solution in $\mathbb{N}$.

If $n=6$, then Ref. 16 implies that $d_1(G)=(q-1)(q^2+q+1)(q^2-q+1)$, $d_2(G)=q(q^4-q^3+q^2-q+1)$, and $d_3(G)=(q^2-q+1)(q^2+q+1)(q^4-q^3+q^2-q+1)$. By the hypothesis, we have that

$$1=(q^4-q^3+q^2-q+1)[(q^2-q+1)(q^2+q+1)-1],$$

and the equation has no solution in $\mathbb{N}$.

If $n=7$, then we have that $d_1(G)=q(q-1)(q^2+q+1)(q^2-q+1)$, $d_2(G)=q^6-q^5+q^4-q^3+q^2-q+1$, and $d_3(G)=q^2(q^2+1)(q^6-q^5+q^4-q^3+q^2-q+1)$. Now $d_2(G)+1=d_3(G)$ by the hypothesis, and we can see that that equation has no solution in $\mathbb{N}$.

Let $n\geqslant 8$. If $2\nmid n$, then

$$1=\left(\frac{q^n+1}{q+1}\right)\left(\frac{q^{n-1}-q^2}{q^2-1}-1\right).$$

It is easy to see that the equation has no solution in $\mathbb{N}$. If $2\mid n$, then we have that $\frac{q^n+q}{q+1}+1=\frac{(q^n-1)(q^{n-1}+1)}{(q+1)(q^2-1)}$ when $q\neq 2$, and, by the hypothesis, that $\frac{q^n+q}{q+1}+1=\frac{(q^n-1)(q^{n-1}-q)}{(q+1)(q^2-1)}$ when $q=2$. If $q\neq 2$, then the equation

$$1=q(q^{2n-2}-q^{n+1}+2q^{n-1}-2q^2+q-2)$$

has no solution in $\mathbb{N}$ since $n\geqslant 8$. If $q=2$, then the left side of the equation

$$(q^n+2q+1)(q^2-1)=q(q^{n-2})(q^n-1)$$

is odd, where the right side of this equation is even. So we rule out this case.

Case 3. Orthogonal groups $B_l(q)$ with $q\geqslant 3$ and Symplectic groups $C_l(q)$.

Let $l=2$:

(i) $q\equiv 0 \pmod 2$. Then by Ref. 19, $d_1(G)=q(q-1)^2/2$, $d_2(G)=q(q^2+1)/2$, and $d_3(G)=q(q+1)^2/2$. By the hypothesis, $\frac{1}{2}q(q-1)^2+1=\frac{1}{2}q(q^2+1)$ and so $q=1\ngeqslant 3$, a contradiction.

(ii) $q \equiv 1 \pmod 2$. Then by Ref. 19, $d_1(G)=q^2+1$, $d_2(G)=q(q-1)^2/2$, and $d_3(G)=q(q+1)^2/2$. By the hypothesis, $\frac{1}{2}q(q-1)^2+1=\frac{1}{2}q(q^2+1)^2$, and so there is no solution in $\mathbb{N}$.

Note that if q is even and $l=2$, then $B_2(q)\cong C_2(q)$.

Let $l=3$. Subcase 1: $B_3(q)$.

If $q\equiv 0 \pmod 2$, then $d_1(G)=q(q^2+q+1)(q-1)^2/2$, $d_2(G)=q(q^2+1)(q^2-q+1)/2$, $d_3(G)=q(q^2-q+1)(q+1)^2/2$. Since $d_3(G)-1=d_2(G)$, we have $1=(q^2-q+1)q^2$, and so the equation has no solution in $\mathbb{N}$.

If $q\equiv 1 \pmod 2$, then $d_1(G)=(q^2+q+1)(q^2-q+1)$, $d_2(G)=q(q^2+q+1)(q-1)^2/2$, $d_3(G)=q(q^2-q+1)(q^2+1)/2$. Since $d_2(G)-d_1(G)=1$, we obtain that $1=(q^2+q+1)[q(q-1)^2/2-(q^2-q+1)]$ has no solution in $\mathbb{N}$ as $q\geqslant 3$.

Subcase 2: $C_3(q)$.

If $q\equiv 0,2 \pmod 4$, then $d_1(G)=q(q^2+q+1)(q-1)^2/2$, $d_2(G)=q(q^2+1)(q^2-q+1)/2$, $d_3(G)=q(q^2-q+1)(q+1)^2/2$. As $d_3(G)-d_2(G)=1$. By the hypothesis, we derive that $1=3q^2(q^2-q+1)/2$ has no solution in $\mathbb{N}$.

If $q\equiv 1 \pmod 4$, then $d_1(G)=(q+1)(q^2-q+1)$, $d_2(G)=q(q^2+q+1)(q-1)^2/2$, $d_3(G)=q(q^2+1)(q^2-q+1)/2$. Since $d_3(G)-d_2(G)=1$, we have that $1=q^3$, a contradiction since q is a prime-power.

If $q\equiv 3 \pmod 4$, then $d_1(G)=(q-1)(q^2+q+1)$, $d_2(G)=q(q^2+q+1)(q-1)^2/2$, $d_3(G)=q(q^2+1)(q^2-q+1)/2$. Since $d_2(G)-d_1(G)=1$, we conclude that $1=(q^3-1)(q(q-1)/2-1)$ has no solution in $\mathbb{N}$ since q is a power of some prime.

Let $l=4$. Subcase 1: $B_4(q)$.

Now $d_1(G)=1/2q(q^2+q+1)(q^2+1)(q-1)^2$, $d_2(G)=1/2q(q^2-q+1)(q^4+1)$, $d_3(G)=1/2q(q^2+q+1)(q^4+1)$ for $q\equiv 0 \pmod 2$ and $d_1(G)=(q^2+1)(q^4+1)$, $d_2(G)=1/2q(q^2+q+1)(q^2+1)(q-1)^2$, $d_3(G)=1/2q(q^2-q+1)(q^4+1)$ for $q\equiv 1 \pmod 2$. By the hypothesis, $1=d_3(G)-d_2(G)$ implies that $1=q^2(q^4+1)$ and $1=q^4(2-q)/2$. Clearly, these equations have no solutions in $\mathbb{N}$.

Subcase 2: $C_4(q)$.

$d_1(G)=1/2q(q^2+q+1)(q^2+1)(q-1)^2$, $d_2(G)=1/2q(q^2-q+1)(q^4+1)$, $d_3(G)=1/2q(q^2+q+1)(q^4+1)$ for $q\equiv 1 \pmod 4$ and $d_1(G)=q^4+1$, $d_2(G)=1/2q(q^2+q+1)(q^2+1)(q-1)^2$, $d_3(G)=1/2q(q^2-q+1)(q^4+1)$ for $q\equiv 2,3 \pmod 4$. Now as with the proof of $B_4(q)$, this case is ruled out.

Let $l=5$. Subcase 1: $B_5(q)$.

$d_1(G)=1/2q(q^2+1)(q^4+q^3+q^2+q+1)(q-1)^2$, $d_2(G)=1/2q(q^4-q^3+q^2-q+1)(q^4+1)$, $d_3(G)=1/2q(q^2+1)(q^4-q^3+q^2-q+1)(q+1)^2$ for $q\equiv 0 \pmod 2$, and $d_1(G)=(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)$, $d_2(G)=1/2q(q^2+1)(q^4+q^3+q^2+q+1)(q-1)^2$, $d_3(G)=1/2q(q^4-q^3+q^2-q+1)(q^4+1)$ for $q\equiv 1 \pmod 2$. Now considering $d_2(G)-d_1(G)=1$, we can rule out this case.

Subcase 2: $C_5(q)$.

Now $d_1(G)=1/2q(q^2+1)(q^4+q^3+q^2+q+1)(q-1)^2$, $d_2(G)=1/2q(q^4-q^3+q^2-q+1)(q^4+1)$, $d_3(G)=1/2q(q^2+1)(q^4-q^3+q^2-q+1)(q+1)^2$ for $q\equiv 0,2 \pmod 4$; $d_1(G)=(q+1)(q^4-q^3+q^2-q+1)$, $d_2(G)=1/2q(q^2+1)(q^4+q^3+q^2+q+1)(q-1)^2$, $d_3(G)=1/2q(q^4-q^3+q^2-q+1)(q^4+1)$ for $q\equiv 1 \pmod 4$; $d_1(G)=(q-1)(q^4+q^3+q^2+q+1)$, $d_2(G)=1/2q(q^2+1)(q^4+q^3+q^2+q+1)(q-1)^2$, $d_3(G)=1/2q(q^4-q^3+q^2-q+1)(q^4+1)$ for $q\equiv 3 \pmod 4$. Now considering the $d_2(G)-d_1(G)=1$, we rule out this case.

Similarly we illustrate that no group occurs when $l=6,7$.

The notation is taken from Ref. 17, and we will consider the case when the group is of a large rank. By Table 2, the following cases will be dealt with for $n\geqslant 6$.

Case 1: $Sp_{2n}(2)$ for $n>2$.

Now $d_1(G)=2^{n-1}(2^n-1)$, $d_2(G)=2^{n-1}(2^n+1)$, and $d_3(G)=P_1$. By the hypothesis, $d_2(G)-d_1(G)=1$, and so $1=2^n$, a contradiction since $n>2$.

Case 2: $Sp_{2n}(q)$ for $2\mid q$, $q>2$, $n\geqslant 2$.

Now $d_1(G)=P_1$, $d_2(G)=q^n(q^n-1)/2$, $d_3(G)=q^n(q^n+1)/2$. Now $d_3(G)-d_2(G)=1$ implies that $1=q^n$, and $n=0$, against $n\geqslant 2$.

Case 3: $PSp_{2n}(q)$ for $2\nmid q$, $n\geqslant 6$.

Now we have that $d_1(G)=\dfrac{q^{2n}-1}{q-1}$, $d_2(G)=\frac{(q^n-1)(q^{2n-2}-1)}{(q-1)(q^2-1)}$ and $d_3(G)=N_1$. By the hypothesis, $d_2(G)-d_1(G)=1$ implies that

$$1=\frac{q^{2n}-1}{q-1}\left(\frac{q^{2n-2}-1}{q^2-1}-1\right),$$

and so this equation has no solution in $\mathbb{N}$.

Case 4: $\Omega_{2n+1}(3)$ for $n\geqslant 4$.

Now $d_1(G)=q^n(q^n-1)/2$, $d_2(G)=\dfrac{q^{2n}-1}{q-1}$ and $d_3(G)=q^n(q^n+1)/2$. By hypothesis, $d_3(G)-d_1(G)=2$ implies that $2=q^n$, and so $q=2$ and $n=1$, against $n\geqslant 4$.

Case 5: $\Omega_{2n+1}(q)$ for $2\nmid q$, $q>3$, $n\geqslant 4$.

Now $d_1(G)=\frac{q^{2n}-1}{q-1}$, $d_2(G)=q^n(q^n-1)/2$, and $d_3(G)=q^n(q^n+1)/2$. By the hypothesis, $d_3(G)-d_2(G)=1$ shows that $1=q^n$, against $n\geqslant 4$.

Case 6: $P\Omega^+_{2n}(2)$, for $n\geqslant 8$ or $P\Omega^+_{2n}(3)$, $n\geqslant 8$.

Now $d_1(G)=2^{n-1}(2^n-1)$, $d_2(G)=\frac{(2^n-1)(2^{n-1}+1)}{q-1}$, $d_3(G)=2^{2n-3}(2^n-1)(2^{n-1}-1)/3$. $d_2(G)-d_1(G)=1$

Table 2 The first three smallest degrees of B_l or C_l.†

G‡	Condition	$d_1(G)$	$d_2(G)$	$d_3(G)$
$Sp_4(2)'$		6	10	15
$Sp_4(3)$		27	36	40
$Sp_{2n}(2)$	$n>2$	$2^{n-1}(2^n-1)$	$2^{n-1}(2^n+1)$	$\mathscr{P}_1$
$Sp_{2n}(q)$	$2\mid q,\ q>2,\ n\geqslant 2$	$\mathscr{P}_1$	$\frac{1}{2}q^n(q^n-1)$	$\frac{1}{2}q^n(q^n+1)$
$PSp_4(q)$	$2\nmid q,\ n=3,4,5$	$\mathscr{P}_1$	$\mathscr{P}_n$	$\mathscr{P}_2$
$PSp_{2n}(q)$	otherwise	$\mathscr{P}_1$	$\mathscr{P}_2$	$\mathscr{N}_1$
$\Omega_{2n+1}(3)$	$n\geqslant 4$	$\mathscr{N}_1^-$	$\mathscr{P}_1$	$\mathscr{N}_1^+$
$\Omega_{2n+1}(q)$	$2\nmid q,\ q>3,\ n\geqslant 4$	$\mathscr{P}_1$	$\mathscr{N}_1^-$	$\mathscr{N}_1^+$
$P\Omega_{2n}^+(2)$	$n\geqslant 8$	$\mathscr{N}_1$	$\mathscr{P}_1$	$\mathscr{N}_2^-$
$P\Omega_{2n}^+(3)$	$n\geqslant 8$	$\mathscr{N}_1$	$\mathscr{P}_1$	$\mathscr{P}_2$
$P\Omega_{2n}^+(q)$	$q>3,\ n\geqslant 8$	$\mathscr{P}_1$	$\mathscr{N}_1$	$\mathscr{P}_2$
$P\Omega_{2n}^-(q)$	$n\geqslant 5$	$\mathscr{P}_1$	$\mathscr{N}_1$	$\mathscr{P}_2$

† Obtained from Tables VII and VIII of Ref. 17.

‡ $\mathscr{P}_1=(q^{2n}-1)/(q-1)$, $\mathscr{P}_2=(q^{2n}-1)(q^{2n-1}-1)/(q-1)(q^2-1)$, $\mathscr{P}_n=(q+1)(q^2+1)\cdots(q^n+1)$ for $PSp_{2n}(q)$ and $\Omega_{2n+1}(q)$; $\mathscr{P}_1=(q^n-\varepsilon)(q^{n-1}+\varepsilon)/(q-1)$, $\mathscr{P}_2=(q^n-\varepsilon)(q^{2n-2}-1)(q^{n-2}+\varepsilon)/(q-1)(q^2-1)$ for $P\Omega_{2n}^{\varepsilon}(q)$; $\mathscr{N}_1=q^{2n-2}(q^{2n}-1)(q^2-1)$, $\dim U=2$ for $PSp_{2n}(q)$; $\mathscr{N}_1^+=q^n(q^n+1)/2$, $\mathscr{N}_1^-=q^n(q^n-1)/2$ for $\Omega_{2n+1}(q)$; $\mathscr{N}_1=q^{n-1}(q^n-\varepsilon)/\gcd(2,q-1)$ for $P\Omega_{2n}^{\varepsilon}(q)$; $\mathscr{N}_2^-=2^{2n-3}(2^n-1)(2^{n-1}-1)/3$ for $\Omega_{2n}^+(2)$.

implies that $1=(2^n-1)[(2^{n-1}+1)/(2-1)-2^{n-1}]$ and so $n=1$, a contradiction.

Similarly we can show that $P\Omega_{2n}^+(3)$ does not satisfy Hypothesis (C).

Case 7: $P\Omega_{2n}^+(q)$ for $q>3$, $n\geqslant 8$ or $P\Omega_{2n}^-(q)$ for $n\geqslant 5$.

Then $d_1(G)=\frac{(q^n-1)(q^{n-1}+1)}{q-1}$, $d_2(G)=q^{n-1}(q^n-1)/\gcd(2,q-1)$, $d_3(G)=\frac{(q^n-1)(q^{2n-2}-1)(q^{n-1}+1)}{(q-1)(q^2-1)}$ for $\varepsilon=+$, and $d_1(G)=\frac{(q^n+1)(q^{n-1}-1)}{q-1}$, $d_2(G)=q^{n-1}(q^n+1)/\gcd(2,q-1)$, $d_3(G)=\frac{(q^n+1)(q^{2n-2}-1)(q^{n-1}-1)}{(q-1)(q^2-1)}$ for $\varepsilon=-$. By the hypothesis, $d_3(G)-d_1(G)=2$ implies that

$$2=\frac{(q^n-1)(q^{n-1}+1)}{q-1}\left(\frac{q^{2n-2}-1}{q^2-1}-1\right)$$

and

$$2=\frac{(q^n+1)(q^{n-1}-1)}{q-1}\left(\frac{q^{2n-2}-1}{q^2-1}-1\right).$$

It is easy to see that these equations have no solution in $\mathbb{N}$ since $n\geqslant 5$.

Case 4: exceptional simple groups of type ${}^2E_n$, E_n, F_4, or ${}^2F_4$.

For ${}^2E_6(q)$, $d_1(G)=q(q^6-q^3+1)(q^4+1)$, $d_2(G)=(q^2+q+1)(q^6-q^3+1)(q^4-q^2+1)(q^2-q+1)^2$, $d_3(G)=q^2(q^2+1)(q^4-q^3+q^2-q+1)(q^4+1)(q^4-q^2+1)$, and $d_2(G)-d_1(G)=1$ implies that $1=(q^6-q^3+1)[(q^2+q+1)(q^4-q^2+1)(q^2-q+1)^2-q(q^6-q^3+1)(q^4+1)]$ has no solution in $\mathbb{N}$.

For $E_6(q)$, $d_1(G)=q(q^6+q^3+1)(q^4+1)$, $d_2(G)=q^2(q^2+1)(q^4+q^3+q^2+q+1)(q^4+1)(q^4-q^2+1)$, $d_3(G)=(q^2-q+1)(q^6+q^3+1)(q^4-q^2+1)(q^2+q+1)^2$ and since $d_3(G)-d_2(G)=1$, we obtain that $1=(q^4-q^2+1)[(q^2-q+1)(q^6+q^3+1)(q^2+q+1)^2-q^2(q^2+1)(q^4+q^3+q^2+q+1)(q^4+1)$, so it has no solution in $\mathbb{N}$.

For $E_7(q)$, $d_1(G)=q(q^4-q^2+1)(q^6+q^5+q^4+q^3+q^2+q+1)(q^6-q^5+q^4-q^3+q^2-q+1)$, $d_2(G)=q^2(q^4-q^2+1)(q^6+q^3+1)(q^6-q^3+1)(q^2-q+1)^2(q^2+q+1)^2$, $d_3(G)=(q^2+q+1)(q^4+q^3+q^2+q+1)(q^6+q^3+1)(q^6+q^5+q^4+q^3+q^2+q+1)(q^6-q^5+q^4-q^3+q^2-q+1)(q-1)^3$. By hypothesis, $1=d_3(G)-d_2(G)$, and so it has no solution in $\mathbb{N}$ since the degree of the determinant $d_3(G)-d_2(G)$ is equal to 27.

For $E_8(q)$, $d_1(G)=q(q^4-q^2+1)(q^8-q^6+q^4-q^2+1)(q^4+1)(q^8-q^4+1)(q^2+1)^2$, $d_2(G)=q^2(q^8-q^6+q^4-q^2+1)(q^4+q^3+q^2+q+1)(q^4-q^3+q^2-q+1)(q^6+q^5+q^4+q^3+q^2+q+1)(q^6-q^5+q^4-q^3+q^2-q+1)(q^8-q^7+q^5-q^4+q^3-q+1)(q^8+q^7-q^5-q^4-q^3+q+1)$, $d_3(G)=(q^4-q^2+1)(q^2-q+1)(q^2+q+1)(q^8-q^6+q^4-q^2+1)(q^4+q^3+q^2+q+1)(q^4-q^3+q^2-q+1)(q^4+1)(q^8-q^4+1)(q^8-q^7+q^5-q^4+q^3-q+1)(q^8+q^7-q^5-q^4-q^3+q+1)(q^2+1)^2$. By the hypothesis, $d_3(G)-d_2(G)=1$, and considering the degree of $d_3(G)-d_2(G)$, we rule out this case.

For $F_4(q)$, $d_1(G)=(q^2+q+1)(q^2-q+1)(q^4-q^2+1)$, $d_2(G)=1/2q(q^4+1)(q-1)^2(q^2+q+1)^2$, $d_3(G)=1/2q(q^2+1)(q^4-q^2+1)(q^4+1)$ for $q\equiv 1,3,5,7,9,11 \pmod{12}$, and $d_1(G)=1/2q(q^4+1)(q-1)^2(q^2+q+1)^2$, $d_2(G)=1/2q(q^2+1)(q^4+1)(q^4-q^2+1)$, $d_3(G)=1/2q(q^4+1)(q+1)^2(q^2-q+1)^2$ for $q\equiv 2,4,8 \pmod{12}$. If the former, considering the degree of

$d_2(G) - d - 1(G)$, we rule out this case. If the latter, then $1 = d_2(G) - d_1(G) = (q^4+1)[1/2q(q^2+1)(q^4 - q^2+1) - 1/2q(q-1)^2(q^2+q+1)^2]$ has no solution in $\mathbb{N}$.

For ${}^2F_4(q^2)$, $d_1(G) = 1/2\sqrt{2}(-q^2+q^4+1)(q+1)(q-1)q(q^2+1)^2$, $d_2(G) = (-q^2+q^4+1)(q^8-q^4+1)q^2$, $d_3(G) = (q-1)(q+1)(q^8-q^4+1)(q^4+1)^2$. Since $d_3(G) - d_2(G)$ is a determinant of degree 18, the equation has no solution in $\mathbb{N}$.

In this case, there is no group satisfying Hypothesis (C).

Case 5: exceptional simple groups of type ${}^3D_4(q)$.

Now $d_1(G) = q(q^4-q^2+1)$, $d_2(G) = 1/2q^3(q^4-q^2+1)(q-1)^2$, $d_3(G) = 1/2q^3(q-1)^2(q^2+q+1)^2$ if $q \equiv 0 \pmod 2$; $d_1(G) = q(q^4-q^2+1)$, $d_2(G) = (q^2-q+1)(q^2+q+1)(q^4-q^2+1)$, $d_3(G) = 1/2q^3(q^4-q^2+1)(q-1)^2$ if $q \equiv 1 \pmod 2$. It is easy to see that there does not exist a prime power q such that $d_1(G)+1 = d_2(G)$.

Case 6: exceptional simple groups of type ${}^2G_2(q^2)$ or $G_2(q)$.

For ${}^2G_2(q^2)$, $d_1 = (q^2+1-q\sqrt{3})(q^2+1+q\sqrt{3})$, $d_2 = 1/6\sqrt{3}q(q-1)(q+1)(q^2+1-q\sqrt{3})$, $d_3 = 1/6\sqrt{3}q(q-1)(q+1)(q^2+1+q\sqrt{3})$. Considering $d_3 - d_1 = 2$, we can rule out this case.

For $G_2(q)$, we have five cases.

Case 1: $d_1 = (q+1)(q^2-q+1)$, $d_2 = (q^2+q+1)(q^2-q+1)$, $d_3 = 1/6q(q^2-q+1)(q-1)^2$ for $q \equiv 1 \pmod 6$.

Case 2: $d_1 = (q-1)(q^2+q+1)$, $d_2 = 1/6q(q^2-q+1)(q-1)^2$, $d_3 = 1/6q(q^2+q+1)(q+1)^2$ for $q \equiv 2 \pmod 6$.

Case 3: $d_1 = (q^2+q+1)(q^2-q+1)$, $d_2 = 1/6q(q^2-q+1)(q-1)^2$, $d_3 = 1/6q(q^2+q+1)(q+1)^2$ for $q \equiv 3 \pmod 6$.

Case 4: $d_1 = (q+1)(q^2-q+1)$, $d_2 = 1/6q(q^2-q+1)(q-1)^2$, $d_3 = 1/6q(q^2+q+1)(q+1)^2$ for $q \equiv 4 \pmod 6$.

Case 5: $d_1 = (q-1)(q^2+q+1)$, $d_2 = (q^2+q+1)(q^2-q+1)$, $d_3 = 1/6q(q^2-q+1)(q-1)^2$ for $q \equiv 5 \pmod 6$.

We will consider only case 1 because the other cases can be considered in a similar manner. By the hypothesis, $d_2(G) - d_1(G) = 1$, and so $1 = (q^2-q+1)q(q-2)$. It is easy to see that it has no solution in $\mathbb{N}$ since q is a prime-power.

Case 6: exceptional simple groups of type ${}^2B_2(q)$.

We can obtain from Ref. 16 that $d_1(G) = \sqrt{2}q(q^2-1)/2$, $d_2(G) = (q^2-1)(q^2-\sqrt{2}q+1)$, and $d_3(G) = q^4$. By the hypothesis, $1 = (q^2-1)[(q^2-\sqrt{2}q+1) - \sqrt{2}q/2]$, and it is easy to see that the equation has no solution in $\mathbb{N}$. □

Lemma 5 *An odd-order group G does not satisfy Hypothesis (C).*

Proof: Assume that the result is wrong. Then the $d_i(G)$ are odd. But by the hypothesis, $d_2(G) = 1 + d_1(G)$ is even. By Theorem 3.11 of Ref. 4, there is a character $\chi \in \mathrm{Irr}(G)$ such that $d_2(G) = \chi(1)$ divides $|G|$. It follows that G is a group of an even order, a contradiction. □

Theorem 4 *Let G be a nonabelian simple group with Hypothesis (C). Then G is isomorphic to $L_2(2^m)$ for $m \geqslant 2$.*

Proof: By Classification Theorem of Finite Simple Groups, G is isomorphic to an alternating group, a simple sporadic group or a simple group of Lie type.

Case 1: if G is isomorphic to a simple group of Lie type, then G is isomorphic to $L_2(2^m)$ with $m \geqslant 2$.

By Lemma 4, G is isomorphic to $L_2(2^m)$ with $m \geqslant 2$.

Case 2: if G is isomorphic to a simple sporadic group, then there exists no group satisfying Hypothesis (C).

By Lemma 3, we can see that there does not exist a group satisfying Hypothesis (C).

Case 3: if G is isomorphic to an alternating group, then $G \cong A_5$ is the desired result.

If $n \geqslant 9$, by Lemma 2, no group occurs.

If $n \leqslant 8$, then by Ref. 15, we obtain $G \cong A_5 \cong L_2(5) \cong L_2(4)$, the needed result. □

Remark 4 There are some groups of an odd order whose three odd smallest degrees are consecutive. For instance, $\mathrm{cd}(M) = \{1, n\}$, $\mathrm{cd}(N) = \{1, n+2\}$ and $\mathrm{cd}(P) = \{1, n+4\}$ where $n > 1$ is an odd number, then $n, n+2, n+4 \in \mathrm{cd}(M \times N \times P)$.

PROOF OF Theorem 3

Lemma 6 *Assume that S_i are nonabelian simple groups and $N = S_1 \times \cdots \times S_l$ is normal in G. Then for all $m \in \mathrm{cd}(S_i)$ and $n \in \mathrm{cd}(S_j)$ with $j \neq i$, $mn \in \mathrm{cd}(G)$.*

Proof: If $m \in \mathrm{cd}(S_i)$ and $n \in \mathrm{cd}(S_j)$, then there are irreducible characters $\chi \in \mathrm{Irr}(S_i)$ and $\beta \in \mathrm{Irr}(S_j)$ such that $\chi(1) = m$ and $\beta(1) = n$. By Theorem 4.21 of Ref. 4, $\chi\beta \in \mathrm{Irr}(N)$ and so $(\chi\beta)(1) = mn \in \mathrm{cd}(N) \subseteq \mathrm{cd}(G)$. □

Proof of Theorem 3

Proof: By hypothesis, $d_i(G) = d_i(L)$ for all $i \in \{1, 2, 3\}$. By Theorem 4, $L \cong L_2(2^m)$ with $m \geqslant 2$ or

$L_2(5)$. Let $q = 2^{2m}$ with $m \geqslant 2$. Then we have that

$$|G| = \frac{1}{2}(q-1)q(q+1)$$

and

$$d_1(G) = q-1, \quad d_2(G) = q, \quad d_3(G) = q+1.$$

It follows that there are irreducible characters χ_i such that $\chi_i(1) = d_i$ for $i \in \{1,2,3\}$. Since G is almost simple, then by Lemma 6,

$$G/K \cong L_2(q)$$

for some subgroup $K \trianglelefteq G$. If $|G| = |L|$, then $K = 1$ and $G \cong L$. □

Theorem 5 *There is no nonabelian simple group G such that $d_i(G)$ are consecutive for all $i \in \{1,2,\dots,n\}$ with $n \geqslant 4$.*

Proof: Assume that there is a nonabelian simple group such that $d_i(G)$ are consecutive for all $i \in \{1,2,\dots,n\}$ with $n \geqslant 4$. Then the group satisfies Hypothesis (C). Now by Theorem 4, G is isomorphic to $L_2(2^m)$ for $m \geqslant 2$. Since $|\mathrm{cd}(L_2(2^m))| = 4$ and $|\mathrm{cd}(G)| \geqslant 5$, we obtain a contradiction. □

APPLICATIONS

We can also obtain from the character degree graph some information about the character degrees of finite groups. As for the character degrees, Huppert[20] illustrated the following conjecture.

Conjecture Let G be a finite group, and let H be a finite nonabelian simple group such that the set of character degrees of G and H are the same. Then $G \cong H \times A$, where A is an abelian group.

Note that $L_2(4) \cong L_2(5)$. As an application of Theorem 3, we prove that the conjecture is true for the groups with Hypothesis (C).

Theorem 6 *Let G be a finite group and $H = L_2(2^m)$. If $\mathrm{cd}(G) = \mathrm{cd}(H)$, then $G \cong H \times A$ with A an abelian group.*

Proof: Since $\mathrm{cd}(G) = \{1, 2^m-1, 2^m, 2^m+1\}$ and $(2^m-1, 2^m+1) = 1$, we have that the degree graph $\Gamma(G)$ has three connected components, i.e., $s(G) = 3$. If G is solvable, then by Ref. 21, $s(G) \leqslant 2$, a contradiction. Thus G is insolvable. Clearly G has Hypothesis (C). Then by Lemma 1 of Ref. 22, there is a normal series $1 \trianglelefteq K \trianglelefteq H \trianglelefteq G$ such that

$$H/K \cong \underbrace{L_2(2^m) \times \cdots \times L_2(2^m)}_{n}$$

for a positive integer n. Assume that $n > 1$. Recall that $d_1 = 2^m - 1$, $d_2 = 2^m$, and $d_3 = 2^m + 1$. Now Lemma 6 implies that for all $d_i \in \mathrm{cd}(L_2(2^m))$, $d_i d_j \in \mathrm{cd}(G)$, a contradiction to the hypothesis. Thus we have $n = 1$, hence H/K is isomorphic to $L_2(2^m)$. It follows from that $L_2(2^m) \leqslant G/K \leqslant SL_2(2^m)$. Note that $L_2(2^m) \cong SL_2(2^m)$. Then $G/K \cong L_2(2^m)$ with $m \geqslant 2$.

Now let $\mathfrak{X}$ be the set of all direct product of mutually non-isomorphic nonabelian simple groups, and let $\mathfrak{D}$ be the set of all solvable groups. Clearly $\mathfrak{X}$ and $\mathfrak{D}$ satisfy Corollary 9.28 of Ref. 23. Now for each $G \in \mathfrak{X}$, $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is solvable. Then every extension of $L_2(2^m)$ by K is isomorphic to $K \times L_2(2^m)$, i.e., $G \cong L_2(2^m) \times K$. If K is nonabelian, then by Theorem 4.21 of Ref. 4, $md_i \in \mathrm{cd}(G)$ for $1 \neq m \in \mathrm{cd}(K)$, a contradiction. Thus K is abelian. □

Note that Theorem 6 is also proved by Huppert[20]. If we further consider group orders, then Theorem 6 implies the following result.

Theorem 7 *Let G be a finite group, and $H = L_2(2^m)$ with $m \geqslant 2$. Then $\mathrm{cd}(G) = \mathrm{cd}(H)$ and $|G| = |H|$ if and only if $G \cong H$.*

Proof: ($\Longrightarrow$) Since $\mathrm{cd}(G) = \{1, 2^m-1, 2^m, 2^m+1\}$, then G has consecutive three smallest degrees of characters, and so, by Theorem 3, G/K is isomorphic to $L_2(2^m)$. Since $|G| = |H|$ and Lemma 6, then we have that $K = 1$. Hence G is isomorphic to $L_2(2^m)$.

($\Longrightarrow$) Since G is isomorphic to H, then by Ref. 19, $\mathrm{cd}(G) = \mathrm{cd}(H)$ and $|G| = |H|$, the desired result. □

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